Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 539213, 10 pages
doi:10.1155/2012/539213

Research Article

Generalized Variational Oscillation Principles for Second-Order Differential Equations with Mixed-Nonlinearities

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Received 14 March 2012; Accepted 4 June 2012

Academic Editor: Mingshu Peng

Using generalized variational principle and Riccati technique, new oscillation criteria are established for forced second-order differential equation with mixed nonlinearities, which improve and generalize some recent papers in the literature.

1. Introduction

In this paper, we consider the second-order forced differential equation with mixed nonlinearities:

$$\left(r(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + p(t)\left|y(t)\right|^{\alpha-1}y(t) + \sum_{j=1}^{m} q_j(t)\left|y(t)\right|^{\beta_j-1}y(t) = e(t), \quad t \geq t_0, \tag{1.1}$$

where r, p, q_j $(1 \leq j \leq m)$, $e \in C([t_0, \infty), \mathbb{R})$ with $r(t) > 0$ and $0 < \alpha < \beta_1 < \beta_2 < \cdots < \beta_m$ are real numbers, p, q_j $(1 \leq j \leq m)$, and e might change signs.

In this paper, we are concerned with the nonhomogeneous equation (1.1). By a solution of (1.1), we mean that a function $y \in C^1[T_y, \infty)$ $(T_y \geq t_0$, where $T_y \geq t_0$ depends on the particular solution) which has the property $p(t)|y'(t)|^{\alpha-1}y'(t) \in C^1[T_y, \infty)$ and satisfies (1.1). A nontrivial solution of (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

When $m = 0$, we have the following second-order half-linear differential equation without or with forcing term:

$$\left(r(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + q(t)\left|y(t)\right|^{\alpha-1}y(t) = 0, \quad t \geq t_0, \tag{1.2}$$

$$\left(r(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + q(t)\left|y(t)\right|^{\alpha-1}y(t) = e(t), \quad t \geq t_0. \tag{1.3}$$

There are a lot of papers involved oscillation (see [1–6]) for these equations since the foundation work of Elbert [2]. In paper [1], using Leighton's variational principle (see [3]) for (1.3), the following result was obtained by Li and Cheng.

Theorem 1.1. *Suppose that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that $e(t) \leq 0$ for $t \in [s_1, t_1]$ and $e(t) \geq 0$ for $t \in [s_2, t_2]$. Let $D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \not\equiv 0, u(s_i) = u(t_i) = 0\}$ for $i = 1, 2$. If there exist $H \in D(s_i, t_i)$ and a positive, nondecreasing function $\rho \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\int_{s_i}^{t_i} H^2(t)\rho(t)q(t)dt > \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \int_{s_i}^{t_i} \frac{r(t)\rho(t)}{|H(t)|^{\alpha-1}} \left(2\left|H'(t)\right| + |H(t)|\frac{\rho'}{\rho}\right)^{\alpha+1} dt \tag{1.4}$$

for $i = 1, 2$. Then, (1.3) *is oscillatory.*

Unfortunately, Theorem 1.1 cannot be applied to the case where $\alpha > 1$, since for $\rho(t) \equiv 1$, the term $|H(t)|^{\alpha-1}$ will appear as a denominator in (1.4) so that the requirement $H(s_i) = H(t_i) = 0$ will cause trouble. This certainly calls for investigation of oscillation criteria that can handle with such cases.

When $\alpha = 1$, (1.2) and (1.3) are reduced to the linear differential equation:

$$\left(r(t)y'(t)\right)' + q(t)y(t) = 0, \quad t \geq t_0, \tag{1.5}$$

$$\left(r(t)y'(t)\right)' + q(t)y(t) = e(t), \quad t \geq t_0. \tag{1.6}$$

In paper [7], Wong proved the following result for (1.6).

Theorem 1.2. *Suppose that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that $e(t) \leq 0$ for $t \in [s_1, t_1]$ and $e(t) \geq 0$ for $t \in [s_2, t_2]$. Let $D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \neq 0, u(s_i) = u(t_i) = 0\}$ for $i = 1, 2$. If there exists $u \in D(s_i, t_i)$ such that*

$$Q_i(u) := \int_{s_i}^{t_i} \left[q(t)u^2(t) - r(t)\left(u'(t)\right)^2\right] dt > 0, \quad i = 1, 2, \tag{1.7}$$

then (1.6) *is oscillatory.*

On the other hand, among the oscillation criteria, Komkov [8] gave a generalized Leighton's variational principle, which also can be applied to oscillation for (1.5).

Theorem 1.3. *Suppose that there exist a C^1 function $u(t)$ defined on $[s_1, t_1]$ and a function $G(u)$ such that $G(u(t))$ is not constant on $[s_1, t_1]$, $G(u(s_1)) = G(u(t_1)) = 0$, $g(u) = G'(u)$ is continuous,*

$$\int_{s_1}^{t_1} \left[q(t)G(u(t)) - r(t)\left(u'(t)\right)^2\right] dt > 0, \tag{1.8}$$

and $(g(u(t)))^2 \leq 4G(u(t))$ for $t \in [s_1, t_1]$. Then, every solution of (1.5) must vanish on $[s_1, t_1]$.

We note that when $G(u) \equiv u^2$, the left-hand side of (1.8) is the energy functional related to (1.5).

When $p(t) \equiv 0$, $m = 1$, (1.1) turns into the quasilinear differential equation:

$$\left(r(t)\left|y'(t)\right|^{\alpha-1} y'(t)\right)' + q(t)\left|y(t)\right|^{\beta-1} y(t) = e(t), \quad t \geq t_0, \tag{1.9}$$

where $p, q, e \in C([t_0, \infty), \mathbb{R})$ with $p(t) > 0$ and $0 < \alpha \leq \beta$ being constants. In paper [9], using the generalized variational principle, Shao proved the following result for (1.9).

Theorem 1.4. *Assume that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that*

$$e(t) \begin{cases} \leq 0, & t \in [s_1, t_1], \\ \geq 0, & t \in [s_2, t_2]. \end{cases} \tag{1.10}$$

Let $u \in C^1[s_i, t_i]$ and nonnegative functions G_1, G_2 satisfying $G_i(u(s_i)) = G_i(u(t_i)) = 0$, $g_i(u) = G_i'(u)$ are continuous and $(g_i(u(t)))^{\alpha+1} \leq (\alpha + 1)^{\alpha+1} G_i^{\alpha}(u(t))$ for $t \in [s_i, t_i]$, $i = 1, 2$. If there exists a positive function $\phi \in C^1([t_0, \infty), \mathbb{R})$ such that

$$Q_i^{\phi}(u) := \int_{s_i}^{t_i} \phi(t) \left[Q_e(t) G_i(u(t)) - r(t)\left(\left|u'(t)\right| + \frac{G_i^{1/(\alpha+1)}(u(t))\left|\phi'(t)\right|}{(\alpha + 1)\phi(t)}\right)^{\alpha+1}\right] dt > 0 \tag{1.11}$$

for $i = 1, 2$. Then (1.9) is oscillatory, where

$$Q_e(t) = \alpha^{-\alpha/\beta} \beta (\beta - \alpha)^{(\alpha-\beta)/\beta} [q(t)]^{\alpha/\beta} |e(t)|^{(\beta-\alpha)/\beta}, \tag{1.12}$$

with the convention that $0^0 = 1$.

Recently, using Riccati transformation, the following oscillation criteria were given for (1.1) by Zheng et al. [10].

Theorem 1.5. *Assume that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that $q_j(t) \geq 0 (1 \leq j \leq m)$ for $t \in [s_1, t_1] \cup [s_2, t_2]$ and*

$$e(t) \begin{cases} \leq 0, & t \in [s_1, t_1], \\ \geq 0, & t \in [s_2, t_2]. \end{cases} \tag{1.13}$$

Let $D(s_i,t_i) = \{u \in C^1[s_i,t_i] : u^{\alpha+1}(t) > 0,\ t \in (s_i,t_i),\ u(s_i) = u(t_i) = 0\}$ *for* $i = 1,2$. *If there exist* $H \in D(s_i,t_i)$ *and a positive function* $\phi \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\int_{s_i}^{t_i} \phi(t)\left[\left(p(t)+\sum_{j=1}^{m} Q_j(t)\right)H^{\alpha+1}(t) - r(t)\left(\left|H'(t)\right| + \frac{\left|H(t)\phi'(t)\right|}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right]dt > 0 \tag{1.14}$$

for $i = 1,2$. *Then* (1.1) *is oscillatory, where*

$$Q_j(t) = \alpha^{-\alpha/\beta_j}\beta_j\left[m(\beta_j-\alpha)\right]^{(\alpha-\beta_j)/\beta_j}\left[q_j(t)\right]^{\alpha/\beta_j}|e(t)|^{(\beta_j-\alpha)/\beta_j}, \quad 1 \le j \le m, \tag{1.15}$$

with the convention that $0^0 = 1$.

The purpose of this paper is to obtain new oscillation criteria for (1.1) based on generalized variational principles. Roughly, if the existence of a "positive" solution of a functional relation implies the "positivity" of an associated functional over a set of "admissible" functions, then we say that a variational oscillation principle is valid. For instance, in Theorem 1.1, $H \in D(s_i,t_i)$ is admissible, and the functional is

$$\int_{s_i}^{t_i}\left\{\left(\frac{1}{\alpha+1}\right)^{\alpha+1}\frac{p(t)\rho(t)}{|H(t)|^{\alpha-1}}\left(2\left|H'(t)\right| + |H(t)|\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1} - H^2(t)\rho(t)q(t)\right\}dt. \tag{1.16}$$

Our emphasis will be directed towards oscillation criteria that are closely related to the generalized energy functional (the generalization of $(\alpha+1)$-degree energy functional) for half-linear equations (see [4, 11–13] for more details on these functionals), which improve the results mentioned above. Examples will also be given to illustrate the effectiveness of our main results.

2. Main Results

Firstly, we give an inequality, which is a transformation of Young's inequality.

Lemma 2.1 (see [14]). *Suppose that* X *and* Y *are nonnegative, then*

$$\gamma X Y^{\gamma-1} - X^{\gamma} \le (\gamma-1)Y^{\gamma}, \quad \gamma > 1, \tag{2.1}$$

where equality holds if and only if $X = Y$.

Now, we will give our main results.

Theorem 2.2. *Assume that for any* $T \ge t_0$, *there exist* $T \le s_1 < t_1 \le s_2 < t_2$ *such that*

$$e(t)\begin{cases} \le 0, & t \in [s_1,t_1], \\ \ge 0, & t \in [s_2,t_2]. \end{cases} \tag{2.2}$$

Let $u \in C^1[s_i, t_i]$ *and nonnegative functions* G_1, G_2 *satisfying* $G_i(u(s_i)) = G_i(u(t_i)) = 0$, $g_i(u) = G_i'(u)$ *are continuous and* $(g_i(u(t)))^{\alpha+1} \le (\alpha+1)^{\alpha+1} G_i^{\alpha}(u(t))$ *for* $t \in [s_i, t_i]$, $i = 1, 2$. *If there exists a positive function* $\phi \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$Q_i^{\phi}(u) := \int_{s_i}^{t_i} \phi(t) \left[G_i(u(t)) \left(p(t) + \sum_{j=1}^{m} Q_j(t) \right) \right. \tag{2.3}$$

$$\left. -r(t) \left(|u'(t)| + \frac{G_i^{1/(\alpha+1)}(u(t)) |\phi'(t)|}{(\alpha+1)\phi(t)} \right)^{\alpha+1} \right] dt > 0 \tag{2.4}$$

for $i = 1, 2$, *where* $Q_j(t)$ *is defined as* (1.15) *with the convention that* $0^0 = 1$. *Then,* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that there is a nontrivial nonoscillatory solution $y = y(t)$. We assume that $y(t) \neq 0$ on $[T_0, \infty)$ for some $T_0 \ge t_0$. Set

$$w(t) = \phi(t) \frac{r(t) |y'(t)|^{\alpha-1} y'(t)}{|y(t)|^{\alpha-1} y(t)}, \quad t \ge T_0. \tag{2.5}$$

Then differentiating (2.5) and making use of (1.1), it follows that for all $t \ge T_0$,

$$w'(t) = \frac{\phi'(t)}{\phi(t)} w(t) - \phi(t) p(t) + \frac{\phi(t) e(t)}{|y(t)|^{\alpha-1} y(t)} - \alpha \frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}} - \phi(t) \sum_{j=1}^{m} q_j(t) |y|^{\beta_j - \alpha}. \tag{2.6}$$

By the assumptions, we can choose $s_i, t_i \ge T_0$ for $i = 1, 2$ so that $e(t) \le 0$ on the interval $I_1 = [s_1, t_1]$, with $s_1 < t_1$ and $y(t) \ge 0$, or $e(t) \ge 0$ on the interval $I_2 = [s_2, t_2]$, with $s_2 < t_2$ and $y(t) \le 0$. For given $t \in I_1$ or $t \in I_2$, set $F_j(x) = q_j(t) x^{\beta_j - \alpha} - e(t)/m x^{\alpha}$, $1 \le j \le m$, we have $F_j'(x_j^*) = 0$, $F_j''(x_j^*) > 0$, where $x_j^* = [-\alpha e(t)/m(\beta_j - \alpha) q_j(t)]^{1/\beta_j}$. So, $F_j(x)$ obtains it minimum on x_j^* and

$$F_j(x) \ge F_j\left(x_j^*\right) = Q_j(t). \tag{2.7}$$

So on the interval I_1 or I_2, (2.6) and (2.2) imply that $w(t)$ satisfies

$$\phi(t) \left(p(t) + \sum_{j=1}^{m} Q_j(t) \right) \le -w'(t) + \frac{\phi'(t)}{\phi(t)} w(t) - \alpha \frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}. \tag{2.8}$$

Multiplying $G_i(u(t))$ through (2.8) and integrating (2.8) from s_i to t_i, using the fact that $G_i(u(s_1)) = G_i(u(t_1)) = 0$, we obtain

$$\begin{aligned}
&\int_{s_i}^{t_i} \phi(t)\left(p(t)+\sum_{j=1}^{m} Q_j(t)\right) G_i(u(t))dt \\
&\quad\le \int_{s_i}^{t_i} G_i(u(t))\left\{-w'(t)+\frac{\phi'(t)}{\phi(t)}w(t)-\alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}\right\}dt \\
&\quad= -G_i(u(t))w(t)\Big|_{s_i}^{t_i}+\int_{s_i}^{t_i} g_i(u(t))u'(t)w(t)dt \\
&\qquad+\int_{s_i}^{t_i} G_i(u(t))\left\{\frac{\phi'(t)}{\phi(t)}w(t)-\alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}\right\}dt \\
&\quad= \int_{s_i}^{t_i}\left[g_i(u(t))u'(t)+G_i(u(t))\frac{\phi'(t)}{\phi(t)}\right]w(t)dt \\
&\qquad-\alpha\int_{s_i}^{t_i} G_i(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}dt \\
&\quad\le \int_{s_i}^{t_i}\left[\left|g_i(u(t))\right|\left|u'(t)\right|+G_i(u(t))\frac{\left|\phi'(t)\right|}{\phi(t)}\right]|w(t)|dt \\
&\qquad-\alpha\int_{s_i}^{t_i} G_i(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}dt \\
&\quad\le (\alpha+1)\int_{s_i}^{t_i}\left[G_i^{\alpha/(\alpha+1)}(u(t))\left|u'(t)\right|+G_i(u(t))\frac{\left|\phi'(t)\right|}{(\alpha+1)\phi(t)}\right]|w(t)|dt \\
&\qquad-\alpha\int_{s_i}^{t_i} G_i(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}dt.
\end{aligned} \tag{2.9}$$

Let

$$\begin{aligned}
&X=\left[\frac{\alpha}{(r(t)\phi(t))^{1/\alpha}}\right]^{\alpha/(\alpha+1)} G_i^{\alpha/(\alpha+1)}|w(t)|, \qquad \gamma = 1+\frac{1}{\alpha}, \\
&Y=(\alpha\phi(t)r(t))^{\alpha/(\alpha+1)}\left[\left|u'(t)\right|+\frac{G_i^{1/(\alpha+1)}\left|\phi'(t)\right|}{(\alpha+1)\phi(t)}\right]^{\alpha},
\end{aligned} \tag{2.10}$$

by Lemma 2.1 and (2.9), we have

$$\int_{s_i}^{t_i} \phi(t)\left(p(t)+\sum_{j=1}^{m} Q_j(t)\right)G_i(u(t))dt \le \int_{s_i}^{t_i} \phi(t)r(t)\left[\left|u'(t)\right| + \frac{G_i^{1/(\alpha+1)}(u(t))\left|\phi'(t)\right|}{(\alpha+1)\phi(t)}\right]^{\alpha+1} dt, \tag{2.11}$$

which contradicts with (2.3). This completes the proof of Theorem 2.2. □

Corollary 2.3. *If $\phi(t) \equiv 1$ in Theorem 2.2, and* (2.3) *is replaced by*

$$Q_i(u) := \int_{s_i}^{t_i} \left[\left(p(t)+\sum_{j=1}^{m} Q_j(t)\right)G_i(u(t)) - r(t)\left|u'(t)\right|^{\alpha+1}\right] dt > 0, \tag{2.12}$$

for $i = 1, 2$. Then, (1.1) *is oscillatory.*

If we choose $G_1(u) = G_2(u) = u^{\alpha+1}$ in Corollary 2.3, then we have the following corollary.

Corollary 2.4. *Suppose that for any $T \ge t_0$, there exist $T \le s_1 < t_1 \le s_2 < t_2$ such that* (2.2) *is true. Let $D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \not\equiv 0,\ u(s_i) = u(t_i) = 0\}$ for $i = 1, 2$. If there exist $u \in D(s_i, t_i)$ such that*

$$\widetilde{Q}_i(u) := \int_{s_i}^{t_i} \left[\left(p(t)+\sum_{j=1}^{m} Q_j(t)\right)|u(t)|^{\alpha+1} - r(t)\left|u'(t)\right|^{\alpha+1}\right] dt > 0, \tag{2.13}$$

for $i = 1, 2$. Then, (1.3) *is oscillatory.*

Remark 2.5. Corollary 2.4 is closely related to the $(\alpha + 1)$-degree functional (1.8), so Theorem 2.2, Corollaries 2.3, and 2.4 are generalizations of Theorem 1.2, and improvement of Theorem 1.1 since the positive constant α in Theorem 2.2 and Corollary 2.3 can be selected as any number lying in $(0, \infty)$. We note further that in most cases, oscillation criteria are obtained using the same auxiliary function on $[s_1, t_1]$ and $[s_2, t_2]$, we note that such functions can be selected differently.

Remark 2.6. If $G(u) \equiv u^{\alpha+1}$, then Theorem 2.2 reduces to Theorem 1.5, and if $p(t) \equiv 0$, $j = 1$, Theorem 2.2 reduces to Theorem 1.4. So Theorem 2.2 and Corollary 2.3 are generalizations of the papers by Zheng et al. [10] and Shao [9].

Remark 2.7. The hypothesis (2.2) in Theorem 2.2 and Corollary 2.3 can be replaced by the following condition:

$$e(t)\begin{cases} \ge 0, & t \in [s_1, t_1], \\ \le 0, & t \in [s_2, t_2]. \end{cases} \tag{2.14}$$

The conclusion is still true for these cases.

Example 2.8. Consider the following forced mixed nonlinearities differential equation:

$$\left(\gamma t^{-\lambda/3} y'(t)\right)' + p(t)y(t) + q(t)\left|y(t)\right|^2 y(t) = -\sin^3 t, \quad t \geq 2\pi, \tag{2.15}$$

where $\gamma, \lambda > 0$ are constants, $q(t) = t^{-\lambda}\exp(3\sin t)$, $p(t) = t^{-\lambda/3}\exp(\sin t)$, for $t \in [2n\pi, (2n+1)\pi)$, and $q(t) = t^{-\lambda}\exp(-3\sin t)$, $p(t) = t^{-\lambda/3}\exp(-\sin t)$, for $t \in [(2n+1)\pi, (2n+2)\pi)$, $n > 0$ is an integer, Shao [9] obtain oscillation for (2.15) when $K(t) \equiv 0$. Using Theorem 2.2, we can easily verify that $Q_1(t) = (3/2)\sqrt[3]{2}t^{-\lambda/3}\exp(\sin t)\sin^2 t$ for $t \in [2n\pi, (2n+1)\pi)$, and $Q_1(t) = (3/2)\sqrt[3]{2}t^{-\lambda/3}\exp(-\sin t)\sin^2 t$ for $t \in [(2n+1)\pi, (2n+2)\pi)$. For any $T \geq 1$, we choose n sufficiently large so that $n\pi = 2k\pi \geq T$ and $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$, we select $u(t) = \sin t \geq 0$, $G_1(u) = u^2\exp(-u)$ (we note that $(G_1'(u))^2 \leq 4G_1(u)$ for $u \geq 0$), $\phi(t) = t^{\lambda/3}$, then we have

$$\begin{aligned}
\int_{s_1}^{t_1} \phi(t)(p(t) + Q_1(t))G_1(u(t))dt &= \int_0^{\pi} \sin^2 t\, dt + \frac{3}{2}\sqrt[3]{2}\int_0^{\pi}\sin^4 t\, dt = \frac{\pi}{2} + \frac{9}{8}\sqrt[3]{2}, \\
\int_{s_1}^{t_1} \phi(t)p(t)\left[\left|u'(t)\right| + \frac{G_1^{1/(\alpha+1)}(u(t))\left|\phi'(t)\right|}{(\alpha+1)\phi(t)}\right]^{\alpha+1} dt \\
&= \gamma \int_{2k\pi}^{(2k+1)\pi}\left[\left|\cos t\right| + \frac{\lambda\left|\sin t\right|\exp(3\sin t/2)}{2t}\right]^2 dt \\
&< \gamma\int_{2k\pi}^{(2k+1)\pi}\left(1 + \frac{\lambda e^{3/2}}{2}\right)^2 dt = \gamma\left(1 + \frac{\lambda e^{3/2}}{2}\right)^2\pi.
\end{aligned} \tag{2.16}$$

So we have $Q_1^{\phi}(u) > 0$ provided, $0 < \gamma < (4\pi + 9\sqrt[3]{2})/2(2 + \lambda e^{3/2})^2\pi$. Similarly, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we select $u(t) = \sin t \leq 0$, $G_2(u) = u^2\exp(u)$ (we note that $(G_2'(u))^2 \leq 4G_2(u)$ for $u \leq 0$), we can show that the integral inequality $Q_2^{\phi}(u) > 0$ for $0 < \gamma < (4\pi + 9\sqrt[3]{2})/2(2 + \lambda e^{3/2})^2\pi$. So (2.15) is oscillatory for $0 < \gamma < (4\pi + 9\sqrt[3]{2})/2(2 + \lambda e^{3/2})^2\pi$ by Theorem 2.2.

Example 2.9. Consider the following forced mixed nonlinearities differential equation:

$$\left(t^{-\lambda}\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + p(t)\left|y(t)\right|^{\alpha-1}y(t) + q(t)y^3(t) = -\sin^{1/3} t, \tag{2.17}$$

for $t \geq 2\pi$, where $p(t) = Kt^{-\lambda}\exp(\sin t)$, $q(t) = t^{-9\lambda/5}\exp(9\sin t/5)$, for $t \in [2n\pi, (2n+1)\pi)$, and $p(t) = Kt^{-\lambda}\exp(-\sin t)$, $q(t) = t^{-9\lambda/5}\exp(-9\sin t/5)$, for $t \in [(2n+1)\pi, (2n+2)\pi)$, $n > 0$ is an integer, $K, \lambda > 0$ are constants and $\alpha = 5/3 > 1$, $\beta = 3$. Obviously, Theorem 1.1 cannot be applied to this case. However, we conclude that (2.17) is oscillatory for $K > (3/4)(1 + 3\lambda e/8)^{8/3}\pi - 9/5^{5/9}4^{4/9}$. Since the zeros of the forcing term $-\sin^{1/3} t$ are $n\pi$, let $u(t) = \sin t$ and $\phi(t) = t^{\lambda}$. Using Theorem 2.2, we can easily verify that $Q(t) = (9/5^{5/9}4^{4/9})t^{-\lambda}\exp(\sin t)\sin^{4/27} t$ for $t \in [2n\pi, (2n+1)\pi)$, and $Q(t) = (9/5^{5/9}4^{4/9})t^{-\lambda}\exp(-\sin t)\sin^{4/27} t$ for $t \in [(2n+1)\pi, (2n+2)\pi)$. For any $T \geq 1$, choose n sufficiently large so that $n\pi = 2k\pi \geq T$ and $s_1 = 2k\pi$ and

$t_1 = (2k+1)\pi$. For $t \in [s_1, t_1]$, we select $G_1(u) = u^{8/3}\exp(-u)$ (we note that $(G_1'(u))^{8/3} \leq (8/3)^{8/3}(G_1(u))^{5/3}$ for $u \geq 0$). It is easy to verify the following estimations:

$$\begin{aligned}
&\int_{s_1}^{t_1} \phi(t)(p(t)+Q(t))G_1(u(t))dt \\
&\quad = \int_{2k\pi}^{(2k+1)\pi} \sin^{8/3} t\left(K + \frac{9}{5^{5/9}4^{4/9}}\sin^{4/27} t\right)dt \\
&\quad > \left(K + \frac{9}{5^{5/9}4^{4/9}}\right)\int_{2k\pi}^{(2k+1)\pi} \sin^3 t\, dt = \frac{4}{3}\left(K + \frac{9}{5^{5/9}4^{4/9}}\right), \\
&\int_{s_1}^{t_1} \phi(t)r(t)\left[\left|u'(t)\right| + \frac{G_1^{1/(\alpha+1)}(u(t))\left|\phi'(t)\right|}{(\alpha+1)\phi(t)}\right]^{\alpha+1} dt \\
&\quad = \int_{2k\pi}^{(2k+1)\pi} \left[|\cos t| + \frac{3\lambda e^{-3\sin t/8}|\sin t|}{8t}\right]^{8/3} dt \\
&\quad < \int_{2k\pi}^{(2k+1)\pi} \left(1 + \frac{3\lambda e}{8}\right)^{8/3} dt = \left(1 + \frac{3\lambda e}{8}\right)^{8/3}\pi.
\end{aligned} \tag{2.18}$$

So we have $Q_1^{\phi}(u) > 0$. Similarly, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we select $u(t) = \sin t < 0$, $G_2(u) = u^{8/3}\exp(u)$ (we note that $(G_2'(u))^{8/3} \leq (8/3)^{8/3}(G_2(u))^{5/3}$ for $u \leq 0$), we can show that the integral inequality $Q_2^{\phi}(u) > 0$. So (2.17) is oscillatory for $K > (3/4)(1+3\lambda e/8)^{8/3}\pi - 9/5^{5/9}4^{4/9}$ by Theorem 2.2.

Acknowledgment

This research was partially supported by the NSF of China (Grants nos. 11171178 and 11271225) and Science and Technology Project of High Schools of Shandong Province (Grant no. J12LI52).

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