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Research Article

Existence for Eventually Positive Solutions of High-Order Nonlinear Neutral Differential Equations with Distributed Delay

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We consider the existence for eventually positive solutions of high-order nonlinear neutral differential equations with distributed delay. We use *Lebesgue's* dominated convergence theorem to obtain new necessary and sufficient condition for the existence of eventually positive solutions.

1. Introduction and Preliminary

In this paper, we consider the high-order nonlinear neutral differential equation:

$$\left[r(t)x(t)-\int_a^b p(t,\tau)x(t-\tau)d\tau\right]^{(n)}+\int_c^d q(t,\sigma)f(x(t-\sigma))d\sigma=0,\quad t\geq t_0, \tag{1.1}$$

and the associated inequality:

$$\left[r(t)x(t)-\int_a^b p(t,\tau)x(t-\tau)d\tau\right]^{(n)}+\int_c^d q(t,\sigma)f(x(t-\sigma))d\sigma\leq 0,\quad t\geq t_0, \tag{1.2}$$

(1) where n is a positive integer, $0<a<b$, $0<c<d$,

(2) $p\in C([t_0,\infty)\times[a,b],R^+)$,

(3) $r \in C([t_0,\infty), R^+)$, $r(t) > 0$, $q \in C([t_0,\infty) \times [c,d], R^+)$,

(4) $f(u)$ are continuously nondecreasing real function with respect to u defined on R such that $uf(u) > 0$, for $u > 0$, for n is positive odd integer; $f(u)$ are continuously decreasing real function with respect to u defined on R such that $uf(u) > 0$, for $u > 0$, for n is positive even integer.

Recently, there has been a lot of activities concerning the existence of eventually positive solutions for nonlinear neutral differential equations. See [1–8]. In [1], Liu et al. have studied the even-order neutral differential equation:

$$\left[a(t)x(t) - \sum_{i=1}^{m} b_i(t)x(t-\tau_i)\right]^{(n)} - \sum_{j=1}^{l} p_j(t)f_j\left(t, x\left(t-\sigma_j\right)\right) = 0, \quad t \geq 0, \tag{1.3}$$

and the associated differential inequality:

$$\left[a(t)x(t) - \sum_{i=1}^{m} b_i(t)x(t-\tau_i)\right]^{(n)} - \sum_{j=1}^{l} p_j(t)f_j\left(t, x\left(t-\sigma_j\right)\right) \geq 0, \quad t \geq 0. \tag{1.4}$$

They have obtained that the existences of eventually positive solutions of (1.3) and (1.4) are equivalent. In [2], Ouyang et al. has studied the odd-order neutral differential equation:

$$\left[a(t)x(t) - \sum_{i=1}^{m} b_i(t)x(t-\tau_i)\right]^{(n)} + \sum_{j=1}^{l} p_j(t)f_j\left(x\left(t-\sigma_j\right)\right) = 0, \quad t \geq 0, \tag{1.5}$$

and the associated differential inequality:

$$\left[a(t)x(t) - \sum_{i=1}^{m} b_i(t)x(t-\tau_i)\right]^{(n)} + \sum_{j=1}^{l} p_j(t)f_j\left(x\left(t-\sigma_j\right)\right) \leq 0, \quad t \geq 0. \tag{1.6}$$

He has obtained that the existences of eventually positive solutions of (1.5) and (1.6) are equivalent.

As usual, a solution of (1.1) is a continuous function $x(t)$ defined on $[-\mu,\infty)$ such that $y(t) := r(t)x(t) - \int_a^b p(t,\tau)x(t-\tau)d\tau$ is n times differentiable and (1.1) holds for all $n \geq 0$. Such a solution $x(t)$ is called an eventually positive solution if there is $T \geq t_0$, such that $x(t) > 0$, for $t \geq T$. Here, $\mu = \max\{b,d\}$.

Lemma 1.1. *Assume* $\sup_{t\geq t_0} r(t) < \infty$, *and* $\int_a^b p(t,\tau)d\tau / r(t) \leq M$, *let* $x(t)$ *is an eventually bounded positive solution of inequality* (1.2), *and set*

$$y(t) = r(t)x(t) - \int_a^b p(t,\tau)x(t-\tau)d\tau. \tag{1.7}$$

If n is a positive odd integer, then eventually,

$$\begin{gathered}
y^{(n)}(t) \leq 0, \qquad (-1)^{k+1} y^{(k)}(t) < 0, \quad (k = 0, 1, \ldots, n-1), \\
\lim_{t \to \infty} y^{(k)}(t) = 0, \quad (k = 1, 2, \ldots, n-1).
\end{gathered} \tag{1.8}$$

If n is a positive even integer, then eventually,

$$\begin{gathered}
y^{(n)}(t) \leq 0, \qquad (-1)^{k} y^{(k)}(t) < 0, \quad (k = 0, 1, \ldots, n-1), \\
\lim_{t \to \infty} y^{(k)}(t) = 0, \quad (k = 1, 2, \ldots, n-1).
\end{gathered} \tag{1.9}$$

Proof. We have the following cases.

Case 1 (If n is a positive odd integer). Because $x(t)$ is bounded, and $\sup_{t \geq t_0} r(t) < \infty$, $\int_a^b p(t,\tau)d\tau / r(t) \leq M$, thus $y(t)$ is bounded. From (1.2), we have $y^{(n)}(t) \leq -\int_c^d q(t,\sigma) f(x(t-\sigma))d\sigma \leq 0$. Assume $y^{(n-1)}(t) \leq 0$, since $y^{(n)}(t) \leq 0$, $y^{(n-1)}(t)$ decreases, set $\lim_{t \to \infty} y^{(n-1)}(t) = L$, then $-\infty \leq L < 0$.
Thus,

$$y^{(n-2)}(t) - y^{(n-2)}(t_0) = \int_{t_0}^{t} y^{(n-1)}(t)dt \leq (L+\varepsilon)(t-t_0) \longrightarrow -\infty, \quad (t \longrightarrow \infty), \tag{1.10}$$

that is, $\lim_{t \to \infty} y^{(n-2)}(t) = -\infty$. Simile, $\lim_{t \to \infty} y^{(k)}(t) = -\infty$, $(k = 0, 1, \ldots, n-2)$, this is a contradiction and $y(t)$ is bounded, therefore, $y^{(n-1)}(t) > 0$.
Again, since it decreases, set $\lim_{t \to \infty} y^{(n-1)}(t) = L$, thus $L \geq 0$. Next, we proof $L = 0$. Assume $L > 0$, then,

$$y^{(n-2)}(t) - y^{(n-2)}(t_0) = \int_{t_0}^{t} y^{(n-1)}(t)dt \geq (L-\varepsilon)(t-t_0) \longrightarrow \infty, \quad (t \longrightarrow \infty), \tag{1.11}$$

then, $\lim_{t \to \infty} y^{(n-2)}(t) = \infty$. Simile, $\lim_{t \to \infty} y^{(k)}(t) = \infty$, $(k = 0, 1, \ldots, n-2)$, this is a contradiction and $y(t)$ is bounded, therefore, $L = 0$. That is, $\lim_{t \to \infty} y^{(n-1)}(t) = 0$. Similarly, we obtain

$$(-1)^{k+1} y^{(k)}(t) < 0, \quad (k = 0, 1, \ldots, n-1), \qquad \lim_{t \to \infty} y^{(k)}(t) = 0, \quad (k = 1, 2, \ldots, n-1). \tag{1.12}$$

Case 2 (If n is a positive even integer). The proof of Case 2 is similar to that of part Case 1, therefore, it is omitted. We obtain

$$(-1)^{k} y^{(k)}(t) < 0, \quad (k = 0, 1, \ldots, n-1), \qquad \lim_{t \to \infty} y^{(k)}(t) = 0, \quad (k = 1, 2, \ldots, n-1). \tag{1.13}$$

The proof is complete. □

Lemma 1.2 (see [3, page 21]). *Let $\eta \in (-\infty, 0)$, $\tau \in (0, \infty)$, $t_0 \in R$, and suppose that a function $x(t) \in C([t_0 - \tau, \infty), R)$ satisfies the inequality:*

$$x(t) \leq \eta + \max_{t-\tau \leq s \leq t} x(s), \quad t \geq t_0. \tag{1.14}$$

Then, $x(t)$ cannot be a nonnegative function.

Lemma 1.3. *Suppose that $\sup_{t \geq t_0} r(t) < \infty$, and $\int_a^b p(t,\tau)d\tau / a(t) \leq 1$. Let $x(t)$ be an eventually positive solution of (1.2) and set*

$$y(t) = r(t)x(t) - \int_a^b p(t,\tau)x(t-\tau)d\tau, \tag{1.15}$$

then eventually

$$y(t) > 0. \tag{1.16}$$

Proof. From (1.2) and (1.7), eventually, we have

$$y^{(n)}(t) \leq -\int_c^d q(t,\sigma)f(x(t-\sigma))d\sigma \leq 0, \tag{1.17}$$

and the hypotheses on $q(t,\sigma)$, $f(x)$, and $x(t)$ yield that $y^{(n)}$ is not eventually zero. Thus, $y^{(i)}(t)$ $(i = 0, l, \ldots, n-1)$ is eventually nonzero. Hence, if $y(t) > 0$ does not hold, then eventually $y(t) < 0$, $y'(t) < 0$, or $y(t) < 0$, $y'(t) > 0$.

Case 1. $y(t) < 0$, $y'(t) < 0$, then there exists $t_1 > 0$ such that $y(t) \leq y(t_1) < 0$ for $t > t_1$. Then, $\alpha := -y(t_1)/\sup_{t \geq t_0} r(t) > 0$. In view of (1.3) and $\int_a^b p(t,\tau)d\tau / a(t) \leq 1$, we obtain

$$\begin{aligned} x(t) &= \frac{y(t)}{r(t)} + \frac{1}{r(t)} \int_a^b p(t,\tau)x(t-\tau)d\tau \\ &\leq -\alpha + \frac{1}{r(t)} \int_a^{\mathrm{b}} p(t,\tau)d\tau \max_{t-b \leq s \leq t} x(s) \\ &\leq -\alpha + \max_{t-b \leq s \leq t} x(s). \end{aligned} \tag{1.18}$$

According to Lemma 1.2, we obtain $x(t) < 0$. This is a contradiction and so $y(t)$ is eventually positive.

Case 2. $y(t) < 0$, $y'(t) > 0$, one argues that $y''(t) < 0$ and repeats the argument to obtain that $y^{(n)}(t) > 0$ which contradicts the offset after $y^{(n)}(t) < 0$.

The proof is complete. □

2. Comparison Theory of Existence for Eventually Positive Solution

Theorem 2.1. *Assume all conditions of Lemma 1.1 hold, n is a positive odd integer. And*

(H_1) $p(t,\tau)+q(t,\sigma)>0$, *for sufficiently large t.*

Then (1.1) *has an eventually bounded positive solution if and only if inequality* (1.2) *has an eventually bounded positive solution.*

Proof. It is clear that an eventually bounded positive solution of (1.1) is also an eventually bounded positive solution of (1.2). So, it suffices to prove that if (1.2) has an eventually bounded positive solution $x(t)$, for $t>t_0$, then so does (1.1). Set

$$y(t)=r(t)x(t)-\int_a^b p(t,\tau)x(t-\tau)d\tau. \tag{2.1}$$

It follows from Lemma 1.1 and (1.2) that eventually,

$$\begin{gathered} y^{(n)}(t)\le 0, \qquad (-1)^{k+1}y^{(k)}(t)>0, \quad (k=0,1,\dots,n-1), \\ \lim_{t\to\infty} y^k(t)=0, \quad (k=1,2,\dots,n-1). \end{gathered} \tag{2.2}$$

By using (1.8) and integrating (1.2) from t to ∞, we obtain

$$\begin{gathered} y^{(n-1)}(\infty)-y^{(n-1)}(t)\le-\int_t^\infty\left[\int_c^d q(t,\sigma)f(x(t-\sigma))d\sigma\right]ds, \\ y^{(n-1)}(t)\ge\int_t^\infty\left[\int_c^d q(s,\sigma)f(x(s-\sigma))d\sigma\right]ds. \end{gathered} \tag{2.3}$$

By repeating the same procedure n times and by using (1.8), we are led to the inequality:

$$y(t)\ge\int_t^\infty dt_n\int_{t_n}^\infty dt_{n-1}\int_{t_{n-1}}^\infty dt_{n-2}\cdots\int_{t_2}^\infty\left[\int_c^d q(s,\sigma)f(x(s-\sigma))d\sigma\right]ds, \quad t\ge t_0. \tag{2.4}$$

Using *Tonelli's* theorem, we reverse the order of integration and obtain

$$=\int_t^\infty\frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma)f(x(s-\sigma))d\sigma\right]ds, \quad t\ge t_0. \tag{2.5}$$

That is,

$$x(t)\ge\frac{1}{r(t)}\int_a^b p(t,\tau)x(t-\tau)d\tau+\frac{1}{r(t)}\int_t^\infty\frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma)f(x(s-\sigma))d\sigma\right]ds, \quad t\ge t_0. \tag{2.6}$$

Let $T \geq t_0$ be such that (2.6) hold, and $x(t-\mu) > 0,\ t \geq T$. Now, we consider the set of functions

$$\Omega = \{z \in C([T-\mu, \infty), R^+) : 0 \leq z(t) \leq 1,\ t \geq T-\mu\} \tag{2.7}$$

and define an operator S on Ω as follows:

$$(Sz)(t) = \begin{cases} \dfrac{t-T+\mu}{\mu}(Sz)(T) + \left(1 - \dfrac{t-T+\mu}{\mu}\right), & T-\mu \leq t < T, \\ \dfrac{1}{x(t)}\left\{\dfrac{1}{r(t)}\displaystyle\int_a^b p(t,\tau)z(t-\tau)x(t-\tau)d\tau \right. & \\ \qquad \left. +\dfrac{1}{r(t)}\displaystyle\int_t^\infty \dfrac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma)f(z(s-\sigma)x(s-\sigma))d\sigma\right]ds\right\}, & t \geq T. \end{cases} \tag{2.8}$$

Then, it follows from *Lebesgue's* dominated convergence theorem that S is continuous. By using (2.6), it is easy to see that S maps Ω into itself, and for any $z \in \Omega$, we have $(Sz)(t) > 0$, for $T-\mu \leq t < T$. Next, we define the sequence $z_k(t) \in \Omega$

$$\begin{gathered} z_0(t) \equiv 1, \quad t \geq T-\mu, \\ z_{k+1}(t) = (Sz_k)(t), \quad t \geq T-\mu,\ k = 0, 1, \ldots. \end{gathered} \tag{2.9}$$

Then, by using (2.6) and a simple induction, we can easily see that

$$0 \leq z_{k+1}(t) \leq z_k(t) \leq 1, \quad t \geq T-\mu. \tag{2.10}$$

Set $\lim_{k\to\infty} z_k(t) = z(t),\ t \geq T-\mu$ then $z(t)$ satisfies

$$z(t) = \begin{cases} \dfrac{t-T+\mu}{\mu}(Sz)(T) + \left(1 - \dfrac{t-T+\mu}{\mu}\right), & T-\mu \leq t < T, \\ \dfrac{1}{x(t)}\left\{\dfrac{1}{r(t)}\displaystyle\int_a^b p(t,\tau)z(t-\tau)x(t-\tau)d\tau \right. & \\ \qquad \left. +\dfrac{1}{r(t)}\displaystyle\int_t^\infty \dfrac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma)f(z(s-\sigma)x(s-\sigma))d\sigma\right]ds\right\}, & t \geq T. \end{cases} \tag{2.11}$$

Again, set $\omega(t) = z(t)x(t)$, then $\omega(t)$ satisfies $\omega(t) > 0,\ T-\mu \leq t \leq T$, and

$$\omega(t) = \frac{1}{r(t)}\int_a^b p(t,\tau)\omega(t-\tau)d\tau + \frac{1}{r(t)}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma)f(\omega(s-\sigma))d\sigma\right]ds, \quad t \geq T. \tag{2.12}$$

Thus, $\omega(t)$ is a positive solution of (1.1) for $t \geq T$.

The following:

$$\omega(t) > 0, \quad t \geq T - \mu. \tag{2.13}$$

Assume that there exists $t^* \geq T - \mu$, such that $\omega(t) > 0$, for $T - \mu \leq t \leq t^*$, and $\omega(t^*) = 0$. Then,

$$\begin{aligned} 0 = \omega(t^*) &= \frac{1}{r(t^*)} \int_a^b p(t^*, \tau)\omega(t^* - \tau)d\tau \\ &\quad + \frac{1}{r(t^*)} \int_{t^*}^{\infty} \frac{(s - t^*)^{n-1}}{(n-1)!} \left[\int_c^d q(s, \sigma) f(\omega(s - \sigma))d\sigma\right] ds, \quad t^* \geq T, \end{aligned} \tag{2.14}$$

which implies

$$\begin{aligned} p(t^*, \tau) &= 0, \\ q(s, \sigma) f(\omega(s - \sigma)) &\equiv 0, \end{aligned} \tag{2.15}$$

which contradicts (H_1). Thus, $\omega(t)$ is an eventually bounded positive solution of (1.1).

The proof is complete. □

Theorem 2.2. *Assume all conditions of Lemma 1.3 hold, n is a positive odd integer. And*

(H_1) $p(t, \tau) + q(t, \sigma) > 0$, *for sufficiently large t.*

Then, (1.1) *has an eventually positive solution if and only if inequality* (1.2) *has an eventually positive solution.*

Proof. It is clear that an eventually positive solution of (1.1) is also an eventually positive solution of (1.2). So, it suffices to prove that if (1.2) has an eventually positive solution $x(t)$, for $t > t_0$, then so does (1.1). Set

$$y(t) = r(t)x(t) - \int_a^b p(t, \tau)x(t - \tau)d\tau. \tag{2.16}$$

It follows from Lemma 1.3 and (1.2) that eventually, $y^{(n)}(t) \leq 0$, $y(t) > 0$, which implies that there exists a nonnegative even integer $n^* \leq n - 1$, such that eventually

$$\begin{aligned} y^{(k)}(t) > 0, &\quad (k = 0, 1, \ldots, n^*), \\ (-1)^{(k)} y^{(k)}(t) > 0, &\quad (k = n^*, \ldots, n - 1). \end{aligned} \tag{2.17}$$

We consider the following possible cases.

Case 1 ($n^* = 0$). Since n is a positive integer, $n - 1$ is an even integer, we can easily see that there exists a $T' > 0$, such $y^{(n-1)}(t) > 0$, and $y^n(t) \leq 0$, $t \geq T'$, $y^{(n-1)}(\infty) \geq 0$.

By using (2.17) and integrating (1.2) from t to ∞, we obtain

$$\begin{gathered}y^{(n-1)}(\infty)-y^{(n-1)}(t)\leq-\int_t^{\infty}\left[\int_c^d q(s,\sigma)f(x(s-\sigma))d\sigma\right]ds,\\ y^{(n-1)}(t)\geq\int_t^{\infty}\left[\int_c^d q(s,\sigma)f(x(s-\sigma))d\sigma\right]ds.\end{gathered}\tag{2.18}$$

By repeating the same procedure n times and by using (2.31), we are led to the inequality:

$$y(t)\geq\int_t^{\infty}dt_n\int_{t_n}^{\infty}dt_{n-1}\int_{t_{n-1}}^{\infty}dt_{n-2}\cdots\int_{t_2}^{\infty}\left[\int_c^d q(s,\sigma)f(x(s-\sigma))d\sigma\right]ds,\quad t\geq t_0.\tag{2.19}$$

Using *Tonelli's* theorem, we reverse the order of integration and obtain

$$=\int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma)f(x(s-\sigma))d\sigma\right]ds,\quad t\geq t_0.\tag{2.20}$$

That is,

$$x(t)\geq\frac{1}{r(t)}\int_a^b p(t,\tau)x(t-\tau)d\tau+\frac{1}{r(t)}\int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(t,\sigma)f(x(t-\sigma))d\sigma\right]ds,\quad t\geq t_0.\tag{2.21}$$

Let $T\geq t_0$ be such that (2.21) hold, and $x(t-\mu)>0$, $t\geq T$. Now, we consider the set of functions

$$\Omega=\left\{z\in C\left(\left[T-\mu,\infty\right),R^{+}\right):0\leq z(t)\leq 1,\ t\geq T-\mu\right\},\tag{2.22}$$

and define an operator S on Ω as follows:

$$(Sz)(t)=\begin{cases}\dfrac{t-T+\mu}{\mu}(Sz)(T)+\left(1-\dfrac{t-T+\mu}{\mu}\right), & T-\mu\leq t<T,\\ \dfrac{1}{x(t)}\left\{\dfrac{1}{r(t)}\displaystyle\int_a^b p(t,\tau)z(t-\tau)x(t-\tau)d\tau\right. & \\ \qquad\left.+\dfrac{1}{r(t)}\displaystyle\int_t^{\infty}\dfrac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma)f(z(s-\sigma)x(s-\sigma))d\sigma\right]ds\right\}, & t\geq T.\end{cases}\tag{2.23}$$

Then, it follows from *Lebesgue's* dominated convergence theorem that S is continuous. By using (2.21), it is easy to see that S maps Ω into itself, and for any $z \in \Omega$, we have $(Sz)(t) > 0$, for $T - \mu \le t < T$. Next, we define the sequence $z_k(t) \in \Omega$

$$\begin{aligned} z_0(t) &\equiv 1, \quad t \ge T - \mu, \\ z_{k+1}(t) &= (Sz_k)(t), \quad t \ge T - \mu, \ k = 0, 1, \ldots. \end{aligned} \tag{2.24}$$

Then, by using (2.21) and a simple induction, we can easily see that

$$0 \le z_{k+1}(t) \le z_k(t) \le 1, \quad t \ge T - \mu. \tag{2.25}$$

Set $\lim_{k\to\infty} z_k(t) = z(t), \ t \ge T - \mu$, then $z(t)$ satisfies:

$$z(t) = \begin{cases} \dfrac{t - T + \mu}{\mu}(Sz)(T) + \left(1 - \dfrac{t - T + \mu}{\mu}\right), & T - \mu \le t < T, \\ \dfrac{1}{x(t)}\left\{ \dfrac{1}{r(t)}\displaystyle\int_a^b p(t,\tau) z(t-\tau) x(t-\tau) d\tau \right. \\ \qquad \left. + \dfrac{1}{r(t)}\displaystyle\int_t^\infty \dfrac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma) f(z(s-\sigma)x(s-\sigma)) d\sigma\right] ds \right\}, & t \ge T. \end{cases} \tag{2.26}$$

Again, set $\omega(t) = z(t)x(t)$, then $\omega(t)$ satisfies $\omega(t) > 0, \ T - \mu \le t \le T$, and

$$\omega(t) = \frac{1}{r(t)}\int_a^b p(t,\tau)\omega(t-\tau)d\tau + \frac{1}{r(t)}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma) f(\omega(s-\sigma)) d\sigma\right] ds, \quad t \ge T. \tag{2.27}$$

Thus, $\omega(t)$ is a positive solution of (1.1) for $t \ge T$.

Consider the following:

$$\omega(t) > 0, \quad t \ge T - \mu. \tag{2.28}$$

Assume that there exists $t^* \ge T - \mu$, such that $\omega(t) > 0$, for $T - \mu \le t \le t^*$, and $\omega(t^*) = 0$. Then,

$$\begin{aligned} 0 = \omega(t^*) &= \frac{1}{r(t^*)}\int_a^b p(t^*,\tau)\omega(t^*-\tau)d\tau \\ &\quad + \frac{1}{r(t^*)}\int_{t^*}^\infty \frac{(s-t^*)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma) f(\omega(s-\sigma)) d\sigma\right] ds, \quad t^* \ge T, \end{aligned} \tag{2.29}$$

which implies

$$p(t^*,\tau) = 0,$$
$$q(s,\sigma)f(\omega(s-\sigma)) \equiv 0, \tag{2.30}$$

which contradicts (H_1). Thus, $\omega(t)$ is an eventually positive solution of equation.

Case 2 $(2 \le n^* \le n-1)$. By using (2.17) and integrating (1.2) from t to ∞, we obtain

$$y^{(n^*)}(t) \ge \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma)f(x(s-\sigma))d\sigma\right]ds, \quad t \ge t_0. \tag{2.31}$$

Let $T \ge 0$ be such that (2.17) and (H_1) hold. Integrating (2.31) from T to t and using (2.17), we have

$$\begin{aligned} x(t) \ge{}& \frac{1}{r(t)}\int_a^b p(t,\tau)x(t-\tau)d\tau \\ &+ \frac{1}{r(t)}\int_T^t \frac{(t-s)^{n^*-1}}{(n^*-1)!}\int_t^\infty \frac{(u-s)^{n-n^*-1}}{(n-n^*-1)!}\left[\int_c^d q(s,\sigma)f(x(s-\sigma))d\sigma\right]du\,ds, \quad t \ge t_0. \end{aligned} \tag{2.32}$$

Using a method similar to the proof of Case 1 yields that (1.1) also has an eventually positive solution.

The proof is complete. □

Theorem 2.3. *Assume all conditions of Lemma 1.1 hold,* n *is a positive even integer, and*

(H_1) $p(t,\tau) + q(t,\sigma) > 0$, *for sufficiently large* t.

Then, (1.1) *has an eventually positive solution if and only if inequality* (1.2) *has an eventually positive solution.*

Proof. It is clear that an eventually bounded positive solution of (1.1) is also an eventually bounded positive solution of (1.2). So, it suffices to prove that if (1.2) has an eventually bounded positive solution $x(t)$, for $t > t_0$, then so does (1.1). Set

$$y(t) = r(t)x(t) - \int_a^b p(t,\tau)x(t-\tau)d\tau. \tag{2.33}$$

It follows from Lemma 1.1 and (1.2) that eventually,

$$y^{(n)}(t) \le 0, (-1)^k y^{(k)}(t) < 0, \quad (k = 0,1,\dots,n-1),$$
$$\lim_{t\to\infty} y^k(t) = 0, \quad (k = 1,2,\dots,n-1). \tag{2.34}$$

By using (1.9) and integrating (1.2) from t to ∞, we obtain

$$\begin{gathered} y^{(n-1)}(\infty) - y^{(n-1)}(t) \le -\int_t^\infty \left[\int_c^d q(s,\sigma) f(x(s-\sigma)) d\sigma\right] ds, \\ y^{(n-1)}(t) \ge \int_t^\infty \left[\int_c^d q(s,\sigma) f(x(s-\sigma)) d\sigma\right] ds. \end{gathered} \tag{2.35}$$

By repeating the same procedure n times and by using (1.9), we are led to the inequality:

$$y(t) \le -\int_t^\infty dt_n \int_{t_n}^\infty dt_{n-1} \int_{t_{n-1}}^\infty dt_{n-2} \cdots \int_{t_2}^\infty \left[\int_c^d q(s,\sigma) f(x(s-\sigma)) d\sigma\right] ds, \quad t \ge t_0. \tag{2.36}$$

Using *Tonelli's* theorem, we reverse the order of integration and obtain

$$= -\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q(s,\sigma) f(x(s-\sigma)) d\sigma\right] ds, \quad t \ge t_0. \tag{2.37}$$

That is,

$$x(t) \le \frac{1}{r(t)} \int_a^b p(t,\tau)\omega(t-\tau) d\tau - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q(s,\sigma) f(x(s-\sigma)) d\sigma\right] ds, \quad t \ge t_0. \tag{2.38}$$

Let $T \ge 0$ be such that (2.38) hold, and $x(t-\mu) > 0$, $t \ge T$. Now, we consider the set of functions:

$$\Omega = \{z \in C([T-\mu,\infty), R) : 0 < z(t) \le 1, \ t \ge T-\mu\}, \tag{2.39}$$

and define an operator S on Ω as follows:

$$(Sz)(t) = \begin{cases} \dfrac{t-T+\mu}{\mu}(Sz)(T) + \left(1 - \dfrac{t-T+\mu}{\mu}\right), \\ \qquad T-\mu \le t < T, \\ \dfrac{x(t)}{\dfrac{1}{r(t)} \int_a^b p(t,\tau) \dfrac{x(t-\tau)}{z(t-\tau)} d\tau - \dfrac{1}{r(t)} \int_t^\infty \dfrac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q(s,\sigma) f\left(\dfrac{x(s-\sigma)}{z(s-\sigma)}\right) d\sigma\right] ds}, \\ \qquad t \ge T. \end{cases} \tag{2.40}$$

Then, it follows from *Lebesgue's* dominated convergence theorem that S is continuous. By using (2.38), it is easy to see that S maps Ω into itself and, for any $z \in \Omega$, we have $(Sz)(t) < 0$, for $T-\mu \le t < T$. Next, we define the sequence $z_k(t) \in \Omega$:

$$\begin{gathered} z_0(t) \equiv 1, \quad t \ge T-\mu, \\ z_{k+1}(t) = (Sz_k)(t), \quad t \ge T-\mu, \; k = 0, 1, \ldots. \end{gathered} \tag{2.41}$$

Then, by using (2.38) and a simple induction, we can easily see that

$$0 < z_{k+1}(t) \le z_k(t) \le 1, \quad t \ge T-\mu. \tag{2.42}$$

Set $\lim_{k\to\infty} z_k(t) = z(t), \; t \ge T-\mu$, then $z(t)$ satisfies

$$z(t) = \begin{cases} \dfrac{t-T+\mu}{\mu}(Sz)(T) + \left(1 - \dfrac{t-T+\mu}{\mu}\right), \\ \qquad\qquad T-\mu \le t < T, \\ \dfrac{x(t)}{\dfrac{1}{r(t)}\int_a^b p(t,\tau)\dfrac{x(t-\tau)}{z(t-\tau)}d\tau - \dfrac{1}{r(t)}\int_t^\infty \dfrac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma) f\left(\dfrac{x(s-\sigma)}{z(s-\sigma)}\right)d\sigma\right]ds}, \\ \qquad\qquad t \ge T. \end{cases} \tag{2.43}$$

Again, set $\omega(t) = x(t)/z(t)$, then $\omega(t)$ satisfies $\omega(t) > 0, \; T-\mu \le t \le T$, and

$$\begin{aligned} \omega(t) = {} & \frac{1}{r(t)}\int_a^b p(t,\tau)\omega(t-\tau)d\tau \\ & - \frac{1}{r(t)}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma) f(\omega(s-\sigma))d\sigma\right]ds, \quad t \ge T. \end{aligned} \tag{2.44}$$

Thus, $\omega(t)$ is a positive solution of (1.1) for $t \ge T$.

Consider the following

$$\omega(t) > 0, \quad t \ge T-\mu. \tag{2.45}$$

Assume that there exists $t^* \ge T-\mu$, such that $\omega(t) > 0$, for $T-\mu \le t \le t^*$, and $\omega(t^*) = 0$. Then,

$$\begin{aligned} 0 = \omega(t^*) = {} & \frac{1}{r(t^*)}\int_a^b p(t^*,\tau)\omega(t^*-\tau)d\tau \\ & - \frac{1}{r(t^*)}\int_{t^*}^\infty \frac{(s-t^*)^{n-1}}{(n-1)!}\left[\int_c^d q(s,\sigma) f(\omega(s-\sigma))d\sigma\right]ds, \quad t^* \ge T, \end{aligned} \tag{2.46}$$

which implies

$$p(t^*, \tau) = 0,$$
$$q(s, \sigma) f(\omega(s - \sigma)) \equiv 0, \tag{2.47}$$

which contradicts (H_1). Thus, $\omega(t)$ is an eventually positive solution of (1.1).

The proof is complete. □

Example 2.4. Consider high-order neutral differential equation with distributed delay

$$\left(x(t) - \int_1^2 e^{-t} x(t-\tau) d\tau\right)^{(n)} + \frac{1}{2e^{-1} - 3e^{-2}} \int_1^2 \sigma x(t-\sigma) d\sigma = 0. \tag{2.48}$$

Here, $r(t) \equiv 1$, $p(t-\tau) = e^{-t}$, $q(t, \sigma) = \sigma / 2e^{-1} - 3e^{-2}$, $a = c = 1$, $b = d = 2$. It is easy to see that

$$r(t) \equiv 1 < \infty, \quad \frac{\int_1^2 e^{-t} d\tau}{1} = e^{-t} < 1, \quad t > 0. \tag{2.49}$$

From Theorem 2.2, we have that (1.1) has an eventually positive solution if and only if inequality (1.2) has an eventually positive solution. In fact, $x(t) = e^t$ is a positive solution of (1.1).

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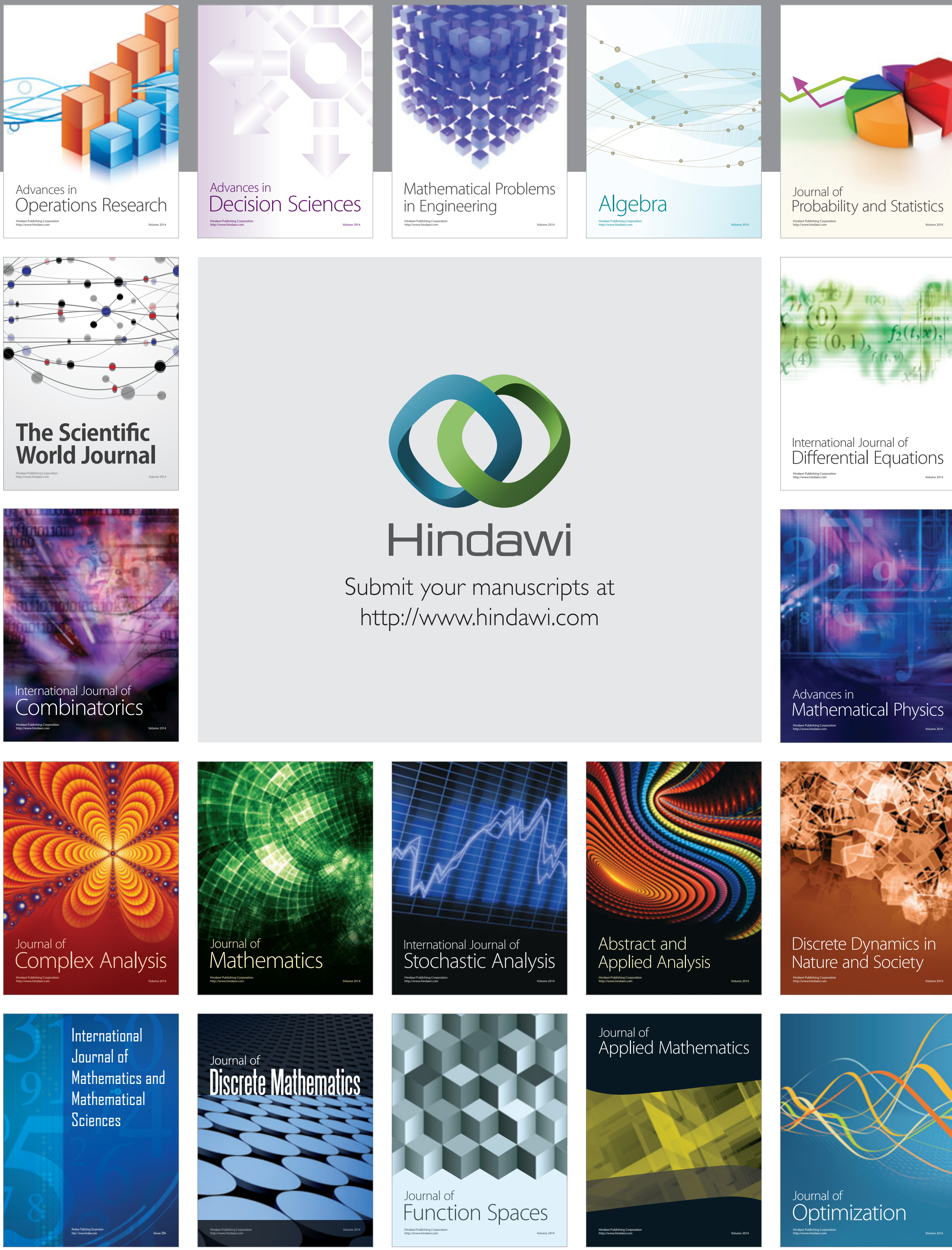
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