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Research Article

Existence of Solutions for Fractional-Order Neutral Differential Inclusions with Impulsive and Nonlocal Conditions

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This paper investigates the existence of solutions for fractional-order neutral impulsive differential inclusions with nonlocal conditions. Utilizing the fractional calculus and fixed point theorem for multivalued maps, new sufficient conditions are derived for ensuring the existence of solutions. The obtained results improve and generalize some existed results. Finally, an illustrative example is given to show the effectiveness of theoretical results.

1. Introduction

This paper deals with the existence of solutions for the following fractional-order differential inclusions with impulsive nonlocal conditions:

$$\begin{aligned} &{}^{c}D^{q}\left[x(t)-g(t,x(t))\right]\in F(t,x(t)),\quad t\in I=[0,T],\ t\neq t_k,\\ &\Delta x|_{t=t_k}\in I_k\left(x\left(t_k^{-}\right)\right),\quad k=1,\ldots,m,\ 0<t_1<\cdots<t_m<T,\\ &x(0)=x_0+h(x), \end{aligned} \tag{1.1}$$

where $0<q<1$ and ${}^{c}D^{q}$ is the Caputo fractional derivative. $I_k : R \to \mathcal{P}(R)$ $(k = 1,\ldots,m,)$ and $F : I \times R \to \mathcal{P}(R)$ are multivalued maps ($\mathcal{P}(R)$ is the family of all nonempty subsets

of R). $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$, $x(t_k^-)$ represents the right and left limits of $x(t)$ at $t = t_k$, respectively. g, h are given functions to be specified later.

The nonlocal problem was more general and has better effect than the classical Cauchy problems. So it has been studied extensively under various conditions in the literature [1–7] and the references contained therein. In [2], Byszewski and Lakshmikantham considered the existence and uniqueness of mild solutions when f and g satisfied Lipschitz conditions. In [3], Ntouyas and Tsamatos studied the case of compactness conditions of g and $T(t)$. Liang et al. in [5] discussed the nonlinear nonlocal Cauchy problems when $T(t)$ was compact and g was Lipschitz and g was not Lipschitz and not compact, respectively. Xue in [6] established existence results of mild solutions for semilinear differential equations under various conditions on f, g, and $T(t)$. Subsequently in [7], the author examined the semilinear differential equations when $T(t)$ was compact and g failed to be compact.

The theory of impulsive differential equations and differential inclusions has received much attention for the past decades because of its wide applicability in control, electrical engineering, mechanics, biology, and so on. For more details on this theory and applications, we refer to the monograph of Lakshmikantham et al. [8], Samoilenko and Perestyuk [9] and the references therein. In [10], Abada et al. discussed the impulsive differential inclusions with delays when A was the infinitesimal generator of a strongly continuous semigroup $T(t)$. In the following work [11], they obtained some existence and controllability results when A was a nondensely defined closed linear operator.

Since fractional-order differential equations have proved to be valuable tools in the modeling of many phenomena in physics and technical sciences, differential equations involving Riemann-Liouville as well as Caputo derivatives have been investigated extensively in the last decades (see [12–28] and the references therein). In [13], Mophou obtained the existence and uniqueness of mild solutions for the semilinear impulsive fractional differential equations by means of the Schauder fixed point theorem in Banach spaces. In [18], Benchohra and Slimani established sufficient conditions for the existence of solutions for a class of initial value problem for impulsive fractional differential equations. In [21], Chang and Nieto studied the existence of solutions for a class of fractional differential inclusions with boundary conditions by means of Bohnenblust-Karlin fixed point theorem. Balachandran et al. in [25] investigated fractional-order impulsive integrodifferential equations in Banach spaces. In [27], Agarwal and Ahmad studied the existence of solutions for nonlinear fractional differential equations and inclusions of order $q \in (3,4]$ with antiperiodic boundary conditions. The existence results were established for convex as well as nonconvex multivalued maps by some fixed point theorems, such as Leray-Schauder degree theory, and nonlinear alternative of Leray-Schauder type.

However, there is little information in the literature on neutral fractional-order impulsive differential inclusions with nonlocal conditions. Motivated by works mentioned above, we consider the existence results of (1.1) by using fixed point theorem for multivalued maps.

This paper is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary results which will be used throughout the paper. In Section 3, based on the fractional calculus and fixed point theorem for multivalued maps, we will prove the existence of solution to the problem (1.1). In Section 4, An example is given to show the effectiveness of our abstract results.

2. Preliminaries

In this section, we introduce some definitions, notes, and preliminary facts which will be used in this paper.

Let $C(I,R)$ be a Banach space, with the norm $\|x\|_C = \sup\{|x(t)|, t \in I\}$. Let $L^p(I,R)$ denote the space of Bochner integrable functions on I with the norm $\|x\|_{L^p} = (\int_0^T |x(t)|^p dt)^{1/p}$.

Let $(X, \|\cdot\|)$ be a Banach space. $P(X)$ is the family of all nonempty subsets of X. $P_{bd}(X), P_{cl}(X), P_{bd,cl}(X), P_{cp,cv}(X)$, and $P_{bd,cl,cv}(X)$ denote, respectively, the family of all nonempty bounded, closed, bounded-closed, compact-convex, and bounded-closed-convex subsets of X. A multivalued map $G : X \to P(X)$ is *convex* (*closed*) valued if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_{bd}(X)$.

A mapping G is called *upper semicontinuous (u.s.c.)* on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X and if, for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is *completely continuous* if $G(V)$ is relatively compact for every $V \in P_{bd}(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

A mapping G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$. A multivalued map $G : I \to P_{cl}(X)$ is said to be *measurable* if, for each $x \in X$, the function

$$t \longmapsto d(x, G(t)) = \inf\{|x - z| : z \in G(t)\} \tag{2.1}$$

is measurable.

Definition 2.1. A multivalued map $F : I \times X \to \mathcal{P}(X)$ is L^1-Carathéodory if

(i) $t \mapsto F(t,u)$ is measurable for each $u \in X$;

(ii) $u \mapsto F(t,u)$ is upper semicontinuous for almost all $t \in I$;

(iii) for each $q > 0$, there exists $\phi_q \in L^1(I, R^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t,u)\} \le \phi_q(t), \tag{2.2}$$

for all $\|u\| \le q$ and a.e. $t \in I$.

Definition 2.2. A multivalued operator $N : X \to P_{cl}(X)$ is called

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad x, y \in X; \tag{2.3}$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

The key tool in our approach is the following fixed point theorem.

Lemma 2.3 (see [29]). *Let $B(0,r)$ and $B[0,r]$ denote, respectively, the open and closed balls in a Banach space X centered at origin and of radius r, and let $\Phi_1 : X \to P_{\mathrm{bd,cl,cv}}(X)$ and $\Phi_2 : B[0,r] \to P_{cp,cv}(X)$ be two multivalued operators satisfying the following:*

(a) Φ_1 *is a contraction,*

(b) Φ_2 *is completely continuous.*

Then either

(i) *the operator inclusion $x \in \Phi_1 x + \Phi_2 x$ has a solution in $B[0,r]$ or*

(ii) *there exists an $u \in X$ with $\|u\| = r$ such that $\lambda u \in \Phi_1 u + \Phi_2 u$, for some $\lambda > 1$.*

Lemma 2.4 (see [30]). *Let I be a compact real interval and X a Banach space. Let F be an L^1-Caratheodory multivalued mapping with compact convex values with $S_{F,y} \neq \emptyset$, where $S_{F,y} = \{f \in L^1(I,X) : f(t) \in F(t,y(t)), a.e.t \in I\}$ and let $\Gamma : L^1(I,X) \to C(I,X)$ be a linear continuous mapping. Then the operator*

$$\Gamma \circ S_F : C(I,X) \longrightarrow P_{\mathrm{cp,cv}}(C(I,X)), \qquad x \longmapsto (\Gamma \circ S_F)(x) = \Gamma(S_{F,x}) \tag{2.4}$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

For more details on multivalued map, see the books of Aubin and Cellina [31], Deimling [32].

Now, we recall some definitions and facts about fractional derivatives and fractional integrals of arbitrary order, see [33].

Definition 2.5. The Riemann-Liouville fractional integral operator of order $q > 0$ of a function h is defined by

$$I_a^q h(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} h(s)\,ds, \tag{2.5}$$

where Γ is the gamma function.

Definition 2.6. The Caputo fractional-order derivative of order $q > 0$ of a function $h(t)$ is defined by

$$\left({}^c D_{a+}^q h\right)(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} h^{(n)}(s)\,ds, \quad n-1 < q < n,\ n \in \mathbb{N}^+. \tag{2.6}$$

Lemma 2.7 (see [34]). *Let $q > 0$; then the differential equation*

$${}^c D^q y(t) = h(t) \tag{2.7}$$

has solutions $y(t) = I^q h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_i \in R$, $n = [q] + 1$.

Lemma 2.8 (see [34]). *Let $\alpha > 0$; then*

$$I^{\alpha c}D^{\alpha}h(t) = h(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1}t^{n-1} \tag{2.8}$$

for some $a_i \in R,\ i = 0, 1, 2, \ldots,\ n-1,\ n = [\alpha] + 1$.

3. Main results

In this section, main results are presented.

Let $PC(I,R) = \{x : I \to R : x \in C((t_{k-1}, t_k], R),\ k = 0, 1, \ldots, m$, and there exist $x(t_k^-),\ x(t_k^+),\ k = 1, \ldots, m$ with $x(t_k^-) = x(t_k)\}$. This set is a Banach space with the norm $\|x\|_{PC} = \sup_{t\in I}|x(t)|$.

As a consequence of Lemmas 2.7 and 2.8, we can define the solution of problem (1.1).

Definition 3.1. The function $x \in PC(I,R)$ is said to be a solution of the problem (1.1) if $x(0) = x_0 + h(x)$, and there exist $f(t) \in S_{F,x}$, $J_k \in I_k(x(t_k^-))$ such that integral equation

$$\begin{aligned} x(t) = {} & x_0 + h(x) - g(0, x(0)) + g(t, x(t)) + \frac{1}{\Gamma(q)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i - s)^{q-1}f(s)ds \\ & + \frac{1}{\Gamma(q)}\int_{t_i}^{t}(t-s)^{q-1}f(s)ds + \sum_{0<t_i<t}J_k, \quad t \in I, \end{aligned} \tag{3.1}$$

is satisfied.

For the study of the system (1.1), the following hypotheses are given:

(H1) F is a Carathéodory map with compact convex values, and for each fixed $x \in R,\ S_{F,x}$ is nonempty;

(H2) there exist a function $a(\cdot) \in L^p(I, R^+)$, where $p > 1/q,\ p \in \mathbb{N}^+$ and a nondecreasing continuous function $\Omega : R^+ \to R^+$ such that

$$\|F(t,x)\|_p = \sup\{|f(t)| : f(t) \in F(t, x(t))\} \le a(t)\Omega(\|x\|), \tag{3.2}$$

for $t \in I$ and each $x \in R$;

(H3) $g : I \times R \to R$ is Lipschtiz continuous with Lipschtiz constant l_g, and there exists $M > 0$ such that $|g(t, x(t))| \le M$;

(H4) $h : PC(I,R) \to R$ is Lipschtiz continuous with Lipschtiz constant l_h;

(H5) $I_k : R \to \mathcal{P}(R)$ $(k = 1, \ldots, m)$ are multivalued maps with bounded, closed, and convex values. And there exist $c_k > 0,\ k = 1, \ldots, m$, such that

$$H_d(I_k(y), I_k(x)) \le c_k\|y - x\|, \tag{3.3}$$

for all $x, y \in R$;

(H6) $C = ((p-1)/(pq-1))^{(p-1)/p} T^{q-1/p}$, $C_0 = |x_0| + |h(0)| + 2M + \sum_{k=1}^{m} |I_k(0)|$,

$$l_h + 2l_g + \sum_{k=1}^{m} c_k < 1. \tag{3.4}$$

The existence of solutions is now presented.

Theorem 3.2. *Assume that (H1)–(H6) are satisfied; then the problem* (1.1) *has at least one solution on I provided that there exists a real number $r > 0$ such that*

$$\frac{C_0\Gamma(q) + C(m+1)\Omega(r)\|a\|_{L^p}}{(1 - l_h - \sum_{k=1}^{m} c_k)\Gamma(q)} < r. \tag{3.5}$$

Proof. We transform the problem (1.1) into a fixed point problem by considering the multivalued operator $N(x) : PC(I,R) \to \mathcal{P}(PC(I,R))$:

$$\begin{aligned} N(x) = \Bigg\{ & \varphi \in PC(I,R) : \varphi(t) = x_0 + h(x) - g(0,x(0)) + g(t,x(t)) + \sum_{0<t_i<t} J_k \\ & + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f(s)ds + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} f(s)ds \Bigg\}, \end{aligned} \tag{3.6}$$

where $f \in S_{F,x}$, $J_k \in I_k(x(t_k^-))$, $k = 1, \ldots, m$.

Consider the multivalued operators α, β :

$$\begin{aligned} \alpha(x) &= \left\{ \varphi \in PC(I,R) : \varphi(t) = x_0 + h(x) - g(0,x(0)) + g(t,x(t)) + \sum_{0<t_i<t} J_k \right\}, \\ \beta(x) &= \left\{ \varphi \in PC(I,R) : \varphi(t) = \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f(s)ds + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} f(s)ds \right\}. \end{aligned} \tag{3.7}$$

It is clear that $N = \alpha + \beta$. This problem of finding a solution of (1.1) is reduced to find a solution of the operator inclusions $x \in \alpha(x) + \beta(x)$. In what follows, we aim to show that the operator $\alpha + \beta$ has a fixed point, which is a solution of the system (1.1). For this purpose, We will show that $\alpha + \beta$ satisfies all the conditions of Lemma 2.3. For better readability, the proof will be given in several steps.

Step 1. A priori bounded on solution: we show that the second assertion of Lemma 2.3 is not true.

Define an open ball $B(0,r) \in PC(I,R)$, where the real number r satisfies the inequality given in condition (3.5). Let x be a possible solution of $\lambda x \in \alpha(x)+\beta(x)$, for some real number $\lambda > 1$ with $\|x\| = r$. Then we have

$$\begin{aligned} x(t) \in \lambda^{-1}&\left(x_0 + h(x) - g(0,x(0)) + g(t,x(t)) + \sum_{0<t_i<t} J_k \right) \\ &+ \frac{\lambda^{-1}}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f(s)ds + \frac{\lambda^{-1}}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} f(s)ds. \end{aligned} \tag{3.8}$$

Hence by (H2)–(H5), we have

$$\begin{aligned} |x(t)| &\le \left| x_0 + h(x) - g(0,x(0)) + g(t,x(t)) + \sum_{0<t_i<t} J_k \right| \\ &\quad + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \left| f(s) \right| ds + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} \left| f(s) \right| ds \\ &\le |x_0| + |h(0)| + l_h \|x\| + 2M + \sum_{k=1}^{m} c_k \left| x\left(t_k^-\right) \right| + \sum_{k=1}^{m} |I_k(0)| \\ &\quad + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} a(s)\Omega(\|x\|)ds + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} a(s)\Omega(\|x\|)ds \\ &\le C_0 + \left(l_h + \sum_{k=1}^{m} c_k \right) \|x\| + \frac{C(m+1)\Omega(\|x\|)\|a\|_{L^p}}{\Gamma(q)}. \end{aligned} \tag{3.9}$$

Taking the supremum over t, we get

$$\|x\| \le C_0 + \left(l_h + \sum_{k=1}^{m} c_k \right) \|x\| + \frac{C(m+1)\Omega(\|x\|)\|a\|_{L^p}}{\Gamma(q)}. \tag{3.10}$$

Substituting $\|x\| = r$ in the above inequality yields

$$r \le \frac{C_0\Gamma(q) + C(m+1)\Omega(\|r\|)\|a\|_{L^p}}{(1 - l_h - \sum_{k=1}^{m} c_k)\Gamma(q)}, \tag{3.11}$$

which is a contradiction to (3.5).

Step 2. $\alpha : PC(I,R) \to \mathcal{P}(PC(I,R))$ is a contraction mapping.
Indeed, let $x_1, x_2 \in PC(I,R)$, and from (H5), we have

$$\begin{aligned} H_d(\alpha(x_1),\alpha(x_2)) &= H_d\Bigg(x_0 + h(x_1) - g(0,x_1(0)) + g(t,x_1(t)) + \sum_{0<t_i<t} I_k\big(x_1(t_k^-)\big), \\ &\qquad x_0 + h(x_2) - g(0,x_2(0)) + g(t,x_2(t)) + \sum_{0<t_i<t} I_k\big(x_2(t_k^-)\big)\Bigg) \\ &\le H_d(h(x_1),h(x_2)) + H_d\big(-g(0,x_1(0)),-g(0,x_2(0))\big) \\ &\quad + H_d\big(g(t,x_1(t)),g(t,x_2(t))\big) + H_d\Bigg(\sum_{0<t_i<t} I_k\big(x_1(t_k^-)\big), \sum_{0<t_i<t} I_k\big(x_2(t_k^-)\big)\Bigg) \\ &\le \Bigg(l_h + 2l_g + \sum_{k=1}^{m} c_k\Bigg)\|x_1 - x_2\|. \end{aligned} \tag{3.12}$$

Hence by (3.4), α is a contraction mapping.

Step 3. β has compact, convex values, and it is completely continuous. □

Claim 1. β is convex for each $x \in PC(I,R)$.

In fact, if φ_1 and φ_2 belong to $\beta(x)$, then there exist $f_1, f_2 \in S_{F,x}$ such that

$$\varphi_i(t) = \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f_i(s)\,ds + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} f_i(s)\,ds, \quad i = 1,2. \tag{3.13}$$

Let $0 \le \lambda \le 1$. Then, for each $t \in I$, we have

$$\begin{aligned} \big(\lambda\varphi_1 + (1-\lambda)\varphi_2\big)(t) &= \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1}\big(\lambda f_1(s) + (1-\lambda) f_2(s)\big)\,ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1}\big(\lambda f_1(s) + (1-\lambda) f_2(s)\big)\,ds. \end{aligned} \tag{3.14}$$

Since $S_{F,x}$ is convex (because F has convex values), we have

$$\lambda\varphi_1 + (1-\lambda)\varphi_2 \in \beta(x). \tag{3.15}$$

Claim 2. β maps bounded sets into bounded sets in $PC(I,R)$. Let $r > 0$, $B_r = \{x \in PC(I,R) : \|x\| \le r\}$ be a bounded set in $PC(I,R)$. For $x \in B_r$, $\varphi \in \beta(x)$, then there exists $f \in S_{F,x}$ such that

$$\varphi(t) = \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f(s)\,ds + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} f(s)\,ds. \tag{3.16}$$

Therefore,

$$\begin{aligned}
\left|\varphi(t)\right| &\leq \frac{1}{\Gamma(q)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\left|f(s)\right|ds+\frac{1}{\Gamma(q)}\int_{t_i}^{t}(t-s)^{q-1}\left|f(s)\right|ds\\
&\leq \frac{1}{\Gamma(q)}\sum_{k=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}a(s)\Omega(\|x\|)ds+\frac{1}{\Gamma(q)}\int_{t_i}^{t}(t-s)^{q-1}a(s)\Omega(\|x\|)ds \qquad (3.17)\\
&\leq \frac{C(m+1)\Omega(r)\|a\|_{L^p}}{\Gamma(q)}.
\end{aligned}$$

Thus

$$\|\varphi\| \leq \frac{C(m+1)\Omega(r)\|a\|_{L^p}}{\Gamma(q)} := m^*. \qquad (3.18)$$

Claim 3. β maps bounded sets into equicontinuous sets. Let $\tau_1, \tau_2 \in I$, $\tau_1 < \tau_2$, B_r be a bounded set of $PC(I,R)$ as in Claim 2 and $x \in B_r$. Then there exists $f \in S_{F,x}$ such that

$$\varphi(t) = \frac{1}{\Gamma(q)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}f(s)ds+\frac{1}{\Gamma(q)}\int_{t_i}^{t}(t-s)^{q-1}f(s)ds. \qquad (3.19)$$

By elementary computation, we have

$$\begin{aligned}
\left|\varphi(\tau_2)-\varphi(\tau_1)\right| &\leq \frac{1}{\Gamma(q)}\int_{t_i}^{\tau_1}\left|(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}\right|\left\|f(s)\right\|ds+\frac{1}{\Gamma(q)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}\left|f(s)\right|ds\\
&\leq \frac{\|a\|_{L^p}\Omega(r)}{\Gamma(q)}\left(\int_{t_i}^{\tau_1}\left[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}\right]^{p/(p-1)}ds\right)^{(p-1)/p}\\
&\quad+\frac{\|a\|_{L^p}\Omega(r)}{\Gamma(q)}\left(\int_{\tau_1}^{\tau_2}(\tau_2-s)^{(pq-p)/(p-1)}ds\right)^{(p-1)/p}\\
&\leq \frac{\|a\|_{L^p}\Omega(r)}{\Gamma(q)}\left(\frac{p-1}{pq-1}\right)^{(p-1)/p}\\
&\quad\times\left[(\tau_2-\tau_1)^{(pq-1)/(p-1)}-(\tau_2-t_i)^{(pq-1)/(p-1)}+(\tau_1-t_i)^{(pq-1)/(p-1)}\right]^{(p-1)/p}\\
&\quad+\frac{\|a\|_{L^p}\Omega(r)}{\Gamma(q)}\left(\frac{p-1}{pq-1}\right)^{(p-1)/p}(\tau_2-\tau_1)^{(pq-1)/p}.
\end{aligned} \qquad (3.20)$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem, we conclude that β is completely continuous.

Claim 4. β has a closed graph. Let $x_n \to x_*$, $\varphi_n \in \beta(x_n)$, and $\varphi_n \to \varphi_*$; we prove that $\varphi_* \in \beta(x_*)$. Now $\varphi_n \in \beta(x_n)$ means that there exists $f_n \in S_{F,x_n}$ such that

$$\varphi_n(t) = \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f_n(s)ds + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} f_n(s)ds, \quad t \in I. \tag{3.21}$$

We will prove that there exists $f_* \in S_{F,x_*}$ such that

$$\varphi_*(t) = \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f_*(s)ds + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} f_*(s)ds. \tag{3.22}$$

Consider the continuous operator $\kappa : L^1(I,X) \to C(I,X)$

$$(\kappa f)(t) = \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f(s)ds + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} f(s)ds. \tag{3.23}$$

From Lemma 2.4, it follows that $\kappa \circ S_F$ is a closed graph operator. Since $\varphi_n \in \kappa \circ S_{F,x_n}$, we have $\varphi_* \in \kappa \circ S_{F,x_*}$. That is, there exists $f_* \in S_{F,x_*}$ such that

$$\varphi_*(t) = \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f_*(s)ds + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} f_*(s)ds. \tag{3.24}$$

Therefore, β has a closed graph.

As a consequence of Lemma 2.3, the operator inclusion $x \in \alpha(x) + \beta(x)$ has a solution which implies that the problem (1.1) has a solution on I.

Remark 3.3. If we take $F(t,x) = \{f(t,x)\}$, where f is a continuous operator, then the result corresponds to a single-value problem. In this case, our result is also new in the present configuration.

Corollary 3.4. *Assume that (H1)–(H6) are satisfied; then the problem* (1.1) *has at least one solution on I provided that*

$$\|a\|_{L^p} < \liminf_{R \to +\infty} \frac{R\left(1 - l_h - \sum_{k=1}^{m} c_k\right) - C_0\Gamma(q)}{C(m+1)\Omega(R)}. \tag{3.25}$$

Proof. From (3.25), we know that there exists a constant $r > 0$, such that

$$\|a\|_{L^p} < \frac{r\left(1 - l_h - \sum_{k=1}^{m} c_k\right) - C_0\Gamma(q)}{C(m+1)\Omega(r)}. \tag{3.26}$$

Since the proof of Corollary 3.4 follows the very same lines as the proof of Theorem 3.2, we do not give details. □

Corollary 3.5. *Assume that (H1), (H3)–(H6), and the following condition are satisfied,*
(H2′) There exists a function $\rho(\cdot) \in L^p(I, R)$, such that

$$\|F(t,x)\|_{\mathcal{D}} = \sup\{|f(t)| : f(t) \in F(t,x(t))\} \leq \rho(t)\|x\|, \tag{3.27}$$

for $t \in I$ and each $x \in R$.
Then the problem (1.1) *has at least one solution on I provided that*

$$\|\rho\|_{L^p} < \frac{1 - l_h - \sum_{k=1}^{m} c_k}{C(m+1)}. \tag{3.28}$$

Proof. In this case, we take $a(t) = \rho(t)$, $\Omega(x) = x$ in Corollary 3.4.
At last we would like to discuss the impulsive condition.

(H5′) $I_k : R \to R$ $(k = 1, \ldots, m)$ are single continuous operator, and there exists $c_k > 0$, $k = 1, \ldots, m$, such that $|I_k(y) - I_k(x)| \leq c_k|y - x|$, for all $x, y \in R$. □

Corollary 3.6. *Assume that (H1)–(H4), (H5′), (H6) are satisfied; then the problem* (1.1) *has at least one solution on I provided that*

$$\|a\|_{L^p} < \liminf_{R \to +\infty} \frac{R(1 - l_h - \sum_{k=1}^{m} c_k) - C_0\Gamma(q)}{C(m+1)\Omega(R)}. \tag{3.29}$$

4. An Illustrative Example

In this section, an example is given to show the effectiveness of our theoretical result. Consider the following fractional-order differential inclusions with impulsive and nonlocal conditions:

$$\begin{aligned} {}^{c}D^{q}\left[x(t) - \frac{e^{-t}|x(t)|}{(9+e^{t})(1+|x(t)|)}\right] &\in F(t,x(t)), \quad t \in I = [0,1],\ t \neq \frac{1}{2}, \\ \Delta x|_{t=1/2} &= \frac{|x(1/2^{-})|}{3 + |x(1/2^{-})|}, \\ x(0) &= h(x), \end{aligned} \tag{4.1}$$

where $h(x) = (1/9)x(0) + (1/10)x(1)$, $g(t,x) = (e^{-t}x)/(9+e^{t})(1+x)$, $I_k(x) = x/(3+x)$, $x \in [0,\infty)$, and $F : I \times R \to P(R)$ is a multivalued map given by

$$x \longrightarrow F(t,x) = \left[\frac{e^{-2t}|x(t)|}{(9+e^{t})(1+|x(t)|)}, \frac{e^{-t}|x(t)|}{(9+e^{t})(1+|x(t)|)}\right]. \tag{4.2}$$

Thus,

$$\|F(t,x)\|_P = \sup\{|f(t)| : f(t) \in F(t,x(t))\} \le a(t)\Omega(\|x\|), \tag{4.3}$$

where $a(t) = 1/(9+e^t)$, $\Omega(\|x\|) = |x(t)|/(1+|x(t)|)$.

By computation, we have

$$\begin{gathered} |h(x)-h(y)| \le \frac{19}{90}|x-y|, \qquad |g(t,x)-g(t,y)| \le \frac{1}{10}|x-y|, \\ |I_k(x)-I_k(y)| \le \frac{1}{3}|x-y|, \qquad |g(t,x)| \le \frac{1}{10}. \end{gathered} \tag{4.4}$$

Suppose that $x_0 = h(0) = I_k(0) = 0$; we have $l_g = 1/10$, $l_h = 2/9$, $M = 1/10$, $c_k = 1/3$, $C_0 = 2M = 1/5$, $C = ((p-1)/(pq-1))^{(p-1)/p}$.

Clearly, for suitable $q \in (0,1)$, all the conditions of Corollary 3.6 are satisfied. So there exists at least one solution of problem (4.1) on I.

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