Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 403209, 11 pages
doi:10.1155/2012/403209

Research Article

On (α, β)-Derivations in BCI-Algebras

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Received 15 September 2012; Accepted 30 October 2012

Academic Editor: Cengiz Çinar

The notion of (regular) (α, β)-derivations of a BCI-algebra X is introduced, some useful examples are discussed, and related properties are investigated. The condition for a (α, β)-derivation to be regular is provided. The concepts of a $d_{(\alpha,\beta)}$-invariant (α, β)-derivation and α-ideal are introduced, and their relations are discussed. Finally, some results on regular (α, β)-derivations are obtained.

1. Introduction

BCK-algebras and BCI-algebras are two classes of nonclassical logic algebras which were introduced by Imai and Iséki in 1966 [1, 2]. They are algebraic formulation of BCK-system and BCI-system in combinatory logic. However, these algebras were not studied any further until 1980. Iséki published a series of notes in 1980 and presented a beautiful exposition of BCI-algebras in these notes (see [3–5]). The notion of a BCI-algebra generalizes the notion of a BCK-algebra in the sense that every BCK-algebra is a BCI-algebra but not vice versa (see [6]). Later on, the notion of BCI-algebras has been extensively investigated by many researchers (see [7–9] and references therein).

Throughout our discussion, X will denote a BCI-algebra unless otherwise mentioned. In the year 2004, Jun and Xin [10] applied the notion of derivation in ring and near-ring theory to BCI-algebras, and as a result they introduced a new concept, called a (regular) derivation, in BCI-algebras. Using this concept as defined, they investigated some of its properties. Using the notion of a regular derivation, they also established characterizations of a p-semisimple BCI-algebra. For a self map d of a BCI-algebra, they defined a d-invariant ideal and gave conditions for an ideal to be d-invariant. According to Jun and Xin, a self-map $d : X \to X$ is called a left-right derivation (briefly (l, r)-derivation) of X if $d(x * y) = d(x) * y \wedge x * d(y)$ holds for all $x, y \in X$. Similarly, a self-map $d : X \to X$ is called a right-left derivation (briefly (r, l)-derivation) of X if $d(x * y) = x * d(y) \wedge d(x) * y$ holds for all $x, y \in X$.

Moreover, if d is both (l, r)- and (r, l)-derivation, it is a derivation on X. After the work of Jun and Xin [10], many research articles have been appeared on the derivations of BCI-algebras and a greater interest has been devoted to the study of derivation in BCI-algebras on various aspects (see [11–15]).

Several authors [16–19] have studied derivations in rings and near-rings. Inspired by the notions of σ-derivation [20], left derivation [21] and generalized derivation [19, 22] in rings and near rings theory, many authors have applied these notions in a similar way to the theory of BCI-algebras (see [11, 14, 15]). For instant, in 2005 [15], Zhan and Liu have given the notion of f-derivation of BCI-algebras as follows: a self-map $d_f : X \to X$ is said to be a left-right f-derivation or (l, r)-f-derivation of X if it satisfies the identity $d_f(x * y) = d_f(x) * f(y) \wedge f(x) * d_f(y)$ for all $x, y \in X$. Similarly, a self map $d_f : X \to X$ is said to be a right-left f-derivation or (r, l)-f-derivation of X if it satisfies the identity $d_f(x * y) = f(x) * d_f(y) \wedge d_f(x) * f(y)$ for all $x, y \in X$. Moreover, if d_f is both (l, r) and (r, l)-f-derivation, it is said that d_f is an f-derivation where f is an endomorphism. In the year 2007, Abujabal and Al-Shehri [11] defined and studied the notion of left derivation of BCI-algebras as follows: a self-map $D : X \to X$ is said to be a left derivation of X if satisfying $D(x * y) = x * D(y) \wedge y * D(x)$ for all $x, y \in X$. Furthermore, in 2009 [14], Öztürk et al. have introduced the notion of generalized derivation in BCI-algebras. A self map $D : X \to X$ is called a generalized (l, r)-derivation if there exists an (l, r)-derivation $d : X \to X$ such that $D(x * y) = D(x) * y \wedge x * d(y)$ for all $x, y \in X$. If there exists an (r, l)-derivation $d : X \to X$ such that $D(x * y) = x * D(y) \wedge d(x) * y$ for all $x, y \in X$, the mapping $D : X \to X$ is called generalized (r, l)-derivation. Moreover, if D is both a generalized (l, r)- (r, l)-derivation, D is a generalized derivation on X.

In fact, the notion of derivation in ring theory is quite old and playsa significant role in analysis, algebraic geometry, and algebra. In his famous book "Structures of Rings" Jacobson [23] introduced the notion of (s_1, s_2)-derivation which was later more commonly known as (σ, τ) or (θ, ϕ)-derivation. After that a number of research articles have been appeared on (σ, τ) or (θ, ϕ)-derivations in the theory of rings (see [16, 24, 25] and references therein).

Motivated by the notion of (σ, τ) or (θ, ϕ)-derivation in the theory of rings, in the present paper, we introduce the notion of (α, β)-derivation in a BCI-algebra X and investigate related properties. We provide a condition for a (α, β)-derivation to be regular. We also introduce the concepts of a $d_{(\alpha,\beta)}$-invariant (α, β)-derivation and α-ideal, and then we investigate their relations. Furthermore, we obtain some results on regular (α, β)-derivations.

2. Preliminaries

We begin with the following definitions and properties that will be needed in the sequel.

A nonempty set X with a constant 0 and a binary operation $*$ is called a BCI-algebra if for all $x, y, z \in X$ the following conditions hold:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Define a binary relation $\leq$ on X by letting $x * y = 0$ if and only if $x \leq y$. Then $(X, \leq)$ is a partially ordered set. A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$, is called BCK-algebra.

A BCI-algebra X has the following properties: for all $x, y, z \in X$

(a1) $x * 0 = x$,

(a2) $(x * y) * z = (x * z) * y$,

(a3) $x \le y$ implies $x * z \le y * z$ and $z * y \le z * x$,

(a4) $(x * z) * (y * z) \le x * y$,

(a5) $x * (x * (x * y)) = x * y$,

(a6) $0 * (x * y) = (0 * x) * (0 * y)$,

(a7) $x * 0 = 0$ implies $x = 0$.

For a BCI-algebra X, denote by X_+ (resp. $G(X)$) the BCK-part (resp. the BCI-G part) of X, that is, X_+ is the set of all $x \in X$ such that $0 \le x$ (resp. $G(X) := \{x \in X \mid 0 * x = x\}$). Note that $G(X) \cap X_+ = \{0\}$ (see [26]). If $X_+ = \{0\}$, then X is called a p-semisimple BCI-algebra. In a p-semisimple BCI-algebra X, the following hold:

(a8) $(x * z) * (y * z) = x * y$,

(a9) $0 * (0 * x) = x$ for all $x \in X$,

(a10) $x * (0 * y) = y * (0 * x)$,

(a11) $x * y = 0$ implies $x = y$,

(a12) $x * a = x * b$ implies $a = b$,

(a13) $a * x = b * x$ implies $a = b$,

(a14) $a * (a * x) = x$.

Let X be a p-semisimple BCI-algebra. We define addition "+" as $x + y = x * (0 * y)$ for all $x, y \in X$. Then $(X, +)$ is an abelian group with identity 0 and $x - y = x * y$. Conversely let $(X, +)$ be an abelian group with identity 0 and let $x * y = x - y$. Then X is a p-semisimple BCI-algebra and $x + y = x * (0 * y)$ for all $x, y \in X$ (see [9]).

For a BCI-algebra X we denote $x \wedge y = y * (y * x)$, in particular $0 * (0 * x) = a_x$, and $L_p(X) := \{a \in X \mid x * a = 0 \Rightarrow x = a, \text{ for all } x \in X\}$. We call the elements of $L_p(X)$ the p-atoms of X. For any $a \in X$, let $V(a) := \{x \in X \mid a * x = 0\}$, which is called the branch of X with respect to a. It follows that $x * y \in V(a * b)$ whenever $x \in V(a)$ and $y \in V(b)$ for all $x, y \in X$ and all $a, b \in L_p(X)$. Note that $L_p(X) = \{x \in X \mid a_x = x\}$, which is the p-semisimple part of X, and X is a p-semisimple BCI-algebra if and only if $L_p(X) = X$ (see [27, Proposition 3.2]). Note also that $a_x \in L_p(X)$, that is, $0 * (0 * a_x) = a_x$, which implies that $a_x * y \in L_p(X)$ for all $y \in X$. It is clear that $G(X) \subset L_p(X)$, and $x * (x * a) = a$ and $a * x \in L_p(X)$ for all $a \in L_p(X)$ and all $x \in X$. A BCI-algebra X is said to be torsion free if $x + x = 0 \Rightarrow x = 0$ for all $x \in X$ [14]. For more details, refer to [7–10, 26, 27].

3. (α, β)-Derivations in BCI-Algebras

In what follows, α and β are endomorphisms of a BCI-algebra X unless otherwise specified.

Definition 3.1. Let X be a BCI-algebra. Then a self map $d_{(\alpha,\beta)} : X \to X$ is called a (α, β)-derivation of X if it satisfies:

$$(\forall x, y \in X) \quad \left(d_{(\alpha,\beta)}(x * y) = \left(d_{(\alpha,\beta)}(x) * \alpha(y)\right) \wedge \left(d_{(\alpha,\beta)}(y) * \beta(x)\right)\right). \tag{3.1}$$

Example 3.2. Consider a BCI-algebra $X = \{0, a, b\}$ with the following Cayley table:

$$\begin{array}{c|ccc} * & 0 & a & b \\ \hline 0 & 0 & 0 & b \\ a & a & 0 & b \\ b & b & b & 0 \end{array} \tag{3.2}$$

(1) Define a map

$$d_{(\alpha,\beta)} : X \longrightarrow X, \quad x \longmapsto \begin{cases} b & \text{if } x \in \{0, a\}, \\ 0 & \text{if } x = b, \end{cases} \tag{3.3}$$

and define two endomorphisms

$$\begin{aligned} \alpha : X \longrightarrow X, \quad x \longmapsto \begin{cases} 0 & \text{if } x \in \{0, a\}, \\ b & \text{if } x = b, \end{cases} \\ \beta : X \longrightarrow X, \quad x \longmapsto \begin{cases} 0 & \text{if } x \in \{0, b\}, \\ a & \text{if } x = a. \end{cases} \end{aligned} \tag{3.4}$$

It is routine to verify that $d_{(\alpha,\beta)}$ is a (α, β)-derivation of X.

(2) Define a map

$$d_{(\alpha,\beta)} : X \longrightarrow X, \quad x \longmapsto \begin{cases} 0 & \text{if } x \in \{0, b\}, \\ a & \text{if } x = a, \end{cases} \tag{3.5}$$

and define two endomorphisms

$$\begin{aligned} \alpha : X \longrightarrow X, \quad x \longmapsto \begin{cases} 0 & \text{if } x \in \{a, b\}, \\ b & \text{if } x = 0, \end{cases} \\ \beta : X \longrightarrow X, \quad x \longmapsto \begin{cases} 0 & \text{if } x \in \{0, a\}, \\ a & \text{if } x = b. \end{cases} \end{aligned} \tag{3.6}$$

It is routine to verify that $d_{(\alpha,\beta)}$ is a (α, β)-derivation of X.

Lemma 3.3 (see [8]). *Let X be a BCI-algebra. For any $x, y \in X$, if $x \leq y$, then x and y are contained in the same branch of X.*

Lemma 3.4 (see [8]). *Let X be a BCI-algebra. For any $x, y \in X$, if x and y are contained in the same branch of X, then $x * y, y * x \in X_+$.*

Proposition 3.5. *Let X be a commutative BCI-algebra. Then every (α,β)-derivation $d_{(\alpha,\beta)}$ of X satisfies the following assertion:*

$$(\forall x,y \in X) \quad (x \leq y \Longrightarrow d_{(\alpha,\beta)}(x) \leq d_{(\alpha,\beta)}(y)), \tag{3.7}$$

that is, every (α,β)-derivation of X is isotone.

Proof. Let $x,y \in X$ be such that $x \leq y$. Since X is commutative, we have $x = x \wedge y$. Hence

$$\begin{aligned} d_{(\alpha,\beta)}(x) &= d_{(\alpha,\beta)}(x \wedge y) \\ &= (d_{(\alpha,\beta)}(y) * \alpha(y * x)) \wedge (d_{(\alpha,\beta)}(y * x) * \beta(y)) \\ &\leq (d_{(\alpha,\beta)}(y) * \alpha(y * x)). \end{aligned} \tag{3.8}$$

Since every endomorphism of X is isotone, we have $\alpha(x) \leq \alpha(y)$. It follows from Lemma 3.3 that $0 = \alpha(x) * \alpha(y) \in X_+$ and $\alpha(y) * \alpha(x) \in X_+$ so that there exists $a(\neq 0) \in X_+$ such that $\alpha(y * x) = \alpha(y) * \alpha(x) = a$. Hence (3.8) implies that $d_{(\alpha,\beta)}(x) \leq d_{(\alpha,\beta)}(y) * a$. Using (a3), (a2), and (III), we have

$$\begin{aligned} d_{(\alpha,\beta)}(x) * d_{(\alpha,\beta)}(y) &\leq (d_{(\alpha,\beta)}(y) * a) * d_{(\alpha,\beta)}(y) \\ &= (d_{(\alpha,\beta)}(y) * d_{(\alpha,\beta)}(y)) * a = 0 * a = 0, \end{aligned} \tag{3.9}$$

and so $d_{(\alpha,\beta)}(x) * d_{(\alpha,\beta)}(y) = 0$, that is, $d_{(\alpha,\beta)}(x) \leq d_{(\alpha,\beta)}(y)$ by (a7). □

Example 3.6. In Example 3.2 (1), the (α,β)-derivation $d_{(\alpha,\beta)}$ does not satisfy the inequality (3.7).

Proposition 3.7. *Every (α,β)-derivation $d_{(\alpha,\beta)}$ of a BCI-algebra X satisfies the following assertion:*

$$(\forall x \in X) \quad (d_{(\alpha,\beta)}(x) = d_{(\alpha,\beta)}(x) \wedge d_{(\alpha,\beta)}(0)). \tag{3.10}$$

Proof. Let $d_{(\alpha,\beta)}$ be an (α,β)-derivation of X. Using (a2) and (a4), we have

$$\begin{aligned} d_{(\alpha,\beta)}(x) = d_{(\alpha,\beta)}(x * 0) &= (d_{(\alpha,\beta)}(x) * \alpha(0)) \wedge (d_{(\alpha,\beta)}(0) * \beta(x)) \\ &= (d_{(\alpha,\beta)}(x) * 0) \wedge (d_{(\alpha,\beta)}(0) * \beta(x)) \\ &= d_{(\alpha,\beta)}(x) \wedge (d_{(\alpha,\beta)}(0) * \beta(x)) \\ &= (d_{(\alpha,\beta)}(0) * \beta(x)) * ((d_{(\alpha,\beta)}(0) * \beta(x)) * d_{(\alpha,\beta)}(x)) \\ &= (d_{(\alpha,\beta)}(0) * \beta(x)) * ((d_{(\alpha,\beta)}(0) * d_{(\alpha,\beta)}(x)) * \beta(x)) \\ &\leq d_{(\alpha,\beta)}(0) * (d_{(\alpha,\beta)}(0) * d_{(\alpha,\beta)}(x)) \\ &= d_{(\alpha,\beta)}(x) \wedge d_{(\alpha,\beta)}(0). \end{aligned} \tag{3.11}$$

Obviously $d_{(\alpha,\beta)}(x) \wedge d_{(\alpha,\beta)}(0) \leq d_{(\alpha,\beta)}(x)$ by (II). Therefore, the equality (3.10) is valid. □

Theorem 3.8. *Let $d_{(\alpha,\beta)}$ be a (α,β)-derivation on a BCI-algebra X. Then*

(1) (*for all* $a \in L_p(X), x \in X$) $(d(a * x) = d(a) * \alpha(x))$,

(2) (*for all* $a \in L_p(X), x \in X$) $(d(a + x) = d(a) + \alpha(x))$,

(3) (*for all* $a, b \in L_p(X)$) $(d(a + b) = d(a) + \alpha(b))$.

Proof. (1) For any $a \in L_p(X)$, we have $a * x \in L_p(X)$ for all $x \in X$. Thus $d(a*x) = d(a)*\alpha(x) \wedge d(x) * \beta(a) = d(a) * \alpha(x)$.

(2) For any $a \in L_p(X)$ and $x \in X$, it follows from (1) that

$$\begin{aligned} d(a + x) &= d(a * (0 * x)) = d(a) * \alpha(0 * x) \\ &= d(a) * (\alpha(0) * \alpha(x)) = d(a) * (0 * \alpha(x)) \\ &= d(a) + \alpha(x). \end{aligned} \tag{3.12}$$

(3) The proof follows directly from (2). □

Definition 3.9. Let X be a BCI-algebra and $d_{(\alpha,\beta)}$, $d'_{(\alpha,\beta)}$ be two self maps of X, we define $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} : X \to X$ by $(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)})(x) = d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x))$ for all $x \in X$.

Theorem 3.10. *Let X be a p-semisimple BCI-algebra. If $d_{(\alpha,\beta)}$ and $d'_{(\alpha,\beta)}$ are two (α,β)-derivations on X such that $\alpha^2 = \alpha$. Then $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)}$ is a (α,β)-derivation on X.*

Proof. For any $x, y \in X$, it follows from (a14) that

$$\begin{aligned} \left(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)}\right)(x * y) &= d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(x * y)\right) \\ &= d_{(\alpha,\beta)}\left(\left(d'_{(\alpha,\beta)}(x) * \alpha(y)\right) \wedge \left(d'_{(\alpha,\beta)}(y) * \beta(x)\right)\right) \\ &= d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(x) * \alpha(y)\right) \\ &= \left(d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(x)\right) * \alpha(\alpha(y))\right) \wedge \left(d_{(\alpha,\beta)}(\alpha(y)) * \beta\left(d'_{(\alpha,\beta)}(x)\right)\right) \\ &= d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(x)\right) * \alpha(y) \end{aligned}$$

$$
\begin{aligned}
&= \left(d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(y) * \beta(x)\right)\right) * \left(\left(d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(y) * \beta(x)\right)\right)\right. \\
&\qquad\qquad \left. * \left(d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(x)\right) * \alpha(y)\right)\right) \\
&= \left(d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(x)\right) * \alpha(y)\right) \wedge \left(d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(y) * \beta(x)\right)\right) \\
&= \left(\left(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)}\right)(x) * \alpha(y)\right) \wedge \left(\left(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)}\right)(y) * \beta(x)\right).
\end{aligned} \tag{3.13}
$$

This completes the proof. □

Theorem 3.11. *Let α, β be two endomorphisms and $d_{(\alpha,\beta)}$ be a self map on a p-semisimple BCI-algebra X such that $d_{(\alpha,\beta)}(x) = \alpha(x)$ for all $x \in X$. Then $d_{(\alpha,\beta)}$ is a (α, β)-derivation on X.*

Proof. Let us take $d_{(\alpha,\beta)}(x) = \alpha(x)$ for all $x \in X$. Since $x, y \in X \Rightarrow x * y \in X$. Using (a14), we have

$$
\begin{aligned}
d_{(\alpha,\beta)}(x * y) &= \alpha(x * y) = \alpha(x) * \alpha(y) = d_{(\alpha,\beta)}(x) * \alpha(y) \\
&= \left(d_{(\alpha,\beta)}(y) * \beta(x)\right) * \left(\left(d_{(\alpha,\beta)}(y) * \beta(x)\right) * \left(d_{(\alpha,\beta)}(x) * \alpha(y)\right)\right) \\
&= \left(d_{(\alpha,\beta)}(x) * \alpha(y)\right) \wedge \left(d_{(\alpha,\beta)}(y) * \beta(x)\right).
\end{aligned} \tag{3.14}
$$

This completes the proof. □

Definition 3.12. A (α, β)-derivation $d_{(\alpha,\beta)}$ of a BCI-algebra X is said to be regular if $d_{(\alpha,\beta)}(0) = 0$.

Example 3.13. (1) The (α, β)-derivation $d_{(\alpha,\beta)}$ of X in Example 3.2 (1) is not regular.

(2) The (α, β)-derivation $d_{(\alpha,\beta)}$ of X in Example 3.2 (2) is regular.

We provide conditions for a (α, β)-derivation to be regular.

Theorem 3.14. *Let $d_{(\alpha,\beta)}$ be a (α, β)-derivation of a BCI-algebra X. If there exists $a \in X$ such that $d_{(\alpha,\beta)}(x) * \alpha(a) = 0$ for all $x \in X$, then $d_{(\alpha,\beta)}$ is regular.*

Proof. Assume that there exists $a \in X$ such that $d_{(\alpha,\beta)}(x) * \alpha(a) = 0$ for all $x \in X$. Then

$$
\begin{aligned}
0 = d_{(\alpha,\beta)}(x * a) * a &= \left(\left(d_{(\alpha,\beta)}(x) * \alpha(a)\right) \wedge \left(d_{(\alpha,\beta)}(a) * \beta(x)\right)\right) * a \\
&= \left(0 \wedge \left(d_{(\alpha,\beta)}(a) * \beta(x)\right)\right) * a = 0 * a,
\end{aligned} \tag{3.15}
$$

and so $d_{(\alpha,\beta)}(0) = d_{(\alpha,\beta)}(0 * a) = (d_{(\alpha,\beta)}(0) * \alpha(a)) \wedge (d_{(\alpha,\beta)}(a) * \beta(0)) = 0$. Hence $d_{(\alpha,\beta)}$ is regular. □

Definition 3.15. For a (α, β)-derivation $d_{(\alpha,\beta)}$ of a BCI-algebra X, we say that an ideal A of X is a α-ideal (resp. β-ideal) if $\alpha(A) \subseteq A$ (resp. $\beta(A) \subseteq A$).

Definition 3.16. For a (α, β)-derivation $d_{(\alpha,\beta)}$ of a BCI-algebra X, we say that an ideal A of X is $d_{(\alpha,\beta)}$-invariant if $d_{(\alpha,\beta)}(A) \subseteq A$.

Example 3.17. (1) Let $d_{(\alpha,\beta)}$ be a (α, β)-derivation of X which is described in Example 3.2 (1). We know that $A := \{0, a\}$ is both a α-ideal and a β-ideal of X. But $A := \{0, a\}$ is an ideal of X which is not $d_{(\alpha,\beta)}$-invariant.

(2) Let $d_{(\alpha,\beta)}$ be a (α, β)-derivation of X which is described in Example 3.2 (2). We know that $A := \{0, a\}$ is both a β-ideal and a $d_{(\alpha,\beta)}$-invariant ideal of X. But $A := \{0, a\}$ is not a α-ideal of X.

Next, we prove some results on regular (α, β)-derivations in a BCI-algebra.

Theorem 3.18. *Let $d_{(\alpha,\beta)}$ be a regular (α, β)-derivation of a BCI-algebra X. Then*

(1) *(for all* $a \in X$) $(a \in L_p(X) \Rightarrow d_{(\alpha,\beta)}(a) \in L_p(X))$,

(2) *(for all* $a \in X$) $(a \in L_p(X) \Rightarrow \alpha(a), \beta(a) \in L_p(X))$,

(3) *(for all* $a \in L_p(X)$) $(d_{(\alpha,\beta)}(a) = d_{(\alpha,\beta)}(0) + \alpha(a))$,

(4) *(for all* $a, b \in L_p(X)$) $(d_{(\alpha,\beta)}(a + b) = d_{(\alpha,\beta)}(a) + d_{(\alpha,\beta)}(b) - d_{(\alpha,\beta)}(0))$.

Proof. (1) Let $d_{(\alpha,\beta)}$ be a regular (α, β)-derivation, that is, $d_{(\alpha,\beta)}(0) = 0$. Then the proof follows directly form Proposition 3.7.

(2) Let $a \in L_p(X)$. Then $a = 0 * (0 * a)$, and so $\alpha(a) = \alpha(0 * (*0 * a)) = 0 * (*0 * \alpha(a))$. Thus $\alpha(a) \in L_p(X)$. Similarly, $\beta(a) \in L_p(X)$.

(3) Let $a \in L_p(X)$. Using (2), (a1) and (a14), we have

$$\begin{aligned}
d_{(\alpha,\beta)}(a) &= d_{(\alpha,\beta)}(0 * (0 * a)) \\
&= \left(d_{(\alpha,\beta)}(0) * \alpha(0 * a)\right) \wedge \left(d_{(\alpha,\beta)}(0 * a) * \beta(0)\right) \\
&= \left(d_{(\alpha,\beta)}(0) * \alpha(0 * a)\right) \wedge \left(d_{(\alpha,\beta)}(0 * a) * 0\right) \\
&= \left(d_{(\alpha,\beta)}(0) * \alpha(0 * a)\right) \wedge d_{(\alpha,\beta)}(0 * a) \\
&= d_{(\alpha,\beta)}(0 * a) * \left(d_{(\alpha,\beta)}(0 * a) * \left(d_{(\alpha,\beta)}(0) * \alpha(0 * a)\right)\right) \\
&= d_{(\alpha,\beta)}(0) * \alpha(0 * a) \\
&= d_{(\alpha,\beta)}(0) * (0 * \alpha(a)) \\
&= d_{(\alpha,\beta)}(0) + \alpha(a).
\end{aligned} \tag{3.16}$$

(4) Let $a, b \in L_p(X)$. Then $a + b \in L_p(X)$. Using (3), we have

$$\begin{aligned}
d_{(\alpha,\beta)}(a + b) &= d_{(\alpha,\beta)}(0) + \alpha(a + b) = d_{(\alpha,\beta)}(0) + \alpha(a) + \alpha(b) \\
&= d_{(\alpha,\beta)}(0) + \alpha(a) + d_{(\alpha,\beta)}(0) + \alpha(b) - d_{(\alpha,\beta)}(0) \\
&= d_{(\alpha,\beta)}(a) + d_{(\alpha,\beta)}(b) - d_{(\alpha,\beta)}(0).
\end{aligned} \tag{3.17}$$

This completes the proof. □

Theorem 3.19. *Let X be a torsion free BCI-algebra and $d_{(\alpha,\beta)}$ be a regular (α,β)-derivation on X such that $\alpha \circ d_{(\alpha,\beta)} = d_{(\alpha,\beta)}$. If $d^2_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $d_{(\alpha,\beta)} = 0$ on $L_p(X)$.*

Proof. Let us suppose $d^2_{(\alpha,\beta)} = 0$ on $L_p(X)$. If $x \in L_p(X)$, then $x + x \in L_p(X)$ and so by using Theorem 3.18 (3) and (4), we have

$$\begin{aligned} 0 &= d^2_{(\alpha,\beta)}(x+x) = d_{(\alpha,\beta)}\left(d_{(\alpha,\beta)}(x+x)\right) \\ &= d_{(\alpha,\beta)}(0) + \alpha\left(d_{(\alpha,\beta)}(x+x)\right) = d_{(\alpha,\beta)}(0) + d_{(\alpha,\beta)}(x+x) \\ &= d_{(\alpha,\beta)}(0) + d_{(\alpha,\beta)}(x) + d_{(\alpha,\beta)}(x) - d_{(\alpha,\beta)}(0) \\ &= d_{(\alpha,\beta)}(x) + d_{(\alpha,\beta)}(x). \end{aligned} \tag{3.18}$$

Since X is a torsion free. Therefore, $d_{(\alpha,\beta)}(x) = 0$ for all $x \in X$ implying thereby $d_{(\alpha,\beta)} = 0$. This completes the proof. □

Theorem 3.20. *Let X be a torsion free BCI-algebra and $d_{(\alpha,\beta)}$, $d'_{(\alpha,\beta)}$ be two regular (α,β)-derivations on X such that $\alpha \circ d'_{(\alpha,\beta)} = d'_{(\alpha,\beta)}$. If $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $d'_{(\alpha,\beta)} = 0$ on $L_p(X)$.*

Proof. Let us suppose $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $L_p(X)$. If $x \in L_p(X)$, then $x + x \in L_p(X)$ and so by using Theorem 3.18 (1) and (2), we have

$$\begin{aligned} 0 &= \left(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)}\right)(x+x) = d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(x+x)\right) = d_{(\alpha,\beta)}(0) + \alpha\left(d'_{(\alpha,\beta)}(x+x)\right) \\ &= d_{(\alpha,\beta)}(0) + d'_{(\alpha,\beta)}(x+x) = d_{(\alpha,\beta)}(0) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) - d'_{(\alpha,\beta)}(0)\right) \\ &= \left(d_{(\alpha,\beta)}(0) - d'_{(\alpha,\beta)}(0)\right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)\right) \\ &= \left(\left(d_{(\alpha,\beta)}(0) * d'_{(\alpha,\beta)}(0)\right)\right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)\right) \\ &= \left(d_{(\alpha,\beta)}(0) * \left(0 * d'_{(\alpha,\beta)}(0)\right)\right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)\right) \\ &= \left(d_{(\alpha,\beta)}(0) + d'_{(\alpha,\beta)}(0)\right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)\right) \\ &= \left(d_{(\alpha,\beta)}(0) + \alpha d'_{(\alpha,\beta)}(0)\right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)\right) \\ &= d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(0)\right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)\right) \\ &= \left(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)}\right)(0) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)\right) = d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x). \end{aligned} \tag{3.19}$$

Since X is a torsion free. Therefore $d'_{(\alpha,\beta)}(x) = 0$ for all $x \in X$ and so $d'_{(\alpha,\beta)} = 0$. This completes the proof. □

Proposition 3.21. *Let $d_{(\alpha,\beta)}$ be a regular (α,β)-derivation of a BCI-algebra X. If $d^2_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $(\alpha \circ d_{(\alpha,\beta)})(x) = (1/2)((\alpha \circ d_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$.*

Proof. Assume that $d^2_{(\alpha,\beta)} = 0$ on $L_p(X)$. If $x \in L_p(X)$, then $x + x \in L_p(X)$ and so by using Theorem 3.18 (3) and (4), we have

$$\begin{aligned} 0 &= d^2_{(\alpha,\beta)}(x+x) = d_{(\alpha,\beta)}\left(d_{(\alpha,\beta)}(x+x)\right) = d_{(\alpha,\beta)}(0) + \alpha\left(d_{(\alpha,\beta)}(x+x)\right) \\ &= d_{(\alpha,\beta)}(0) + \alpha\left(d_{(\alpha,\beta)}(x) + d_{(\alpha,\beta)}(x) - d_{(\alpha,\beta)}(0)\right) \\ &= d_{(\alpha,\beta)}(0) + 2\alpha\left(d_{(\alpha,\beta)}(x)\right) - \alpha\left(d_{(\alpha,\beta)}(0)\right). \end{aligned} \tag{3.20}$$

Hence $(\alpha \circ d_{(\alpha,\beta)})(x) = (1/2)((\alpha \circ d_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$.

This completes the proof. □

Proposition 3.22. *Let $d_{(\alpha,\beta)}$ and $d'_{(\alpha,\beta)}$ be two regular (α,β)-derivations of a BCI-algebra X. If $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $(\alpha \circ d'_{(\alpha,\beta)})(x) = (1/2)((\alpha \circ d'_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$.*

Proof. Let $x \in L_p(X)$. Then $x + x \in L_p(X)$, and so $d'_{(\alpha,\beta)}(x+x) \in L_p(X)$ by Theorem 3.18 (1). It follows from Theorem 3.18 (3) and (4) that

$$\begin{aligned} 0 &= \left(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)}\right)(x+x) = d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(x+x)\right) \\ &= d_{(\alpha,\beta)}(0) + \alpha\left(d'_{(\alpha,\beta)}(x+x)\right) \\ &= d_{(\alpha,\beta)}(0) + \alpha\left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) - d'_{(\alpha,\beta)}(0)\right) \\ &= d_{(\alpha,\beta)}(0) + 2\alpha\left(d'_{(\alpha,\beta)}(x)\right) - \alpha\left(d'_{(\alpha,\beta)}(0)\right) \end{aligned} \tag{3.21}$$

so that $\alpha(d'_{(\alpha,\beta)}(x)) = (1/2)((\alpha \circ d'_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$.

This completes the proof. □

Acknowledgments

The authors would like to express their sincere thanks to the anonymous referees. This research is supported by the Deanship of Scientific Research, University of Tabuk, Tabuk, Kingdom of Saudi Arabia.

References

[1] Y. Imai and K. Iséki, "On axiom systems of propositional calculi. XIV," *Proceedings of the Japan Academy*, vol. 42, pp. 19–22, 1966.

[2] K. Iséki, "An algebra related with a propositional calculus," *Proceedings of the Japan Academy*, vol. 42, pp. 26–29, 1966.

[3] K. Iséki, "On BCI-algebras," *Kobe University. Mathematics Seminar Notes*, vol. 8, no. 1, pp. 125–130, 1980.
[4] K. Iséki, "A variety of *BCI*-algebras," *Kobe University. Mathematics Seminar Notes*, vol. 8, no. 1, pp. 225–226, 1980.
[5] K. Iséki, "Some examples of *BCI*-algebras," *Kobe University. Mathematics Seminar Notes*, vol. 8, no. 1, pp. 237–240, 1980.
[6] Q. P. Hu and K. Iséki, "On some classes of BCI-algebras," *Mathematica Japonica*, vol. 29, no. 2, pp. 251–253, 1984.
[7] M. Aslam and A. B. Thaheem, "A note on p-semisimple *BCI*-algebras," *Mathematica Japonica*, vol. 36, no. 1, pp. 39–45, 1991.
[8] S. A. Bhatti, M. A. Chaudhry, and B. Ahmad, "On classification of BCI-algebras," *Mathematica Japonica*, vol. 34, no. 6, pp. 865–876, 1989.
[9] M. Daoji, "*BCI*-algebras and abelian groups," *Mathematica Japonica*, vol. 32, no. 5, pp. 693–696, 1987.
[10] Y. B. Jun and X. L. Xin, "On derivations of *BCI*-algebras," *Information Sciences*, vol. 159, no. 3-4, pp. 167–176, 2004.
[11] H. A. S. Abujabal and N. O. Al-Shehri, "On left derivations of *BCI*-algebras," *Soochow Journal of Mathematics*, vol. 33, no. 3, pp. 435–444, 2007.
[12] S. Ilbira, A. Firat, and Y. B. Jun, "On symmetric bi-derivations of BCI-algebras," *Applied Mathematical Sciences*, vol. 5, no. 57–60, pp. 2957–2966, 2011.
[13] G. Mudiuddin and A. M. Al-roqi, "On t-derivations of *BCI*-algebras," *Abstract and Applied Analysis*, vol. 2012, Article ID 872784, 12 pages, 2012.
[14] M. A. Öztürk, Y. Çeven, and Y. B. Jun, "Generalized derivations of *BCI*-algebras," *Honam Mathematical Journal*, vol. 31, no. 4, pp. 601–609, 2009.
[15] J. Zhan and Y. L. Liu, "On f-derivations of *BCI*-algebras," *International Journal of Mathematics and Mathematical Sciences*, no. 11, pp. 1675–1684, 2005.
[16] N. Aydın and K. Kaya, "Some generalizations in prime rings with (σ, τ)-derivation," *Doga Turkish Journal of Mathematics*, vol. 16, no. 3, pp. 169–176, 1992.
[17] H. E. Bell and L.-C. Kappe, "Rings in which derivations satisfy certain algebraic conditions," *Acta Mathematica Hungarica*, vol. 53, no. 3-4, pp. 339–346, 1989.
[18] H. E. Bell and G. Mason, "On derivations in near-rings, near-rings and Near-fields," *North-Holland Mathematics Studies*, vol. 137, pp. 31–35, 1987.
[19] B. Hvala, "Generalized derivations in rings," *Communications in Algebra*, vol. 26, no. 4, pp. 1147–1166, 1998.
[20] A. A. M. Kamal, "σ-derivations on prime near-rings," *Tamkang Journal of Mathematics*, vol. 32, no. 2, pp. 89–93, 2001.
[21] M. Brešar and J. Vukman, "On left derivations and related mappings," *Proceedings of the American Mathematical Society*, vol. 110, no. 1, pp. 7–16, 1990.
[22] M. Brešar, "On the distance of the composition of two derivations to the generalized derivations," *Glasgow Mathematical Journal*, vol. 33, no. 1, pp. 89–93, 1991.
[23] N. Jacobson, *Structures of Rings*, vol. 37, American Mathematical Society, Colloquium Publications, Providence, RI, USA.
[24] O. Golbasi and N. Aydin, "Results on prime near-ring with (σ, τ)-derivation," *Mathematical Journal of Okayama University*, vol. 46, pp. 1–7, 2004.
[25] K. Kaya, "On (σ, τ)-derivations of prime rings," *Doga Turkish Journal of Mathematics*, vol. 12, no. 2, pp. 42–45, 1988.
[26] Y. B. Jun and E. H. Roh, "On the *BCI*-G part of *BCI*-algebras," *Mathematica Japonica*, vol. 38, no. 4, pp. 697–702, 1993.
[27] Y. B. Jun, X. Xin, and E. H. Roh, "The role of atoms in *BCI*-algebras," *Soochow Journal of Mathematics*, vol. 30, no. 4, pp. 491–506, 2004.