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*Research Article*

# Qualitative Analysis for a Predator Prey System with Holling Type III Functional Response and Prey Refuge

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A predator prey system with Holling III functional response and constant prey refuge is considered. By using the Dulac criterion, we discuss the global stability of the positive equilibrium of the system. By transforming the system to a Liénard system, the conditions for the existence of exactly one limit cycle for the system are given. Some numerical simulations are presented.

## 1. Introduction

Recently, the qualitative analysis of predator prey systems with Holling II or III types functional response and prey refuge has been done by several papers, see [1–5]. Their main objective is to discuss under what conditions the positive equilibrium of the corresponding system is stable or unstable and the existence of exactly one limit cycles. In general, the prey refuge has two types, one is the so-called constant proportion prey refuge: $(1-m)x$, where $m \in (0,1)$, the other type is called constant prey refuge: $(x-m)$.

In [2], the authors considered the following system with a constant proportion prey refuge:

$$\frac{dx}{dt} = ax - bx^2 - \frac{\alpha(1-m)^2x^2y}{\beta^2+(1-m)^2x^2},$$
$$\frac{dy}{dt} = -cy + \frac{k\alpha(1-m)^2x^2y}{\beta^2+(1-m)^2x^2}, \tag{1.1}$$

where $x$ and $y$ denote the prey and predator density, respectively, at time $t$, the parameters $a, b, \alpha, \beta, c, k$ are positive constants, and their biological meanings can be seen in [2]. The main result is that when $0 < m < (1 + 2bc\beta/(a(k\alpha - 2c)))\sqrt{c/(k\alpha - c)}$ system (1.1) admits only one limit cycle which is globally asymptotically stable.

In paper [4], the authors only gave the local stability analysis to the following system with a constant prey refuge:

$$\frac{dx}{dt} = ax - bx^2 - \frac{\alpha(x-m)^2 y}{\beta^2 + (x-m)^2},$$
$$\frac{dy}{dt} = -cy + \frac{k\alpha(x-m)^2 y}{\beta^2 + (x-m)^2}. \tag{1.2}$$

In this paper, we will research under what conditions that the positive equilibrium is globally asymptotically stable and the existence of exactly one stable limit cycle of system (1.2). For ecological reason, we only consider system (1.2) in $\Omega_0 = \{(x, y) \mid x > m, y > 0\}$ or $\overline{\Omega}_0$.

It easy to obtain the following lemma.

**Lemma 1.1.** *Any solution* $(x(t), y(t))$ *of system* (1.2) *with initial condition* $x(0) > m,\ y(0) > 0$ *is positive and bounded for all* $t \geq 0$.

## 2. Basic Results

Let $\overline{x} = x - m,\ \overline{y} = \alpha y, dt = (\beta^2 + \overline{x}^2)d\overline{t}$, then system (1.2) changes (still denote $\overline{x}, \overline{y},\ \overline{t}$ as $x, y, t$)

$$\frac{dx}{dt} = (x+m)(a - b(x+m))\left(\beta^2 + x^2\right) - x^2 y,$$
$$\frac{dy}{dt} = -c\beta^2 y + (k\alpha - c)x^2 y. \tag{2.1}$$

Then $\Omega_0$ transforms to $\Omega = \{(x, y) \mid\ x > 0,\ y > 0\}$ and system (2.1) is bounded.

Clearly, if $(H_1)\ 0 < m < a/b$ holds, system (2.1) has positive boundary equilibrium $E_0((a/b) - m, 0)$; if $(H_2)\ k\alpha > c,\ 0 < m < (a - bx_*)/b$, system (2.1) has a positive equilibrium $E_*(x_*, y_*)$, where

$$x_* = \beta\sqrt{\frac{c}{k\alpha - c}}, \qquad y_* = \frac{k\alpha}{c}(x_* + m)(a - b(x_* + m)). \tag{2.2}$$

It is easy to obtain the following lemma.

**Lemma 2.1.** *Let* $(H_1)$ *hold. Further assume that* $(H_3)\ k\alpha \leq c$ *and* $(H_4) k\alpha > c, m > \max\{0, (a - bx_*)/b\}$. *Then* $E_0$ *is locally asymptotically stable, if any of* $(H_3)$ *and* $(H_4)$ *holds. When* $k\alpha > c,\ 0 < m < (a - bx_*)/b,\ E_0$ *is unstable, furthermore,* $E_0$ *is a saddle point.*

About the properties of the positive equilibrium, we have the following theorem.

**Theorem 2.2.** *Assume $k\alpha > c$. Then*

(I) $E_*$ *is locally asymptotically stable for* $0 < m < (a - bx_*)/b$ *if* $a(2c - k\alpha) \leq 2bcx_*$ *holds.*

(II) $E_*$ *is locally asymptotically stable for* $m_1 < m < (a - bx_*)/b$ *and* $E_*$ *is locally unstable for* $0 < m < m_1$ *if* $a(2c - k\alpha) > 2bcx_*$ *holds, where*

$$m_1 = \frac{bx_*^3 + \beta^2(a - bx_*) - \sqrt{\Delta/4}}{2b\beta^2}, \tag{2.3}$$

(III) *system* (2.1) *undergoes Hopf bifurcation at* $m = m_1$ *if* $a(2c - k\alpha) > 2bcx_*$ *holds.*

*Proof.* The Jacobian matrix of system (2.1) at $E_*$ is

$$J(E_*) = \begin{pmatrix} -\dfrac{P}{x_*} & -x_*^2 \\ 2(k\alpha - c)x_*y_* & 0 \end{pmatrix}, \tag{2.4}$$

where $P = 2bx_*^4 + (2bm - a)x_*^3 + \beta^2(a - 2bm)x_* + 2m\beta^2(a - bm)$. Then $\operatorname{tr}(J(E_*)) = -P/x_* = R(m)/x_*$, where $R(m) = 2b\beta^2m^2 + 2(b\beta^2x_* - bx_*^3 - a\beta^2)m - a\beta^2x_* + ax_*^3 - 2bx_*^4$, the discriminant of $R(m) = 0$ is $\Delta = 4(b^2x_*^6 + 2b^2x_*^4\beta^2 + b^2\beta^4x_*^2 + 4a^2\beta^4) > 0$. Hence, the equation $R(m) = 0$ has two roots $m_1$ and $m_2$, where $m_1 = (bx_*^3 + \beta^2\ (a - bx_*) - \sqrt{\Delta/4})\ /2b\beta^2,\ m_2 = (bx_*^3 + \beta^2\ (a - bx_*) + \sqrt{\Delta/4})/2b\beta^2$.

Note that

$$\begin{aligned} \left(bx_*^3 + \beta^2(a - bx_*)\right)^2 - \frac{\Delta}{4} &= 2bx_*\beta^2\left(ax_*^2 - a\beta^2 - 2bx_*^3\right) \\ &= 2bx_*\beta^2\left(a\beta^2\frac{2c - k\alpha}{k\alpha - c} - 2bx_*^3\right) \\ &= 2bx_*\beta^4\frac{a(2c - k\alpha) - 2bcx_*}{k\alpha - c}, \end{aligned} \tag{2.5}$$

and $a(2c - k\alpha) > (\leq)2bcx_*$ implies $m_1 > (\leq)0$. Consider

$$m_2 > \frac{a - bx_*}{2b} + \frac{\beta^2\sqrt{a^2 + b^2x_*^2 - 2abx_* + 2abx_*}}{2b\beta^2} > \frac{a - bx_*}{b}. \tag{2.6}$$

Then

(I) If $a(2c - k\alpha) \leq 2bcx_*$ holds, then $m_1 \leq 0$, $R(m) < 0$ holds for $m_1 < m < m_2$. Considering ($H_2$) and $m_2 > (a - bx_*)/b$, for $0 < m < (a - bx_*)/b$, $\operatorname{tr}(J(E_*)) < 0$, which implies $E_*$ is locally asymptotically stable.

(II) If $a(2c - k\alpha) > 2bcx_*$ holds, then $m_1 > 0$, for $m_1 < m < m_2$, $R(m) < 0$, since $m_1 - (a - bx_*)/b = (bx_*^3 - \beta^2(a - bx_*) - \sqrt{\Delta/4})/2b\beta^2$, by $(bx_*^3 - \beta^2(a - bx_*))^2 - (\Delta/4) = -2ab\beta^2x_*(x_*^2 + \beta^2) < 0$, we obtain $m_1 < (a - bx_*)/b$. Together with ($H_2$), for $m_1 < m < (a - bx_*)/b$, $\operatorname{tr}(J(E_*)) < 0$, which means $E_*$ is locally asymptotically stable. On the other hand, for $0 < m < m_1$, $\operatorname{tr}(J(E_*)) > 0$, $E_*$ is locally unstable.

(III) We have

$$\det(J(E_*)) > 0, \qquad \operatorname{tr}(J(E_*)|_{m_1}) = \frac{R(m_1)}{x_*} = 0, \qquad \left.\frac{\partial tr(J(E_*))}{\partial m}\right|_{m_1} \neq 0, \tag{2.7}$$

these satisfy Liu's Hopf bifurcation criterion (see [6], page 255); hence, the Hopf bifurcation occurs at $m = m_1$. This ends the proof. □

## 3. Global Stability of the Positive Equilibrium

Denote $m_3 := (9a - 2\sqrt{3}\beta b - \sqrt{81a^2 + 12b^2\beta^2})/18b < 0$, $m_4 := (9a - 2\sqrt{3}\beta b + \sqrt{81a^2 + 12b^2\beta^2})/18b > 0$.

**Theorem 3.1.** *If $E_*(x_*, y_*)$ is locally stable. Further assume that* $\max\{0, (a - 4b\beta)/2b\} < m < m_4$, *then the positive equilibrium $E_*(x_*, y_*)$ of system* (2.1) *is globally asymptotically stable.*

*Proof.* Take the Dulac function $B(x, y) = x^{-2}y^{-1}$, for system (2.1) we have

$$T = \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} = -\frac{\phi(x)}{x^3 y}, \tag{3.1}$$

where

$$\phi(x) = 2bx^4 + (2bm - a)x^3 + \beta^2(a - 2bm)x + 2m\beta^2(a - bm). \tag{3.2}$$

If $a = 2bm, \phi(x) = 2b(x^4 + m^2\beta^2) > 0$ for $x > 0$.
On the other hand, there exist

$$\begin{aligned} \phi'(x) &= 8bx^3 + 3(2bm - a)x^2 + \beta^2(a - 2bm), \\ \phi''(x) &= 24bx^2 + 6(2bm - a)x. \end{aligned} \tag{3.3}$$

The equation $\phi''(x) = 0$ has two roots $x_1 = 0$, $x_2 = (a - 2bm)/4b$.

*Case 1*. If $a - 2bm > 0$, then for $0 < x < x_2$, $\phi''(x) < 0$; for $x > x_2$, $\phi''(x) > 0$. Hence, $x = x_2$ is the least value of the function $\phi'(x)$. If $\beta > x_2, \phi'(x_2) = (a - 2bm)(\beta - x_2)(\beta + x_2) > 0$, it has $\phi'(x) > 0$ for all $x > 0$, then $\phi(x)$ is increasing for $x > 0$, notice that $\phi(0) > 0$. Therefore, $\phi(x) > 0$ for $x > 0$. Since, $T < 0$ for $x > 0$, system (2.1) does not exist limit cycle.

*Case 2*. If $bm < a < 2bm$, then $x_2 < 0$, for $x > 0$, $\phi''(x) > 0$, hence, for $x > 0$, $\phi'(x)$ is increasing. Evidently, $\phi'(0) < 0$, $\phi'(\beta/\sqrt{3}) > 0$, then there exists $0 < x_0 < \beta/\sqrt{3}$ such that $\phi'(x_0) = 0$, where $\phi'(x_0) = 8bx_0^3 + 3(2bm - a)x_0^2 + \beta^2(a - 2bm)$, hence, when $0 < x < x_0, \phi'(x) < 0$, when $x > x_0$, $\phi'(x) > 0$. We know that $\phi(x)$ takes the least value at $x = x_0$, that is, $\phi(x) > \phi(x_0)$. According to $\phi'(x_0) = 0$, for $x > 0$ we obtain $\phi(x) > \phi(x_0) = (2bm - a)(x_0^3 - 3\beta^2 x_0 + (8m\beta^2(a - bm))/(2bm - a))$, where $0 < x_0 < \beta/\sqrt{3}$.

To prove $\phi(x) > 0$ for $x > 0$, it suffices to prove $\tilde{\phi}(x) = x^3 - 3\beta^2 x + (8m\beta^2(a-bm))/(2bm-a) > 0$ for $0 < x < \beta/\sqrt{3}$. Clearly, $\tilde{\phi}(x)$ takes the least value at $x = \beta$, and $\tilde{\phi}(x)$ is strictly decreasing at the interval $(0, \beta)$. Hence, for $0 < x < \beta/\sqrt{3}$, $\tilde{\phi}(x) > \tilde{\phi}(\beta/\sqrt{3})$ holds. Since $\tilde{\phi}\beta/\sqrt{3} > 0 \Leftrightarrow (m(a-bm))/(2bm-a) > \beta/3\sqrt{3} \Leftrightarrow -3\sqrt{3}bm^2 + (3\sqrt{3}a - 2\beta b)m + \beta a > 0 \Leftrightarrow m_3 < m < m_4$. Therefore, for $0 < x_0 < \beta/\sqrt{3}$, $\tilde{\phi}(x) > 0$ holds if $m_3 < m < m_4$ holds, then for $x > 0$, $\phi(x) > 0$ holds.

In sum, if one of the following three conditions holds (1) $m = a/2b$; (2) $0 < m < a/2b$, $x_2 < \beta \Rightarrow \max\{0, (a-4b\beta)/2b\} < m < a/2b$; (3) $a/2b < m < a/b$, $m_3 < m < m_4 \Rightarrow a/2b < m < m_4$, the function $T$ does not change the sign for $x > 0$, then system (2.1) does not exist limit cycle. It is easy to see that the conditions (1), (2), and (3) are equal to $\max\{0, (a-4b\beta)/2b\} < m < m_4$. The proof is completed. □

## 4. Existence and Uniqueness of Limit Cycle

**Theorem 4.1.** *If $a(2c - k\alpha) > 2bcx_*$ holds, for $0 < m < m_1$ system (2.1) admits at least one limit cycle in $\Omega$.*

*Proof.* We construct a Bendixson loop $\widehat{OABCD}$ which includes $E_*$ of system (2.1). Let $\overline{OA}$ be a length of the line $L_1 : y = 0$, $\overline{AB}$ be a length of line $L_2 : b(x+m) - a = 0$. Define

$$\dot{x} = x^2(a_0 - y), \qquad \dot{y} = \left(-c\beta^2 + (k\alpha - c)x^2\right)y, \tag{4.1}$$

where $a_0 = \max_{x_* \le x \le (a/b)-m}\{((x+m)(a-b(x+m))(\beta^2+x^2))/x^2\}$. The orbit of system (4.1) with initial value $((a/b)-m, a_0)$ intersects with the line $x = x_*$ and the intersection point $C(x_*, y_1)$, we obtain the orbit arc $\widehat{BC}$. Let $\overline{CD}$ be a length of line $L_3 : y = y_1$, $\overline{DO}$ be a length of line $L_4 : x = 0$. Because $\overline{OA}$ is a length of orbit line of system (2.1) and $(dL_2/dt)|_{(2.1)} = -b((a/b)-m)^2 y < 0 (y > 0)$, $(dL_3/dt)|_{(2.1)} = y_1(-c\beta^2 + (k\alpha - c)x^2) < 0$ $(0 < x < x_*)$, $(dL_4/dt)|_{(2.1)} = m\beta^2(a - bm) > 0$, the orbits of system (2.1) tend to the interior of the Bendixson loop from the outer of $\overline{AB}$, $\overline{CD}$, and $\widehat{BC}$, by comparing system (2.1) to system (4.1): $dx/dt|_{(2.1)} < dx/dt|_{(4.1)} < 0$ and $dy/dt|_{(2.1)} = dy/dt|_{(4.1)} > 0$. Then the orbits of system (2.1) tend to the interior of the Bendixson loop from the outer of $\widehat{BC}$. On the other hand, under the condition of Theorem 4.1, $E_*(x_*, y_*)$ is unstable, by Poincaré-Bendixson Theorem, system (2.1) admits at least one limit cycle in the region $\widehat{OABCD} \in \Omega$. This ends the proof. □

**Lemma 4.2** (see [7]). *Let $f(x)$, $g(x)$ be continuously differentiable functions on the open interval $(r1, r2)$, and $\varphi(y)$ be continuously differentiable functions on $R$ in*

$$\frac{dx}{dt} = \varphi(y) - \int_{x_0}^{x} f(u)\,du, \qquad \frac{dy}{dt} = -g(x), \tag{4.2}$$

*such that*

(1) $d\varphi(y)/dy > 0$,

(2) *having a unique* $x_0 \in (r_1, r_2)$, *such that* $(x - x_0)g(x - x_0) > 0$ *for* $x \neq x_0$ *and* $g(x_0) = 0$,

(3) $f(x_0)d/dx(f(x)/g(x)) < 0$ *for* $x \neq x_0$,

*then system* (4.1) *has at most one limit cycle.*

**Theorem 4.3.** *If* $a(2c - k\alpha) > 2bcx_*$ *holds, for* $0 < m < \min\{m_1, a/2b - (8\sqrt{3}\, x_*^3)/(9(x_*^2 - \beta^2))\}$ *system* (2.1) *exists exactly one limit cycle which is globally asymptotically stable in* $\Omega$.

*Proof.* Let $u = x,\ v = \ln y,\ \tau = -x^2 t$, still denote $u, v, \tau$, as $x,\ y, t$, then system (2.1) becomes

$$\frac{dx}{dt} = e^y - \frac{(x+m)(a - b(x+m))(\beta^2 + x^2)}{x^2},$$
$$\frac{dy}{dt} = -\frac{(k\alpha - c)x^2 - c\beta^2}{x^2}, \tag{4.3}$$

the positive equilibrium $E_*(x_*, y_*)$ changes $\widetilde{E}_*(x_*, \ln y_*)$.

Let $\overline{x} = x - x_*,\ \overline{y} = y - \ln y_*$, then $\widetilde{E}_*$ transform to the origin $O(0,0)$, still denote $\overline{x},\ \overline{y}$, as $x,\ y$ yield

$$\frac{dx}{dt} = y_* e^y - y_* - \frac{(x + x_* + m)(a - b(x + x_* + m))\left(\beta^2 + (x + x_*)^2\right)}{(x + x_*)^2} + y_*$$
$$:= \varphi(y) - F(x), \quad (x > -x_*), \tag{4.4}$$
$$\frac{dy}{dt} = -\frac{(k\alpha - c)(x + x_*)^2 - c\beta^2}{(x + x_*)^2} := -g(x),$$

where $F(x) = ((x + x_* + m)(a - b(x + x_* + m))(\beta^2 + (x + x_*)^2))/(x + x_*)^2 - y_*$.

Clearly, $F(0) = 0$. It is easy to see that the conditions (1) and (2) of Lemma 4.2 for $x_0 = 0$ are satisfied. Consider

$$f(x) = F'(x) = \frac{2b(x + x_*)^4 + (2bm - a)(x + x_*)^3 + \beta^2(a - 2bm)(x + x_*) + 2m\beta^2(a - bm)}{(x + x_*)^3}. \tag{4.5}$$

Note that by the assumption of Theorem 4.3, $E_*$ is unstable equilibrium and

$$\operatorname{tr}(J(E_*)) = -\frac{1}{x_*}\left[2bx_*^4 + (2bm - a)x_*^3 + \beta^2(a - 2bm)x_* + 2m\beta^2(a - bm)\right] > 0, \tag{4.6}$$

then $f(0) = -x_*^{-2}\,\mathrm{tr}(J(E_*)) < 0$. Consider

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{2\varphi(x)}{(x+x_*)^2\left((k\alpha-c)(x+x_*)^2 - c\beta^2\right)^2}, \tag{4.7}$$

where

$$\begin{aligned}\varphi(x) &= b(k\alpha-c)(x+x_*)^6 - 3bc\beta^2(x+x_*)^4 + \beta^2(2c-k\alpha)(a-2bm)(x+x_*)^3 \\ &\quad - 3m\beta^2(k\alpha-c)(a-bm)(x+x_*)^2 + mc\beta^4(a-bm) \\ &= (k\alpha-c)\widetilde{\varphi}(x),\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}\widetilde{\varphi}(x) &= b(x+x_*)^6 - 3bx_*^2(x+x_*)^4 + \frac{\beta^2(2c-k\alpha)(a-2bm)}{k\alpha-c}(x+x_*)^3 \\ &\quad - 3m\beta^2(a-bm)(x+x_*)^2 + m\beta^2x_*^2(a-bm).\end{aligned} \tag{4.9}$$

Then, we have

$$\begin{aligned}\widetilde{\varphi}'(x) &= 6b(x+x_*)^5 - 12bx_*^2(x+x_*)^3 + \frac{3\beta^2(2c-k\alpha)(a-2bm)}{k\alpha-c}(x+x_*)^2 \\ &\quad - 6m\beta^2(a-bm)(x+x_*) = (x+x^*)\widetilde{\phi}(x),\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}\widetilde{\phi}(x) &= 6b(x+x_*)^4 - 12bx_*^2(x+x_*)^2 \\ &\quad + \frac{3\beta^2(2c-k\alpha)(a-2bm)}{k\alpha-c}(x+x_*) - 6m\beta^2(a-bm).\end{aligned} \tag{4.11}$$

By a simple computation, we obtain

$$\begin{aligned}\widetilde{\phi}'(x) &= 24b(x+x_*)^3 - 24bx_*^2(x+x_*) + \frac{3\beta^2(2c-k\alpha)(a-2bm)}{k\alpha-c}, \\ \widetilde{\phi}''(x) &= 24b\left(3(x+x_*)^2 - x_*^2\right).\end{aligned} \tag{4.12}$$

It is easy to verify that $\widetilde{\phi}''(-x_*) < 0$ and $\widetilde{\phi}''(x) = 0$ has two roots $\overline{x}_1$ and $\overline{x}_2$ defined by, respectively,

$$\overline{x}_1 = \left(-1 - \frac{\sqrt{3}}{3}\right)x_*, \qquad \overline{x}_2 = \left(-1 + \frac{\sqrt{3}}{3}\right)x_*. \tag{4.13}$$

Obviously, $\overline{x}_1 < -x_* < \overline{x}_2$. Therefore, $\widetilde{\phi}''(x) < 0$ for $-x_* \leqslant x < \overline{x}_2$ and $\widetilde{\phi}''(x) > 0$ for $x > \overline{x}_2$ which indicates that $\overline{x}_2$ is the minimum point of the function $\widetilde{\phi}'(x)$ when $x \geqslant -x_*$. Substituting $\overline{x}_2$ into $\widetilde{\phi}'(x)$, we obtain

$$\begin{aligned}\min_{x\geqslant -x_*}\widetilde{\phi}'(x) &= \widetilde{\phi}'(\overline{x}_2)\\ &= -\frac{16\sqrt{3}bx_*^3}{3}+\frac{3\beta^2(2c-k\alpha)(a-2bm)}{k\alpha-c}\\ &= -\frac{16\sqrt{3}bx_*^3}{3}+3\left(x_*^2-\beta^2\right)(a-2bm).\end{aligned}\tag{4.14}$$

It is easy to see that if $0 < m < (a/2b) - (8\sqrt{3}\,x_*^3)/(9(x_*^2-\beta^2))$, then $\widetilde{\phi}'(\overline{x}_2) > 0$, which implies $\widetilde{\phi}'(x) > 0$ for all $x \geqslant -x_*$. That is, the function $\widetilde{\phi}(x)$ is a strictly increasing function for $x \geqslant -x_*$.

Note that $\widetilde{\phi}(-x_*) = -6m\beta^2(a-bm) < 0$ for $0 < m < a/b$ and $\lim_{x\to+\infty}\widetilde{\phi}(x) = +\infty$. It follows from (4.6) that

$$\widetilde{\phi}(0) = -3\left(2bx_*^4+(2bm-a)x_*^3+(a-2bm)\beta^2x_*+2m\beta^2(a-bm)\right) > 0.\tag{4.15}$$

Hence, there exists a point $-x_* < \widehat{x} < 0$, such that $\widetilde{\phi}(\widehat{x}) = 0$, that is,

$$b(\widehat{x}+x_*)^4-2bx_*^2(\widehat{x}+x_*)^2+\frac{\beta^2(2c-k\alpha)(a-2bm)}{2(k\alpha-c)}(\widehat{x}+x_*)-m\beta^2(a-bm)=0.\tag{4.16}$$

This, together with the monotonicity of $\widetilde{\phi}(x)$ when $x \geqslant -x_*$, we may conclude that $\widetilde{\varphi}'(x) = (x+x_*)\widetilde{\phi}(x) < 0$ for $x \in (-x_*, \widehat{x})$ and $\widetilde{\varphi}'(x) > 0$ for $x \in (\widehat{x}, \infty)$. Therefore, $\widehat{x}$ is the minimum point of the function $\widetilde{\varphi}(x)$ for $-x_* < x < \infty$.

Together with (4.16), we obtain

$$\begin{aligned}\min_{x\geqslant -x_*}\widetilde{\varphi}(x) = \widetilde{\varphi}(\widehat{x}) &= -bx_*^2(\widehat{x}+x_*)^4+\frac{\beta^2(2c-k\alpha)(a-2bm)}{2(k\alpha-c)}(\widehat{x}+x_*)^3\\ &\quad-2m\beta^2(a-2bm)(\widehat{x}+x_*)^2+m\beta^2(a-bm)x_*^2\\ &= -bx_*^2(\widehat{x}+x_*)^4+\frac{1}{2}\left(x_*^2-\beta^2\right)(a-2bm)(\widehat{x}+x_*)^3\\ &\quad-2m\beta^2(a-2bm)(\widehat{x}+x_*)^2+m\beta^2(a-bm)x_*^2\end{aligned}$$

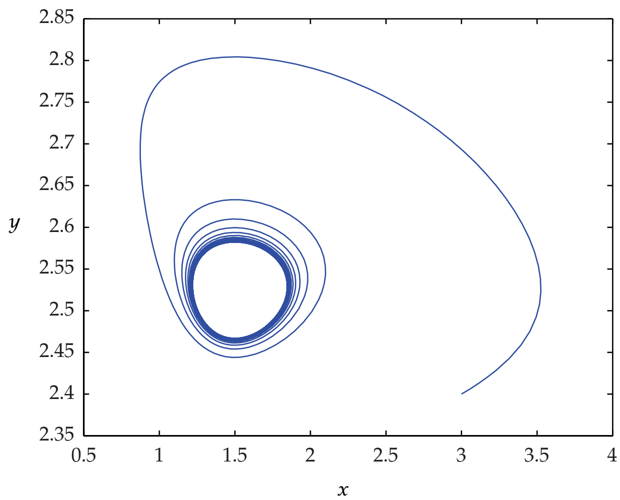


**Figure 1:** The bifurcated periodic solution is stable.

$$
\begin{aligned}
&= -2bx_*^4(\hat{x}+x_*)^2 + \frac{1}{2}\left(x_*^2-\beta^2\right)(a-2bm)(\hat{x}+x_*)\left(2x_*^2+2\hat{x}x_*+\hat{x}^2\right) \\
&\quad -2m\beta^2(a-2bm)(\hat{x}+x_*)^2 \\
&= -2bx_*^4(\hat{x}+x_*)^2 + \left(x_*^2-\beta^2\right)(a-2bm)x_*(\hat{x}+x_*)^2 \\
&\quad -2m\beta^2(a-2bm)(\hat{x}+x_*)^2 + \frac{1}{2}\left(x_*^2-\beta^2\right)(a-2bm)(\hat{x}+x_*)\hat{x}^2 \\
&= -(\hat{x}+x_*)^2\Big(2bx_*^4+(2bm-a)x_*^3+\beta^2(a-2bm)x_* \\
&\quad +2m\beta^2(a-2bm)\Big) + \frac{1}{2}\left(x_*^2-\beta^2\right)(a-2bm)(\hat{x}+x_*)\hat{x}^2.
\end{aligned}
\tag{4.17}
$$

It follows from (4.6), we have $\min_{x\geqslant -x_*}\tilde{\varphi}(x) > 0$. This indicates $\tilde{\varphi}(x) > 0$ for all $x > -x_*$.

Then all the conditions of Lemma 4.2 are satisfied, considering Theorem 4.1, we obtain the conclusion of this theorem. The proof is completed. □

## 5. Numerical Simulations

Take $\alpha = 0.5$, $k = 0.2$, $\beta = 0.5$, $a = 1$, $b = 0.1$, and $c = 0.09$. Then $a(2c - k\alpha) - 2bcx_* = 0.053$, and $m_1 \approx 1.986121812$. One can see a Hopf bifurcation occurring at $m = 1.955$ and the bifurcated periodic solution is stable in Figure 1.

When taking $m = 4.5$, then $x_* = 1.5$, $y_* \approx 2.666666667$, $a(2c - k\alpha) - 2bcx_* = 0.053$, $m_1 \approx 1.986121812$,$(a - bx_*)/b = 8.5$, $(a - 4b\beta)/2b = 4$, $a/2b = 5$. Theorem 3.1 is satisfied; the equilibrium $E_*$ of system (2.1) is globally asymptotically stable. See Figure 2.

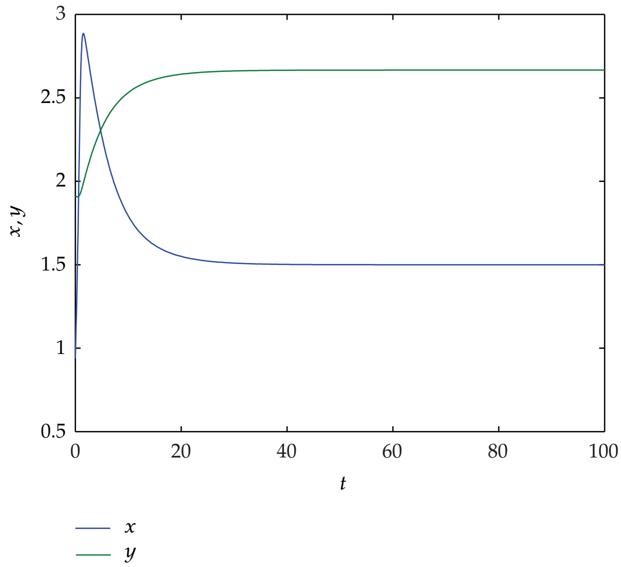


**Figure 2:** The positive equilibrium $E_*$ of system (2.1) is globally asymptotically stable.

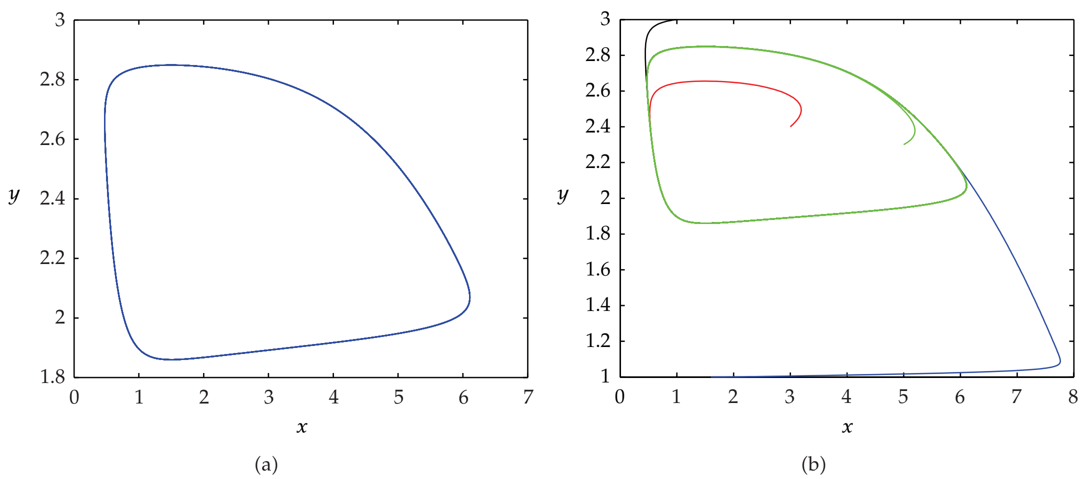


**Figure 3:** The dynamical behaviors of system (2.1) when $\alpha = 0.5$, $k = 0.2$, $\beta = 0.5$, $a = 1$, $b = 0.1$, $c = 0.09$, $m = 1$. (a) The existence of unique limit cycle. (b) The global stability of the limit cycle.

Take $m = 1$, we obtain $E_*(1.5, 2.083333333)$, $a(2c - k\alpha) - 2bcx_* = 0.053$, $m_1 \approx 1.986121812$, $(a/2b) - (8\sqrt{3}x_*^3/9(x_*^2 - \beta^2)) \approx 2.401923788$. The conditions in Theorem 4.1 are satisfied; hence, system (2.1) exists exactly one limit cycle which is globally asymptotically stable. One can see Figure 3.

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