Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 562838, 8 pages
doi:10.1155/2012/562838

Research Article

Strong Convergence Properties for Asymptotically Almost Negatively Associated Sequence

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Received 22 June 2012; Accepted 10 September 2012

Academic Editor: Garyfalos Papaschinopoulos

By applying the moment inequality for asymptotically almost negatively associated (in short *AANA*) random sequence and truncated method, we get the three series theorems for *AANA* random variables. Moreover, a strong convergence property for the partial sums of *AANA* random sequence is obtained. In addition, we also study strong convergence property for weighted sums of *AANA* random sequence.

1. Introduction

A finite family of random variables $\{X_k, 1 \le k \le n, n \ge 2\}$ is said to be negatively associated (in short *NA*) if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \ldots, n\}$

$$\operatorname{Cov}\big(f(X_i : i \in A_1), g\big(X_j : j \in A_2\big)\big) \le 0, \tag{1.1}$$

whenever f, g are coordinate-wise nondecreasing such that the covariance exists. An infinite sequence of random variables $\{X_n, n \ge 1\}$ is said to be *NA* if every finite subfamily is *NA*.

The notion of *NA* was first introduced by Block et al. (1982) [1]. Joag-Dev and Proschan (1983) [2] showed that many well-known multivariate distributions possess the *NA* property. By inspecting the proof of maximal inequality for *NA* random variables in Matuła [3], Chandra and Ghosal discovered that one can also allow negative correlations provided they are small. Primarily motivated by this, Chandra and Ghosal [4, 5] introduced the following dependence.

Definition 1.1. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be asymptotically almost negatively associated, if there exists a nonnegative sequence $q(n) \to 0$ as $n \to \infty$ such that

$$\operatorname{Cov}\left(f(X_n), g(X_{n+1}, X_{n+2}, \ldots, X_{n+k})\right) \leq q(n)\left[\operatorname{Var} f(X_n) \operatorname{Var} g(X_{n+1}, X_{n+2}, \ldots, X_{n+k})\right]^{1/2}, \tag{1.2}$$

for all $n, k \geq 1$ and for all coordinatewise nondecreasing continuous functions f and g whenever the variances exit.

Obviously, the family of *AANA* sequences contain *NA* (in particular, independent) sequences (with $q(n) = 0$, $n \geq 1$) and some more sequences of random variables which are not much deviated from being *NA*. An example of an *AANA* sequence which is not *NA* was introduced by Chandra and Ghosal [4].

Since the notion of *AANA* sequence was introduced by Chandra and Ghosal [4], the *AANA* properties have aroused wide interest because of numerous applications in reliability theory, percolation theory, and multivariate statistical analysis. In the past decades, a lot of effort was dedicated to proving the limit theorems of *AANA* random variables; we can refer to [4–10]. Hence, extending the limit properties of *AANA* random variables has very important significance in the theory and application.

In this paper, we mainly study the strong convergence property for the partial sums of *AANA* random variables; furthermore the strong convergence property for weighted sums of *AANA* random variables is also obtained.

Throughout the paper, let $I(A)$ be the indicator function of the set A, and let $X^c = -cI(X < -c) + XI(|X| \leq c) + cI(X > c)$ for some $c > 0$. The $a_n = O(b_n)$ denotes that there exits a positive constant C such that $|a_n/b_n| \leq C$. The symbol C represents a positive constant which may be different in various places. The main results of this paper are dependent on the following lemmas.

Lemma 1.2 (Yuan and An [6]). *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$, and let $f_1, f_2, \ldots$ be all nondecreasing (or nonincreasing) functions; then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$.*

Lemma 1.3 (Wang et al. [7]). *For $1 < p \leq 2$, let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$ and $EX_n = 0$ for each $n \geq 1$. If $\sum_{n=1}^{\infty} q^2(n) < \infty$, then there exists a positive constant C_p depending only on p such that*

$$E\left(\max_{1\leq i\leq n}|S_i|^p\right) \leq C_p \sum_{i=1}^{n} E|X_i|^p, \tag{1.3}$$

for all $n \geq 1$ where $S_i = \sum_{j=1}^{i} X_j$, $C_p = 2^p[2^{2-p}p + (6p)^p(\sum_{n=1}^{\infty} q^2(n))^{p/q}]$, and $q = p/(p-1)$ is the dual number of p.

Lemma 1.4 (Wu [11]). *Let $\{X_n, n \geq 1\}$ be a sequence of random variables. For each $n \geq 1$, there exists a random variable X such that*

$$P(|X_n| \geq x) \leq CP(|X| \geq x) \tag{1.4}$$

then, for any $r > 0$, $x > 0$, *the following two statements hold:*

$$E|X_n|^r I(|X_n| \le x) \le C\left[E|X|^r I(|X| \le x) + x^r P(|X| > x)\right],$$
$$E|X_n|^r I(|X_n| > x) \le C\left[E|X|^r I(|X| > x)\right]. \tag{1.5}$$

Lemma 1.5 (Sung [12]). *Let* $\phi(x)$ *be a positive increasing function on* $(0, +\infty)$ *satisfying* $\phi(x) \uparrow \infty$ *as* $n \to \infty$, *and let* $\psi(x)$ *be the inverse function of* $\phi(x)$. *If* $\psi(x)$ *and* $\phi(x)$ *satisfy, respectively,*

$$\psi(n)\sum_{i=1}^{n}\frac{1}{\psi(i)} = O(n), \qquad E\left[\phi(|X|)\right] < \infty, \tag{1.6}$$

then

$$\sum_{i=1}^{\infty}\frac{1}{\psi(n)}E|X|I\left(|X| > \psi(n)\right) < \infty. \tag{1.7}$$

2. Strong Convergence for the Partial Sums of *AANA* Random Variables

Theorem 2.1. *Let* $\{X_n, n \ge 1\}$ *be a sequence of AANA random variables with* $\sum_{n=1}^{\infty} q^2(n) < \infty$, *if the following assumptions holds:*

$$\sum_{n=1}^{\infty}P(|X_n| > c) < \infty, \qquad \sum_{n=1}^{\infty}EX_n^c < \infty, \qquad \sum_{n=1}^{\infty}\operatorname{Var} X_n^c < \infty; \tag{2.1}$$

then $\sum_{n=1}^{\infty} X_n$ *almost surely convergence.*

Remark 2.2. The proof of Theorem 2.1 is similar to the proof of Theorem 4.3.4 in [11], and by Lemmas 1.2 and 1.3, we omit it.

Theorem 2.3. *Let* $\{X_n, n \ge 1\}$ *be a sequence of AANA random variables with* $\sum_{n=1}^{\infty} q^2(n) < \infty$.

Assume that $\{g_n(x), n \ge 1\}$ *is a sequence of even functions in* R^1, *for each* $n \ge 1$, $g_n(x)$ *is a positive nondecreasing function in* $(0, +\infty)$ *and satisfies one of the following conditions:*

(i) *for* $x \in (0, 1]$ *there exists a constant* $\alpha > 0$ *such that* $g_n(x) \ge \alpha x$;

(ii) *for* $x \in (0, 1]$, *there exists a constant* $r \in (1, 2]$ *and* $\alpha > 0$ *such that* $g_n(x) \ge \alpha x^r$; *however, for* $x \in (1, \infty)$, $g_n(x) \ge \alpha x$, *furthermore assume that* $EX_n = 0$, *for each* $n \ge 1$.

Let $\{a_n, n \ge 1\}$ *be a constant sequence satisfying* $0 < a_n \uparrow \infty$ *such that*

$$\sum_{n=1}^{\infty}Eg_n\left(\frac{X_n}{a_n}\right) < \infty, \tag{2.2}$$

then $\sum_{n=1}^{\infty}(X_n/a_n)$ *almost surely convergence, and further it follows from the "Kronecker lemma" that*

$$a_n^{-1}\sum_{k=1}^{n}X_k \longrightarrow 0 \ a.s., \quad as\ n \longrightarrow \infty. \tag{2.3}$$

Proof. For each $n \geq 1$, denote $X_n^{a_n} \triangleq -a_n I(X_n < -a_n) + X_n I(|X_n| \leq a_n) + a_n I(X_n > a_n)$.

By Lemma 1.2, we can see that, for fixed $n \geq 1$, $\{X_n^{a_n}\}$ is still a sequence of *AANA* random variables. To verity the Theorem 2.3, for $c = 1$ we only need to prove the convergence of three series of (2.1) under condition (i) or (ii). The proof of Theorem 2.3 includes the following three steps.

(1) *We prove* $\sum_{n=1}^{\infty} P(|X_n/a_n| > 1) < \infty$ *under condition (i) or (ii).*

For each $n \geq 1$, if $g_n(x)$ satisfies condition (i), noting that $g_n(x)$ is a positive nondecreasing even function in $(0, +\infty)$, it is obvious that

$$P\left(\left|\frac{X_n}{a_n}\right| > 1\right) = EI\left(\left|\frac{X_n}{a_n}\right| > 1\right) \leq \alpha^{-1} E g_n\left(\frac{X_n}{a_n}\right). \tag{2.4}$$

By (2.2), we can get

$$\sum_{n=1}^{\infty} P\left(\left|\frac{X_n}{a_n}\right| > 1\right) \leq \alpha^{-1} \sum_{n=1}^{\infty} E g_n\left(\frac{X_n}{a_n}\right) < \infty. \tag{2.5}$$

If $g_n(x)$ satisfies condition (ii), it is easy to prove that (2.5) also holds when $|X_n| > a_n > 0$.

(2) *Next we will show* $\sum_{n=1}^{\infty} E|X_n^{a_n}/a_n| < \infty$.

If $g_n(x)$ satisfies condition (i), it follows that

$$\begin{aligned}
\left|E\frac{X_n^{a_n}}{a_n}\right| &= \left|-EI(X_n < -a_n) + E\frac{X_n}{a_n} I(|X_n| \leq a_n) + EI(X_n > a_n)\right| \\
&\leq EI(|X_n| > a_n) + \left|E\frac{X_n}{a_n} I(|X_n| \leq a_n)\right| \\
&\leq \alpha^{-1} E g_n\left(\frac{X_n}{a_n}\right) + \left|\int_{|X_n| \leq a_n} \frac{X_n}{a_n} dP\right| \\
&\leq 2\alpha^{-1} E g_n\left(\frac{X_n}{a_n}\right).
\end{aligned} \tag{2.6}$$

On the other hand, if condition (ii) holds, according to $EX_n = 0$, for each $n \geq 1$, we have

$$\begin{aligned}\left|E\frac{X_n^{a_n}}{a_n}\right| &\leq EI(|X_n| > a_n) + \left|E\frac{X_n}{a_n}I(|X_n| \leq a_n)\right| \\ &= EI(|X_n| > a_n) + \left|E\frac{X_n}{a_n}I(|X_n| > a_n)\right| \\ &\leq 2\alpha^{-1}Eg_n\left(\frac{X_n}{a_n}\right).\end{aligned} \tag{2.7}$$

Hence, it follows from (2.2) that

$$\sum_{n=1}^{\infty}E\left|\frac{X_n^{a_n}}{a_n}\right| < 2\alpha^{-1}\sum_{n=1}^{\infty}Eg_n\left(\frac{X_n}{a_n}\right) < \infty. \tag{2.8}$$

(3) Finally we prove $\sum_{n=1}^{\infty} E(X_n^{a_n}/a_n)^2 < \infty$.

If $g_n(x)$ satisfies condition (i), for each $n \geq 1$, it is easy to show that by the C_r−inequality

$$\begin{aligned}E\left(\frac{X_n^{a_n}}{a_n}\right)^2 &= E\left|-I(X_n < -a_n) + \frac{X_n}{a_n}I(|X_n| \leq a_n) + I(X_n > a_n)\right|^2 \\ &\leq 3E\left[I(|X_n| > a_n) + \left[\frac{X_n}{a_n}\right]^2 I(|X_n| \leq a_n)\right] \\ &\leq C\alpha^{-1}Eg_n\left(\frac{X_n}{a_n}\right) + CE\left|\frac{X_n}{a_n}\right|I(|X_n| \leq a_n) \\ &\leq C\alpha^{-1}Eg_n\left(\frac{X_n}{a_n}\right).\end{aligned} \tag{2.9}$$

If condition (ii) holds, according to the C_r-inequality, for each $n \geq 1$, we get

$$\begin{aligned}E\left(\frac{X_n^{a_n}}{a_n}\right)^2 &= E\left|-I(X_n < -a_n) + \frac{X_n}{a_n}I(|X_n| \leq a_n) + I(X_n > a_n)\right|^2 \\ &\leq 3E\left[I(|X_n| > a_n) + \left(\frac{X_n}{a_n}\right)^2 I(|X_n| \leq a_n)\right] \\ &\leq C\alpha^{-1}Eg_n\left(\frac{X_n}{a_n}\right) + CE\left|\frac{X_n}{a_n}\right|^r I(|X_n| \leq a_n) \\ &\leq C\alpha^{-1}Eg_n\left(\frac{X_n}{a_n}\right).\end{aligned} \tag{2.10}$$

Therefore, it also follows from (2.2) that

$$\sum_{n=1}^{\infty} E\left(\frac{X_n^{a_n}}{a_n}\right)^2 < C\alpha^{-1}\sum_{n=1}^{\infty} Eg_n\left(\frac{X_n}{a_n}\right) < \infty. \tag{2.11}$$

The proof of the Theorem 2.3 is completed by (2.5), (2.8), and (2.11). □

Corollary 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$, and let $\{a_n, n \geq 1\}$ be a constant sequence satisfying $0 < a_n \uparrow \infty$. For $\theta \in (0,1]$, let $g_n(x) = |x|^\theta/(1+|x|^\theta)$, and if $\{X_n/a_n, n \geq 1\}$ satisfies (2.2), then $a_n^{-1}\sum_{k=1}^{n} X_k \to 0$ a.s., as $n \to \infty$.*

Proof. It is easy to check that $\{g_n(x), n \geq 1\}$ is a sequence of even functions in R^1, for each $n \geq 1$, $g_n(x)$ is a positive nondecreasing function in $(0,+\infty)$, and the following condition holds:

$$g_n(x) \geq \frac{1}{2}x^\theta \geq \frac{1}{2}x, \quad 0 < x \leq 1, \ 0 < \theta \leq 1. \tag{2.12}$$

□

3. Strong Convergence for the Weighted Sums of *AANA* Random Variables

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ be a different distribution sequence of AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$ and $EX_n = 0$, for each $n \geq 1$. There exists a random variable X satisfying $E|X|^r < \infty, 0 < r \leq 2$, such that*

$$P(|X_n| > x) \leq CP(|X| > x), \quad n \geq 1, \ x > 0. \tag{3.1}$$

Assume that the following conditions hold for the constant arrays $\{a_{ni}, n \geq 1, \ 1 \leq i \leq n\}$.

(i) $\max_{1\leq i\leq n}|a_{ni}| = O(\psi^{-1}(n))$; (ii) for some constant $\delta > 0$, $\sum_{i=1}^{n} |a_{ni}|^r = O(n^{-1}\log^{-1-\delta} n)$, where $\phi(x), \psi(x)$ satisfy Lemma 1.5; then

$$T_n = \sum_{i=1}^{n} a_{ni}X_i \xrightarrow{a.s.} 0, n \longrightarrow \infty. \tag{3.2}$$

Proof. Let $Y_i = -\psi(n)I(X_i < -\psi(n)) + X_iI(|X_i| \leq \psi(n)) + \psi(n)I(X_i > \psi(n))$, $\overline{Y}_i = Y_i - EY_i$:

$$T_n = \sum_{i=1}^{n} a_{ni}(X_i - Y_i) + \sum_{i=1}^{n} a_{ni}\overline{Y}_i + \sum_{i=1}^{n} a_{ni}EY_i \triangleq T_{n1} + T_{n2} + T_{n3}. \tag{3.3}$$

It suffices to prove that $T_{ni} \to 0$ a.s., as $n \to \infty, i = 1,2,3$. We will estimate each of these terms separately.

To verity $T_{n1} \to 0$ a.s., as $n \to \infty$, we can get from (3.1) and $E\phi(|X|) < \infty$ that

$$\begin{aligned}\sum_{n=1}^{\infty} P(X_i \neq Y_i) &= \sum_{n=1}^{\infty} P(|X_i| > \psi(n)) \\ &\leq C\sum_{n=1}^{\infty} P(|X| > \psi(n)) \\ &= C\sum_{n=1}^{\infty} P(\phi|X| > n) \\ &\leq CE\phi(|X|) < \infty.\end{aligned} \tag{3.4}$$

Hence, by the Borel-Cantelli Lemma it is obvious that $T_{n1} \to 0$ a.s., as $n \to \infty$.

Next we will show that $T_{n2} \to 0$ as $n \to \infty$ almost surely. For any $\varepsilon > 0$, $0 < r \leq 2$, note that $E|X|^r < \infty$, and it follows from the Markov inequality, Lemma 1.2, Lemma 1.3, C_r-inequality, and Lemma 1.5 that

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}\overline{Y}_i > \varepsilon\right) &\leq C\sum_{n=1}^{\infty} E\left|\sum_{i=1}^{n} a_{ni}\overline{Y}_i\right|^r \leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\left|a_{ni}\overline{Y}_i\right|^r \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} |a_{ni}|^r \left[E|X_i|^r I(|X_i| \leq \psi(n)) + \psi^r(n) EI(|X_i| > \psi(n))\right] \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} |a_{ni}|^r \left(E|X|^r I(|X| \leq \psi(n)) + \psi^r(n) EI(|X| > \psi(n))\right) \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} |a_{ni}|^r \left(EX|^r I(|X| \leq \psi(n)) + E|X|^r\right) \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} |a_{ni}|^r \\ &\leq C\sum_{n=1}^{\infty} \frac{1}{n\log^{1+\delta} n} < \infty.\end{aligned} \tag{3.5}$$

the last series converges using condition (ii), and by Borel-Cantelli lemma we get $T_{n2} \to 0$ a.s., as $n \to \infty$.

Finally we will prove that $T_{n3} \to 0$ a.s., as $n \to \infty$. Note that $EX_n = 0$; for each $n \geq 1$, it is easy to show that by Lemma 1.5, Lemma 1.4, and theKronecker lemma

$$\begin{aligned}\left|\sum_{i=1}^{n} Ea_{ni}Y_i\right| &\leq \left|\sum_{i=1}^{n} Ea_{ni}X_i I(|X_i| \leq \psi(n))\right| + \left|\sum_{i=1}^{n} a_{ni}\psi(n) EI(|X_i| > \psi(n))\right| \\ &\leq \left|\sum_{i=1}^{n} Ea_{ni}X_i I(|X_i| > \psi(n))\right| + \left|\sum_{i=1}^{n} a_{ni}\psi(n) EI(|X_i| > \psi(n))\right| \\ &\leq C\sum_{i=1}^{n} E|a_{ni}X_i| I(|X_i| > \psi(i))\end{aligned}$$

$$\leq C\sum_{i=1}^{n}E|a_{ni}X|I\left(|X| > \psi(i)\right)$$
$$\leq \frac{1}{\psi(n)}\sum_{i=1}^{n}E|X|I\left(|X| > \psi(i)\right) \longrightarrow 0, n \longrightarrow \infty. \tag{3.6}$$

The proof of Theorem 3.1 is completed. □

Acknowledgments

This paper is supported by the National Natural Science Foundantion of China (10901003) and the Natural Science Foundation of Anhui Province (KJ2012ZD001, KJ2013A126, KJ2012Z233).

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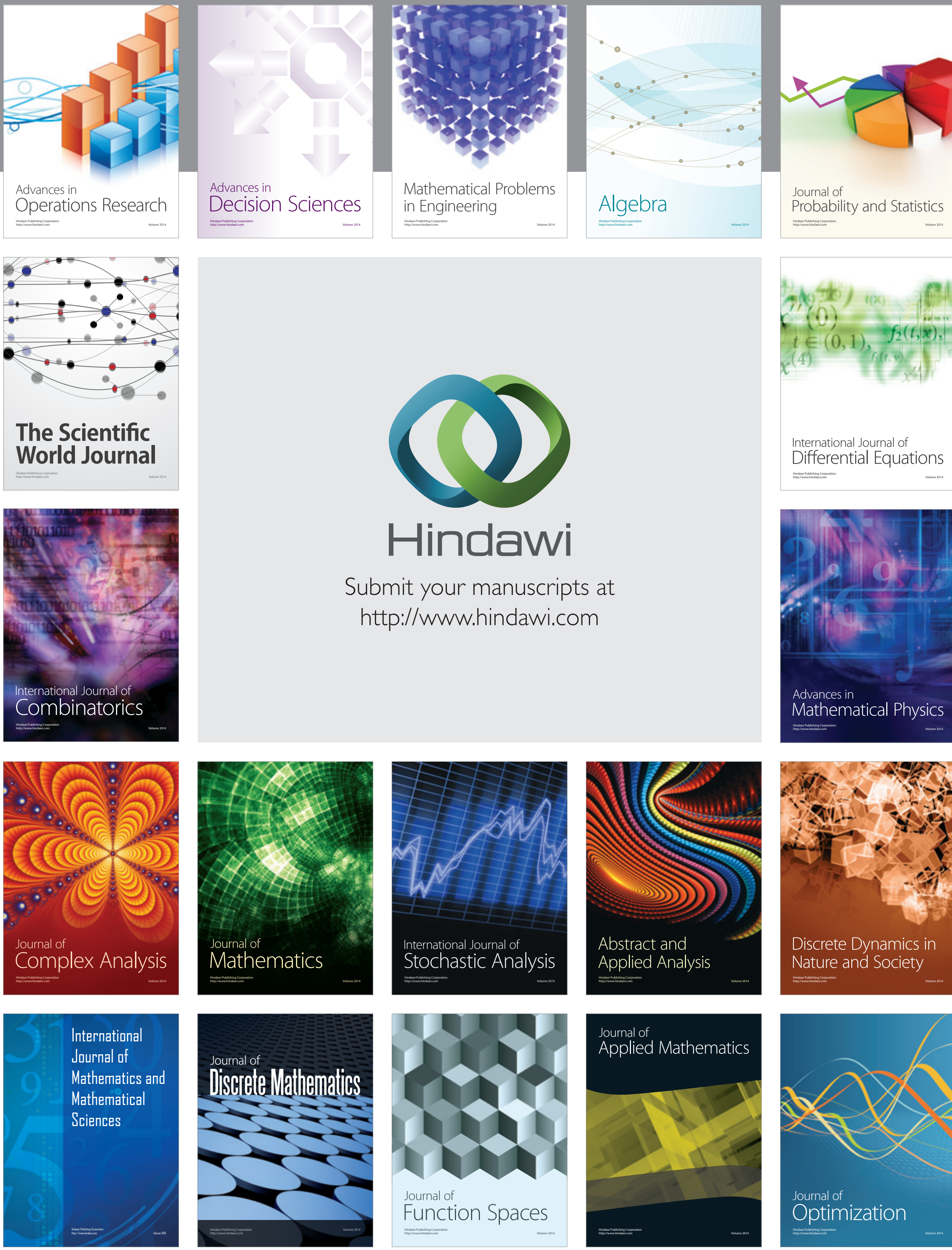

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