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Research Article

Blow-Up Criteria for Three-Dimensional Boussinesq Equations in Triebel-Lizorkin Spaces

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We establish a new blow-up criteria for solution of the three-dimensional Boussinesq equations in Triebel-Lizorkin spaces by using Littlewood-Paley decomposition.

1. Introduction and Main Results

In this paper, we consider the regularity of the following three-dimensional incompressible Boussinesq equations:

$$\begin{gathered} u_t - \mu\Delta u + u\cdot\nabla u + \nabla P = \theta e_3, \quad (x,t)\in\mathbb{R}^3\times(0,\infty), \\ \theta_t - \kappa\Delta\theta + u\cdot\nabla\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0, \qquad \theta(x,0) = \theta_0, \end{gathered} \tag{1.1}$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ denotes the fluid velocity vector field, $P = P(x,t)$ is the scalar pressure, $\theta(x,t)$ is the scalar temperature, $\mu > 0$ is the constant kinematic viscosity, $\kappa > 0$ is the thermal diffusivity, and $e_3 = (0,0,1)^T$, while u_0 and θ_0 are the given initial velocity and initial temperature, respectively, with $\nabla\cdot u_0 = 0$. Boussinesq systems are widely used to model the dynamics of the ocean or the atmosphere. They arise from the density-dependent fluid equations by using the so-called Boussinesq approximation which consists in neglecting the density dependence in all the terms but the one involving the gravity. This approximation can be justified from compressible fluid equations by a simultaneous low Mach number/Froude number limit; we refer to [1] for a rigorous justification. It is well known that the question of global existence or finite-time blow-up of smooth solutions for the 3D incompressible Boussinesq equations. This challenging problem has attracted significant attention. Therefore, it is interesting to study the blow-up criterion of the solutions for system (1.1).

Recently, Fan and Zhou [2] and Ishimura and Morimoto [3] proved the following blow-up criterion, respectively:

$$\operatorname{curl} u \in L^1\left(0, T; \dot{B}^0_{\infty,\infty}\left(\mathbb{R}^3\right)\right), \tag{1.2}$$

$$\nabla u \in L^1\left(0, T; L^\infty\left(\mathbb{R}^3\right)\right). \tag{1.3}$$

Subsequently, Qiu et al. [4] obtained Serrin-type regularity condition for the three-dimensional Boussinesq equations under the incompressibility condition. Furthermore, Xu et al. [5] obtained the similar regularity criteria of smooth solution for the 3D Boussinesq equations in the Morrey-Campanato space.

Our purpose in this paper is to establish a blow-up criteria of smooth solution for the three-dimensional Boussinesq equations under the incompressibility condition $\nabla \cdot u_0 = 0$ in Triebel-Lizorkin spaces.

Now we state our main results as follows.

Theorem 1.1. *Let* $(u_0, \theta_0) \in H^1(\mathbb{R}^3)$, $(u(\cdot,t), \theta(\cdot,t))$ *be the smooth solution to the problem* (1.1) *with the initial data* (u_0, θ_0) *for* $0 \leqslant t < T$. *If the solution* u *satisfies the following condition*

$$\nabla u \in L^p\left(0, T; \dot{F}_{q,(2q/3)}\left(\mathbb{R}^3\right)\right), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \leqslant \infty, \tag{1.4}$$

then the solution (u, θ) *can be extended smoothly beyond* $t = T$.

Corollary 1.2. *Let* $(u_0, \theta_0) \in H^1(\mathbb{R}^3)$, $(u(\cdot,t), \theta(\cdot,t))$ *be the smooth solution to the problem* (1.1) *with the initial data* (u_0, θ_0) *for* $0 \leqslant t < T$. *If the solution* u *satisfies the following condition*

$$\operatorname{curl} u \in L^1\left(0, T; \dot{B}_{\infty,\infty}\left(\mathbb{R}^3\right)\right), \tag{1.5}$$

then the solution (u, θ) *can be extended smoothly beyond* $t = T$.

Remark 1.3. By Corollary 1.2, we can see that our main result is an improvement of (1.2).

2. Preliminaries and Lemmas

The proof of the results presented in this paper is based on a dyadic partition of unity in Fourier variables, the so-called homogeneous Littlewood-Paley decomposition. So, we first introduce the Littlewood-Paley decomposition and Triebel-Lizorkin spaces.

Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing function. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \widehat{f}$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx. \tag{2.1}$$

Let (χ, φ) be a couple of smooth functions valued in $[0,1]$ such that χ is supported in the ball $\{\xi \in \mathbb{R}^3 : |\xi| \leqslant 4/3\}$, φ is supported in the shell $\{\xi \in \mathbb{R}^3 : 3/4 \leqslant |\xi| \leqslant 8/3\}$, and

$$\begin{aligned} &\chi(\xi) + \sum_{j\geqslant 0} \varphi\left(2^{-j}\xi\right) = 1, \quad \forall \xi \in \mathbb{R}^3, \\ &\sum_{j\in Z} \varphi\left(2^{-j}\xi = 1\right), \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}. \end{aligned} \tag{2.2}$$

Denoting $\varphi_j = \varphi(2^{-j}\xi)$, $h = F^{-1}\varphi$, and $\widetilde{h} = F^{-1}\chi$, we define the dyadic blocks as

$$\begin{aligned} \dot{\Delta}_j f &= \varphi\left(2^{-j}D\right) = 2^{3j} \int_{\mathbb{R}^3} h\left(2^j y\right) f(x-y)dy, \quad j \in \mathbb{Z}, \\ \dot{S}_j f &= \sum_{k\leqslant j-1} \Delta_k f = 2^{3j} \int_{\mathbb{R}^3} \widetilde{h}\left(2^j y\right) f(x-y)dy, \quad j \in \mathbb{Z}. \end{aligned} \tag{2.3}$$

Definition 2.1. Let $\mathcal{S}'_h$ be the space of temperate distribution u such that

$$\lim_{j\to -\infty} \dot{S}_j f = 0, \quad \text{in } \mathcal{S}'. \tag{2.4}$$

The formal equality

$$f = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f \tag{2.5}$$

holds in S'_h and is called the homogeneous Littlewood-Paley decomposition. It has nice properties of quasi-orthogonality

$$\dot{\Delta}_j \dot{\Delta}_q f \equiv 0, \quad \left|j - q\right| \geqslant 2. \tag{2.6}$$

Let us now define the homogeneous Besov spaces and Triebel-Lizorkin spaces; we refer to [6, 7] for more detailed properties.

Definition 2.2. Letting $s \in \mathbb{R}$, $p, q \in [1, \infty]$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \left\{ f \in \mathcal{Z}'\left(R^3\right) \mid \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}. \tag{2.7}$$

Here

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|\dot{\Delta}_j f\|_p^q \right)^{1/q}, & q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_p, & q = \infty, \end{cases} \tag{2.8}$$

and $\mathcal{Z}'(\mathbb{R}^3)$ denotes the dual space of $\mathcal{Z}(\mathbb{R}^3) = \{f \in \mathcal{S}(\mathbb{R}^3) \mid D^\alpha \widehat{f}(0) = 0, \forall \alpha \in \mathbb{N}^3$ multi-index$\}$.

Definition 2.3. Let $s \in \mathbb{R}$, $p \in [1,\infty)$, and $q \in [1,\infty]$, and for $s \in \mathbb{R}$, $p = \infty$, and $q = \infty$, the homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is defined by

$$\dot{F}^s_{p,q} = \left\{ f \in \mathcal{Z}'\left(\mathbb{R}^3\right) \mid \|f\|_{\dot{F}^s_{p,q}} < \infty \right\}. \tag{2.9}$$

Here

$$\|f\|_{\dot{F}^s_{p,q}} = \begin{cases} \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jsq} |\dot{\Delta}_j f|^q \right)^{1/q} \right\|_{L^p}, & q < \infty, \\ \left\| \sup_{j\in\mathbb{Z}} 2^{js} |\dot{\Delta}_j f| \right\|_p, & q = \infty, \end{cases} \tag{2.10}$$

for $p = \infty$ and $q \in [1,\infty)$, the space $\dot{F}^s_{p,q}$ is defined by means of Carleson measures which is not treated in this paper. Notice that by Minkowski's inequality, we have the following inclusions:

$$\begin{aligned} \dot{B}^s_{p,q} &\subset \dot{F}^s_{p,q}, \quad \text{if } q \leqslant p, \\ \dot{F}^s_{p,q} &\subset \dot{B}^s_{p,q}, \quad \text{if } q \geqslant p. \end{aligned} \tag{2.11}$$

Also it is well known that

$$\dot{B}^s_{p,p} = \dot{F}^s_{p,p}, \qquad L^\infty \subset \dot{B}^0_{\infty,\infty} = \dot{F}^0_{\infty,\infty}, \qquad \dot{B}^s_{2,2} = \dot{F}^s_{2,2} = \dot{H}^s. \tag{2.12}$$

Throughout the proof of Theorem 1.1 in Section 3, we will use the following interpolation inequality frequently:

$$\|f\|_{L^q} \leqslant C\|f\|_{L^2}^{3/q-1/2} C\|\nabla f\|_{L^2}^{3/2-3/q}, \quad 2 \leqslant q \leqslant 6, \ f \in L^2\left(\mathbb{R}^3\right) \cap \dot{H}^1\left(\mathbb{R}^3\right). \tag{2.13}$$

Lemma 2.4. *Let $k \in N$. Then there exists a constant C independent of f, j such that for $1 \leqslant p \leqslant q \leqslant \infty$*

$$\sup_{|\alpha|=k} \|\partial^{\alpha} \dot{\Delta}_j f\|_q \leqslant C 2^{jk+3j(1/p-1/q)} \|\dot{\Delta}_j f\|_p. \tag{2.14}$$

Remark 2.5. From the above Beinstein estimate, we easily know that

$$\|\dot{\Delta}_j f\|_q \leqslant C 2^{3j(1/p-1/q)} \|\dot{\Delta}_j f\|_p. \tag{2.15}$$

3. Proofs of the Main Results

In this section, we prove Theorem 1.1. For simplicity, without loss of generality, we assume $\mu = \kappa = 1$.

Proof of Theorem 1.1. Differentiating the first equation and the second equation of (1.1) with respect to x_k ($1 \leqslant k \leqslant 3$), and multiplying the resulting equations by $\partial u/\partial x_k = \partial_k u$ and $\partial \theta/\partial x_k = \partial_k \theta$, respectively, then by integrating by parts over $\mathbb{R}^3$ we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\partial_k u\|_{L^2}^2 + \|\nabla \partial_k u\|_{L^2}^2 &= -\int \partial_k[(u\cdot\nabla)u]\cdot\partial_k u\,dx - \int \partial_k \nabla P \cdot \partial_k u\,dx + \int \partial_k(\theta e_3)\partial_k u\,dx, \\ \frac{1}{2}\frac{d}{dt}\|\partial_k \theta\|_{L^2}^2 + \|\nabla \partial_k \theta\|_{L^2}^2 &= -\int \partial_k[(u\cdot\nabla)\theta]\cdot\partial_k \theta\,dx. \end{aligned} \tag{3.1}$$

Noting the incompressibility condition $\nabla \cdot u = 0$, since

$$\begin{gathered} \int \partial_k[(u\cdot\nabla)u]\cdot\partial_k u\,dx = \int (\partial_k u\cdot\nabla)u\cdot\partial_k u\,dx, \\ \int \partial_k \nabla P\cdot\partial_k u\,dx = 0, \\ \int \partial_k[(u\cdot\nabla)\theta]\cdot\partial_k\theta\,dx = \int (\partial_k u\cdot\nabla)\theta\cdot\partial_k\theta\,dx, \end{gathered} \tag{3.2}$$

then the above equations (3.1) can be rewritten as

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\partial_k u\|_{L^2}^2 + \|\nabla \partial_k u\|_{L^2}^2 &= -\int (\partial_k u\cdot\nabla)u\cdot\partial_k u\,dx + \int \partial_k(\theta e_3)\partial_k u\,dx, \\ \frac{1}{2}\frac{d}{dt}\|\partial_k \theta\|_{L^2}^2 + \|\nabla \partial_k \theta\|_{L^2}^2 &= -\int (\partial_k u\cdot\nabla)\theta\cdot\partial_k\theta\,dx. \end{aligned} \tag{3.3}$$

Adding up (3.3), then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\partial_k u\|_{L^2}^2+\|\partial_k \theta\|_{L^2}^2\right)+\|\nabla\partial_k u\|_{L^2}^2+\|\nabla\partial_k\theta\|_{L^2}^2\\
&\quad=-\int(\partial_k u\cdot\nabla)u\cdot\partial_k u\,dx-\int(\partial_k u\cdot\nabla)\theta\cdot\partial_k\theta\,dx+\int\partial_k(\theta e_3)\cdot\partial_k u\,dx\\
&\quad\triangleq I_1+I_2+I_3.
\end{aligned}\tag{3.4}$$

Firstly, for the third term I_3, by Hölder's inequality and Young's inequality, we get

$$I_3=\int\partial_k(\theta e_3)\cdot\partial_k u\,dx\leqslant\frac{1}{2}\|\nabla\theta\|_{L^2}^2+\frac{1}{2}\|\nabla u\|_{L^2}^2.\tag{3.5}$$

The other terms are bounded similarly. For simplicity, we detail the term I_2. Using the Littlewood-Paley decomposition (2.5), we decompose ∇u as follows:

$$\nabla u=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j(\nabla u)=\sum_{j<-N}\dot{\Delta}_j(\nabla u)+\sum_{j=-N}^{j=N}\dot{\Delta}_j(\nabla u)+\sum_{j>N}\dot{\Delta}_j(\nabla u).\tag{3.6}$$

Here N is a positive integer to be chosen later. Plugging (3.6) into I_2 produces that

$$\begin{aligned}
I_2&=\sum_{j<-N}\int_{\mathbb{R}^3}\left|\dot{\Delta}_j(\nabla u)\right||\nabla\theta|^2dx\\
&\quad+\sum_{j=-N}^{j=N}\int_{\mathbb{R}^3}\left|\dot{\Delta}_j(\nabla u)\right||\nabla\theta|^2dx\\
&\quad+\sum_{j>N}\int_{\mathbb{R}^3}\left|\dot{\Delta}_j(\nabla u)\right||\nabla\theta|^2dx\\
&\equiv I_2^1+I_2^2+I_2^3.
\end{aligned}\tag{3.7}$$

For I_2^1, using the Hölder inequality, (2.12), and (2.15), we obtain that

$$\begin{aligned}
I_2^1&\leqslant\|\nabla\theta\|_{L^2}^2\sum_{j<-N}\left\|\dot{\Delta}_j\nabla u\right\|_{L^\infty}\\
&\leqslant C\|\nabla\theta\|_{L^2}^2\sum_{j<-N}2^{(3/2)j}\left\|\dot{\Delta}_j\nabla u\right\|_{L^2}\\
&\leqslant C2^{-(3/2)N}\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2\\
&\leqslant C2^{-(3/2)N}\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)^{3/2}.
\end{aligned}\tag{3.8}$$

For I_2^2, from the Hölder inequality and (2.15), it follows that

$$\begin{aligned} I_2^2 &= \sum_{j=-N}^{j=N} \int_{\mathbb{R}^3} |\nabla\theta|^2 \left|\dot{\Delta}_j(\nabla u)\right| dx = \int_{\mathbb{R}^3} |\nabla\theta|^2 \sum_{j=-N}^{j=N} \left|\dot{\Delta}_j(\nabla u)\right| dx \\ &\leqslant \int_{\mathbb{R}^3} |\nabla\theta|^2 \left(\sum_{j=-N}^{j=N} \left|\dot{\Delta}_j(\nabla u)\right|^{2q/3}\right)^{3/2q} N^{1-3/2q} dx \\ &\leqslant CN^{(2q-3)/2q} \int_{\mathbb{R}^3} |\nabla\theta|^2 \left(\sum_{j=-N}^{j=N} \left|\dot{\Delta}_j(\nabla u)\right|^{2q/3}\right)^{3/2q} dx \\ &\leqslant CN^{(2q-3)/2q} \|\nabla\theta\|_{L^{2q'}}^2 \|\nabla u\|_{\dot{F}^0_{q,(2q/3)}}. \end{aligned} \tag{3.9}$$

Here q' denotes the conjugate exponent of q. Since $2q > 3$ by the Gagliardo-Nirenberg inequality and the Young inequality, we have

$$\begin{aligned} I_2^2 &\leqslant CN^{(2q-3)/2q} \|\nabla\theta\|_{L^2}^{(2q-3)/q} \left\|\nabla^2\theta\right\|_{L^2}^{3/q} \|\nabla u\|_{\dot{F}^0_{q,(2q/3)}} \\ &\leqslant \frac{1}{2}\left\|\nabla^2\theta\right\|_{L^2}^2 + CN\|\nabla\theta\|_{L^2}^2 \|\nabla u\|_{\dot{F}^0_{q,(2q/3)}}^p. \end{aligned} \tag{3.10}$$

For I_2^3, from the Hölder and Young inequalities, (2.12), (2.15), and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} I_2^3 &= \sum_{j>N} \int_{\mathbb{R}^3} \left|\dot{\Delta}_j(\nabla u)\right| |\nabla\theta|^2 dx \\ &\leqslant \|\nabla\theta\|_{L^3}^2 \sum_{j>N} \left\|\dot{\Delta}_j(\nabla u)\right\|_{L^3} \\ &\leqslant C\|\nabla\theta\|_{L^3}^2 \sum_{j>N} 2^{(j/2)} \left\|\dot{\Delta}_j(\nabla u)\right\|_{L^2}, \\ C &\leqslant \|\nabla\theta\|_{L^2} \left\|\nabla^2\theta\right\|_{L^2} \left(\sum_{j>N} 2^{-j}\right)^{1/2} \left(\sum_{j>N} 2^{2j} \left\|\dot{\Delta}_j(\nabla u)\right\|_2^2\right)^{1/2} \\ &\leqslant C2^{-(N/2)} \|\nabla\theta\|_{L^2} \left\|\nabla^2\theta\right\|_{L^2} \left\|\nabla^2 u\right\|_2 \\ &\leqslant C2^{-(N/2)} \|\nabla\theta\|_{L^2} \left(\left\|\nabla^2\theta\right\|_{L^2}^2 + \left\|\nabla^2 u\right\|_{L^2}^2\right). \end{aligned} \tag{3.11}$$

Plugging (3.8), (3.10), and (3.11) into (3.7) yields

$$I_2 \leqslant C2^{-(3/2)N}\left(\|\nabla u\|_2^2+\|\nabla\theta\|_2^2\right)^{3/2}+\frac{1}{2}\left\|\nabla^2\theta\right\|_2^2+CN\|\nabla\theta\|_{L^2}^2\|\nabla u\|_{\dot{F}^0_{q,(2q/3)}}^p$$
$$+C2^{-N/2}\|\nabla\theta\|_{L^2}\left(\left\|\nabla^2\theta\right\|_{L^2}^2+\left\|\nabla^2 u\right\|_{L^2}^2\right). \tag{3.12}$$

Similarly, we also obtain the estimate

$$I_1 \leqslant C2^{-(3/2)N}\left(\|\nabla u\|_2^2+\|\nabla\theta\|_2^2\right)^{3/2}+\frac{1}{2}\left\|\nabla^2 u\right\|_2^2+CN\|\nabla u\|_{L^2}^2\|\nabla u\|_{\dot{F}^0_{q,(2q/3)}}^p$$
$$+C2^{-N/2}\|\nabla u\|_{L^2}\left(\left\|\nabla^2\theta\right\|_{L^2}^2+\left\|\nabla^2 u\right\|_{L^2}^2\right). \tag{3.13}$$

Putting (3.5), (3.12), and (3.13) into (3.4) yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla u,\nabla\theta\|_{L^2}^2+\frac{1}{2}\left\|\left(\nabla^2 u,\nabla^2\theta\right)\right\|_{L^2}^2$$
$$\leqslant\left\{C2^{-N}\|(\nabla u,\nabla\theta)\|_2^2\right\}^{3/2}+CN\|(\nabla u,\nabla\theta)\|_{L^2}^2\|\nabla u\|_{\dot{F}^0_{q,(2q/3)}}^p$$
$$+\left\{C2^{-N}\|(\nabla u,\nabla\theta)\|_{L^2}^2\right\}^{1/2}\left\|\left(\nabla^2 u,\nabla^2\theta\right)\right\|_{L^2}^2. \tag{3.14}$$

Now we take N in (3.14) such that

$$C2^{-N}\|(\nabla u,\nabla\theta)\|_{L^2}^2\leqslant\frac{1}{16}, \tag{3.15}$$

that is,

$$N\geqslant C\frac{\log\left(e+\|(\nabla u,\nabla\theta)\|_{L^2}^2\right)}{\log 2}+4. \tag{3.16}$$

Then (3.14) implies that

$$\frac{d}{dt}\|(\nabla u,\nabla\theta)\|_{L^2}^2\leqslant C+C\log\left(e+\|(\nabla u,\nabla\theta)\|_{L^2}^2\right)\|(\nabla u,\nabla\theta)\|_{L^2}^2\|\nabla u\|_{\dot{F}^0_{q,(2q/3)}}^p. \tag{3.17}$$

Applying the Gronwall inequality twice, we have

$$\|(\nabla u,\nabla\theta)\|_{L^2}^2\leqslant C\exp\left\{\exp\left(C\int_0^T\|\nabla u\|_{\dot{F}^0_{q,(2q/3)}}^p(s)ds\right)\right\}, \tag{3.18}$$

for all $t\in(0,T)$. This completes the proof of Theorem 1.1. □

Proof of Corollary 1.2. In Theorem 1.1, taking $p = 1$, and combining (2.12) with the classical Riesz transformation is bounded in $\dot{B}_{\infty,\infty}(\mathbb{R}^3)$, we can prove it. □

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