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Research Article

The Number of Chains of Subgroups in the Lattice of Subgroups of the Dicyclic Group

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We give an explicit formula for the number of chains of subgroups in the lattice of subgroups of the dicyclic group B_{4n} of order $4n$ by finding its generating function of multivariables.

1. Introduction

Throughout this paper, all groups are assumed to be finite. The *lattice of subgroups* of a given group G is the lattice $(L(G), \leq)$ where $L(G)$ is the set of all subgroups of G and the partial order $\leq$ is the set inclusion. In this lattice $(L(G), \leq)$, a *chain of subgroups* of G is a subset of $L(G)$ linearly ordered by set inclusion. A chain of subgroups of G is called *G-rooted* (or *rooted*) if it contains G. Otherwise, it is called *unrooted*.

The problem of counting chains of subgroups of a given group G has received attention by researchers with related to classifying *fuzzy subgroups* of G under a certain type of equivalence relation. Some works have been done on the particular families of finite abelian groups (e.g., see [1–4]). As a step of this problem toward non-abelian groups, the first author [5] has found an explicit formula for the number of chains of subgroups in the lattice of subgroups of the dihedral group D_{2n} of order $2n$ where n is an arbitrary positive integer. As a continuation of this work, we give an explicit formula for the number of chains of subgroups in the lattice of subgroups of the dicyclic group B_{4n} of order $4n$ by finding its generating function of multivariables where n is an arbitrary integer.

2. Preliminaries

Given a group G, let $\mathcal{C}(G)$, $\mathcal{U}(G)$, and $\mathcal{R}(G)$ be the collection of chains of subgroups of G, of unrooted chains of subgroups of G, and of G-rooted chains of subgroups of G, respectively. Let $C(G) := |\mathcal{C}(G)|$, $U(G) := |\mathcal{U}(G)|$, and $R(G) := |\mathcal{R}(G)|$.

The following simple observation is useful for enumerating chains of subgroups of a given group.

Proposition 2.1. *Let G be a finite group. Then $R(G) = U(G) + 1$ and $C(G) = R(G) + U(G) = 2R(G) - 1$.*

For a fixed positive integer k, we define a function λ as follows:

$$\begin{aligned} \lambda(x_k) &:= 1 - 2x_k, \\ \lambda(x_k, x_{k-1}, \ldots, x_j) &:= \lambda(x_k, x_{k-1}, \ldots, x_{j+1}) - (1 + \lambda(x_k, x_{k-1}, \ldots, x_{j+1}))x_j \end{aligned} \tag{2.1}$$

for any $j = k-1, k-2, \ldots, 1$.

Proposition 2.2 (see [5]). *Let $\mathbb{Z}_n$ be the cyclic group of order*

$$n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}, \tag{2.2}$$

where $p_1, \ldots, p_k$ are distinct prime numbers and $\beta_1, \ldots, \beta_k$ are positive integers. Then the number $R(\mathbb{Z}_n)$ of rooted chains of subgroups in the lattice of subgroups of $\mathbb{Z}_n$ is the coefficient of $x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ of

$$\phi_{\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) = \frac{1}{\lambda(x_k, \ldots, x_1)}. \tag{2.3}$$

Let $\mathbb{Z}$ be the set of all integer numbers. Given distinct positive integers $i_1, \ldots, i_t$, we define a function

$$\pi_{i_1 \cdots i_t} : \mathbb{Z}^k \longmapsto \mathbb{Z}^k, \qquad (x_1, \ldots, x_k) \longmapsto (y_1, \ldots, y_k), \tag{2.4}$$

where

$$y_\ell = \begin{cases} x_\ell, & \text{if } \ell \neq i_j \ \forall j = 1, \ldots, t, \\ x_\ell - 1, & \ell = i_j \text{ for some } j \text{ such that } 1 \leq j \leq t. \end{cases} \tag{2.5}$$

Most of our notations are standard and for undefined group theoretical terminologies we refer the reader to [6, 7]. For a general theory of solving a recurrence relation using a generating function, we refer the reader to [8, 9].

3. The Number of Chains of Subgroups of the Dicyclic Group B_{4n}

Throughout the section, we assume that

$$n := p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}, \tag{3.1}$$

is a positive integer, where $p_1, \ldots, p_k$ are distinct prime numbers and $\beta_1, \ldots, \beta_k$ are nonnegative integers and the dicyclic group B_{4n} of order $4n$ is defined by the following presentation:

$$B_{4n} := \left\langle a, b \mid a^{2n} = e,\ b^2 = a^n,\ bab^{-1} = a^{-1} \right\rangle, \tag{3.2}$$

where e is the identity element.

By the elementary group theory, the following is wellknown.

Lemma 3.1. *The dicyclic group B_{4n} has an index 2 subgroup $\langle a \rangle$, which is isomorphic to $\mathbb{Z}_{2n}$, and has p_i index p_i subgroups*

$$\langle a^{p_i}, b \rangle, \langle a^{p_i}, ab \rangle, \ldots, \left\langle a^{p_i}, a^{p_i - 1} b \right\rangle, \tag{3.3}$$

which are isomorphic to the dicyclic group B_{4n/p_i} of order $4n/p_i$ where $i = 1, 2, \ldots, k$.

Lemma 3.2. *(1) For any $i = 1, 2, \ldots, k$,*

$$\langle a^{p_i}, a^r b \rangle \cap \langle a^{p_i}, a^s b \rangle = \langle a^{p_i} \rangle \cong \mathbb{Z}_{2n/p_i}, \tag{3.4}$$

where $0 \le r < s \le p_i - 1$.

(2) For any distinct prime factors $p_{i_1}, p_{i_2}, \ldots, p_{i_t}$ of n,

$$\langle a^{p_{i_1}}, a^{r_1} b \rangle \cap \langle a^{p_{i_2}}, a^{r_2} b \rangle \cap \cdots \cap \langle a^{p_{i_t}}, a^{r_t} b \rangle \cong B_{4n/p_{i_1} \cdots p_{i_t}}, \tag{3.5}$$

where $r_1, \ldots, r_t$ are nonnegative integers.

Proof. (1) To the contrary suppose that

$$\langle a^{p_i}, a^r b \rangle \cap \langle a^{p_i}, a^s b \rangle \neq \langle a^{p_i} \rangle. \tag{3.6}$$

Then $a^{p_i u + r} b = a^{p_i v + s} b$ for some integers u and v. This implies $p_i \mid s - r$. Since $0 \le r < s \le p_i - 1$, we have $s = r$, a contradiction.

(2) We only give its proof when $t = 2$. The general case can be proved by the inductive process. Let

$$K := \langle a^{p_{i_1}}, a^{r_1} b \rangle \cap \langle a^{p_{i_2}}, a^{r_2} b \rangle. \tag{3.7}$$

Clearly, $a^{p_{i_1}p_{i_2}} \in K$. Since $\gcd(p_{i_1}, p_{i_2}) = 1$, there exist integers u and v such that $p_{i_1}u + p_{i_2}v = 1$. Note that $a^{p_{i_1}(-u(r_1-r_2))+r_1}b = a^{p_{i_1}(-u(r_1-r_2))}a^{r_1}b \in \langle a^{p_{i_1}}, a^{r_1}b\rangle$. On the other hand,

$$\begin{aligned}
a^{p_{i_1}(-u(r_1-r_2))+r_1}b &= a^{-p_{i_1}u(r_1-r_2)+r_1}b \\
&= a^{p_{i_2}v(r_1-r_2)-(r_1-r_2)+r_1}b \quad \text{since} \quad p_{i_1}u + p_{i_2}v = 1 \\
&= a^{p_{i_2}v(r_1-r_2)\ +r_2}b \in \langle a^{p_{i_2}}, a^{r_2}b\rangle.
\end{aligned} \tag{3.8}$$

Considering the order of K, one can see that $K = \langle a^{p_{i_1}p_{i_2}}, a^{p_{i_1}(-u(r_1-r_2))+r_1}b\rangle$. Since

$$\begin{gathered}
\left(a^{p_{i_1}p_{i_2}}\right)^{4n/p_{i_1}p_{i_2}} = e, \qquad \left(a^{p_{i_1}(-u(r_1-r_2))+r_1}b\right)^2 = b^2 = a^n = \left(a^{p_{i_1}p_{i_2}}\right)^{n/p_{i_1}p_{i_2}}, \\
\left(a^{p_{i_1}(-u(r_1-r_2))+r_1}b\right)\left(a^{p_{i_1}p_{i_2}}\right)\left(a^{p_{i_1}(-u(r_1-r_2))+r_1}b\right)^{-1} = \left(a^{p_{i_1}p_{i_2}}\right)^{-1},
\end{gathered} \tag{3.9}$$

we have $K \cong B_{4n/p_{i_1}p_{i_2}}$. □

By Lemma 3.1, we have

$$\mathcal{U}(B_{4n}) = \mathcal{C}(\langle a\rangle \cong\ \mathbb{Z}_{2n})\bigcup_{i=0}^{k}\bigcup_{j=0}^{p_i-1}\mathcal{C}\left(\left\langle a^{p_i}, a^j b\right\rangle \cong B_{4n/p_i}\right). \tag{3.10}$$

Using the inclusion-exclusion principle and Lemma 3.2, one can see that the number $U(B_{4n})$ has the following form:

$$\begin{aligned}
U(B_{4n}) = C(\mathbb{Z}_{2n}) &+ \sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}} z_{i_1,\ldots,i_t} C\left(\mathbb{Z}_{2n/p_{i_1}\cdots p_{i_t}}\right) \\
&+ \sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}} b_{i_1,\ldots,i_t} C\left(B_{4n/p_{i_1}\cdots p_{i_t}}\right)
\end{aligned} \tag{3.11}$$

for suitable integers $z_{i_1,\ldots,i_t}$ and $b_{i_1,\ldots,i_t}$. In the following, we determine the numbers $z_{i_1,\ldots,i_t}$ and $b_{i_1,\ldots,i_t}$ explicitly.

Lemma 3.3. *(1)* $b_{i_1,i_2,\ldots,i_t} = (-1)^{t+1}p_{i_1}p_{i_2}\cdots p_{i_t}$.
(2) $z_{i_1,i_2,\ldots,i_t} = (-1)^{t}p_{i_1}p_{i_2}\cdots p_{i_t}$.

Proof. (1) Clearly $b_{i_1} = (-1)^{1+1}p_{i_1} = p_{i_1}$ for any $i_1 = 1,\ldots,k$. For any integer $t \ge 2$, one can see by Lemma 3.2 that among intersections of the subgroups of the right-hand side of (3.10), the group isomorphic to $B_{4n/p_{i_1}p_{i_2}\cdots p_{i_t}}$ only appears in t-intersection of the subgroups

$$\left\langle a^{p_{i_1}}, a^{j_1}b\right\rangle, \left\langle a^{p_{i_2}}, a^{j_2}b\right\rangle, \ldots, \left\langle a^{p_{i_t}}, a^{j_t}b\right\rangle, \tag{3.12}$$

where $0 \le j_r \le p_{i_r} - 1$ and $1 \le r \le t$. Since there are $\binom{p_{i_1}}{1}\binom{p_{i_2}}{1}\cdots\binom{p_{i_t}}{1} = p_{i_1}p_{i_2}\cdots p_{i_t}$ such choices, we have $b_{i_1,i_2,\ldots,i_t} = (-1)^{t+1}p_{i_1}p_{i_2}\cdots p_{i_t}$.

(2) By Lemma 3.2, one can see that among intersections of the subgroups of the right-hand side of (3.10), the group isomorphic to $\mathbb{Z}_{2n/p_{i_1}p_{i_2}\cdots p_{i_t}}$ only appears one of the following two forms:

$$\begin{gathered}\langle a\rangle \cap \left\langle a^{p_{i_1}}, a^{j_1}b\right\rangle \cap \left\langle a^{p_{i_2}}, a^{j_2}b\right\rangle \cap \cdots \cap \left\langle a^{p_{i_t}}, a^{j_t}b\right\rangle, \\ \left\langle a^{p_{i_1}}, a^{j_1}b\right\rangle \cap \left\langle a^{p_{i_2}}, a^{j_2}b\right\rangle \cap \cdots \cap \left\langle a^{p_{i_t}}, a^{j_t}b\right\rangle,\end{gathered} \tag{3.13}$$

where $0 \le j_r \le p_{i_r} - 1$ and $1 \le r \le t$, and each subgroup type in the first form must appear at least once, and it can appear more than once, while each subgroup type in the second form must appear at least once, and one of the subgroup types must appear more than once. Let γ be the number of the groups isomorphic to $\mathbb{Z}_{2n/p_{i_1}p_{i_2}\cdots p_{i_t}}$ obtained from the first form, and let δ be the number of the groups isomorphic to $\mathbb{Z}_{2n/p_{i_1}p_{i_2}\cdots p_{i_t}}$ obtained from the second form. Then clearly $z_{i_1,i_2,\cdots,i_t} = \gamma + \delta$. Note that

$$\begin{aligned}\gamma &= \sum_{k=0}^{p_{i_1}+\cdots+p_{i_t}-t} (-1)^{t+2+k} \sum_{\substack{j_1+\cdots+j_t=t+k,\\ 1\le j_r\le p_{i_r},1\le r\le k}} \prod_{r=1}^{t}\binom{p_{i_r}}{j_r} \\ &= \sum_{k\ge 0}\sum_{\substack{j_1+\cdots+j_t=t+k,\\ 1\le j_r\le p_{i_r},1\le r\le t}} (-1)^{t+k}\prod_{r=1}^{t}\binom{p_{i_r}}{j_r} \\ &= \sum_{1\le j_r\le p_{i_r},1\le r\le t} (-1)^{j_1+\cdots+j_t}\prod_{r=1}^{t}\binom{p_{i_r}}{j_r} \\ &= \prod_{r=1}^{t}\sum_{1\le j_r\le p_{i_r}} (-1)^{j_1+\cdots+j_t}\binom{p_{i_r}}{j_r} = (-1)^t.\end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned}\delta &= \sum_{k=0}^{p_{i_1}+\cdots+p_{i_t}-t-1} (-1)^{t+2+k} \sum_{\substack{j_1+\cdots+j_t=t+1+k,\\ 1\le j_r\le p_{i_r},1\le r\le t}} \prod_{r=1}^{t}\binom{p_{i_r}}{j_r} \\ &= \sum_{k=0}^{p_{i_1}+\cdots+p_{i_t}-t-1} (-1)^{t+2+k} \sum_{\substack{j_1+\cdots+j_t=t+1+k,\\ 1\le j_r\le p_{i_r},1\le r\le t}} \prod_{r=1}^{t}\binom{p_{i_r}}{j_r} + (-1)^{t+1}\sum_{\substack{j_1+\cdots+j_t=t,\\ 1\le j_r\le p_{i_r},1\le r\le t}} \prod_{r=1}^{t}\binom{p_{i_r}}{j_r} \\ &\quad - (-1)^{t+1}\sum_{\substack{j_1+\cdots+j_t=t,\\ 1\le j_r\le p_{i_r},1\le r\le t}} \prod_{r=1}^{t}\binom{p_{i_r}}{j_r}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{p_{i_1}+\cdots+p_{i_t}-t} (-1)^{t+1+k} \sum_{\substack{j_1+\cdots+j_t=t+k,\\ 1\le j_r\le p_{j_r},1\le r\le t}} \prod_{r=1}^{t} \binom{p_{i_r}}{j_r} - (-1)^{t+1} \sum_{\substack{j_1+\cdots+j_t=t,\\ 1\le j_r\le p_{i_r},1\le r\le t}} \prod_{r=1}^{t} \binom{p_{i_r}}{j_r} \\
&= (-1)^t - (-1)^{t+1} p_{i_1}\cdots p_{i_t}.
\end{aligned}
\tag{3.15}
$$

Therefore, we have $z_{i_1,i_2,\ldots,i_t} = (-1)^t p_{i_1}\cdots p_{i_t}$. □

By Proposition 2.1 and Lemma 3.3, (3.11) becomes

$$
\begin{aligned}
R(B_{4n}) = 2R(\mathbb{Z}_{2n}) &+ 2 \sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}} (-1)^t p_{i_1}\cdots p_{i_t} R\left(\mathbb{Z}_{2n/p_{i_1}\cdots p_{i_t}}\right) \\
&+ 2 \sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}} (-1)^{t+1} p_{i_1}\cdots p_{i_t} R\left(B_{4n/p_{i_1}\cdots p_{i_t}}\right).
\end{aligned}
\tag{3.16}
$$

Let $a_{\beta_1,\ldots,\beta_k} := R(B_{4n})$ and let $b_{\beta_1,\ldots,\beta_k} := R(\mathbb{Z}_{2n})$. Then (3.16) becomes

$$
\begin{aligned}
a_{\beta_1,\ldots,\beta_k} = 2b_{\beta_1,\ldots,\beta_k} &+ 2 \sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}} (-1)^t p_{i_1}\cdots p_{i_t} b_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)} \\
&+ 2 \sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}} (-1)^{t+1} p_{i_1}\cdots p_{i_t} a_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}.
\end{aligned}
\tag{3.17}
$$

Throughout the remaining part of the section, we solve the recurrence relation of (3.17) by using generating function technique. From now on, we allow each β_i to be zero for computational convenience.

Let

$$
\begin{aligned}
\psi_{\beta_1,\ldots,\beta_k}(x_k, x_{k-1},\ldots,x_j) &:= \sum_{\beta_j=0}^{\infty}\cdots\sum_{\beta_{k-1}=0}^{\infty}\sum_{\beta_k=0}^{\infty} a_{\beta_1,\ldots,\beta_k} x_k^{\beta_k} x_{k-1}^{\beta_{k-1}}\cdots x_j^{\beta_j}, \\
\phi_{\beta_1,\ldots,\beta_k}(x_k, x_{k-1},\ldots,x_j) &:= \sum_{\beta_j=0}^{\infty}\cdots\sum_{\beta_{k-1}=0}^{\infty}\sum_{\beta_k=0}^{\infty} b_{\beta_1,\ldots,\beta_k} x_k^{\beta_k} x_{k-1}^{\beta_{k-1}}\cdots x_j^{\beta_j},
\end{aligned}
\tag{3.18}
$$

where $j = k, k-1,\ldots,1$.

For a fixed integer $n = p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$ such that $p_1,\ldots,p_k$ are distinct prime numbers and $\beta_1,\ldots,\beta_k$ are non-negative integers, we define a function μ as follows.

$$
\begin{aligned}
\mu(x_k) &:= 1 - 2p_k x_k, \\
\mu(x_k,\ldots,x_j) &:= \mu(x_k,\ldots,x_{j+1}) - (1+\mu(x_k,\ldots,x_{j+1}))p_j x_j
\end{aligned}
\tag{3.19}
$$

for any $j = k-1,\ k-2,\ldots,1$.

Lemma 3.4. *Let k be a positive integer. If $k = 1$, then*

$$\mu(x_1)\psi_{\beta_1}(x_1) = (1 + \mu(x_1))\phi_{\beta_1}(x_1). \tag{3.20}$$

If $k \geq 2$, then

$$\begin{aligned}
&\mu(x_k, \ldots, x_j)\psi_{\beta_1,\ldots,\beta_k}(x_k, \ldots, x_j) \\
&\quad = (1 + \mu(x_k, \ldots, x_j)) \\
&\qquad \times \Bigg[\phi_{\beta_1,\ldots,\beta_k}(x_k, \ldots, x_j) + \sum_{\substack{1\leq i_1<\cdots<i_t\leq j-1, \\ 1\leq t\leq j-1}} (-1)^t p_{i_1}\cdots p_{i_t}\phi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k, \ldots, x_j) \\
&\qquad\quad + \sum_{\substack{1\leq i_1<\cdots<i_t\leq j-1, \\ 1\leq t\leq j-1}} (-1)^{t+1} p_{i_1}\cdots p_{i_t}\psi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k, \ldots, x_j)\Bigg]
\end{aligned} \tag{3.21}$$

for any $j = k, k-1, \ldots, 2$.

Proof. Assume first that $k = 1$. Then (3.17) with $k = 1$ gives us that

$$a_{\beta_1} = 2b_{\beta_1} + 2p_1 a_{\beta_1-1} - 2p_1 b_{\beta_1-1}. \tag{3.22}$$

Taking $\sum_{\beta_1=1}^{\infty}$ to both sides of (3.22), we have

$$(1 - 2p_1x_1)\psi_{\beta_1}(x_1) = (2 - 2p_1x_1)\phi_{\beta_1}(x_1) \tag{3.23}$$

because $a_0 = R(B_{4p_1^0}) = R(\mathbb{Z}_{2^2}) = 2^2$ and $b_0 = R(\mathbb{Z}_{2p_1^0}) = R(\mathbb{Z}_2) = 2$ by a direct computation.

From now on, we assume that $k \geq 2$. We prove (3.21) by double induction on k and j. Equation (3.17) with $k = 2$ gives us that

$$\begin{aligned}
a_{\beta_1,\beta_2} &= 2b_{\beta_1,\beta_2} - 2p_1 b_{\beta_1-1,\beta_2} - 2p_2 b_{\beta_1,\beta_2-1} + 2p_1p_2 b_{\beta_1-1,\beta_2-1} \\
&\quad + 2p_1 a_{\beta_1-1,\beta_2} + 2p_2 a_{\beta_1,\beta_2-1} - 2p_1p_2 a_{\beta_1-1,\beta_2-1}.
\end{aligned} \tag{3.24}$$

Taking $\sum_{\beta_2=1}^{\infty} x_2^{\beta_2}$ of both sides of (3.24), we have

$$(1 - 2p_2x_2)\psi_{\beta_1,\beta_2}(x_2) = (2 - 2p_2x_2)\left[p_1\psi_{\beta_1-1,\beta_2}(x_2) + \phi_{\beta_1,\beta_2}(x_2) - p_1\phi_{\beta_1-1,\beta_2}(x_2)\right] \tag{3.25}$$

because $a_{\beta_1,0} = a_{\beta_1}$ and $b_{\beta_1,0} = b_{\beta_1}$ by the definition, and

$$a_{\beta_1,0} - 2b_{\beta_1,0} - 2p_1 a_{\beta_1,0} + 2p_1 b_{\beta_1-1,0} = 0 \tag{3.26}$$

by (3.17) with $k = 1$. That is,

$$\mu(x_2)\psi_{\beta_1,\beta_2}(x_2) = \left(1+\mu(x_2)\right)\left[\phi_{\beta_1,\beta_2}(x_2) - p_1\phi_{\beta_1-1,\beta_2}(x_2) + p_1\psi_{\beta_1-1,\beta_2}(x_2)\right]. \tag{3.27}$$

Thus (3.21) holds for $k = 2$.

Assume now that (3.21) holds from 2 to $k-1$ and consider the case for k. Note that the last two terms of the right-hand side of (3.17) can be divided into three terms, respectively, as follows:

$$\begin{aligned}
&2\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}}(-1)^t p_{i_1}\cdots p_{i_t} b_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}\\
&\quad= -2p_k b_{\beta_1,\ldots,\beta_{k-1},\beta_k-1} - 2p_k\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^t p_{i_1}\cdots p_{i_t} b_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},\beta_k-1)}\\
&\qquad+2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^t p_{i_1}\cdots p_{i_t} b_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)},\\
&2\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}}(-1)^{t+1} p_{i_1}\cdots p_{i_t} a_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}\\
&\quad= 2p_k a_{\beta_1,\ldots,\beta_{k-1},\beta_k-1} - 2p_k\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t+1} p_{i_1}\cdots p_{i_t} a_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},\beta_k-1)}\\
&\qquad+2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t+1} p_{i_1}\cdots p_{i_t} a_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}.
\end{aligned} \tag{3.28}$$

Taking $\sum_{\beta_k=1}^{\infty} x_k^{\beta_k}$ of both sides of (3.17) and using (3.28), one can see that

$$\begin{aligned}
&(1-2p_k x_k)\psi_{\beta_1,\ldots,\beta_k}(x_k)\\
&\quad= (2-2p_k x_k)\\
&\qquad\times\left[\phi_{\beta_1,\ldots,\beta_k}(x_k) + \sum_{\substack{1\le i<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^t p_{i_1}\cdots p_{i_t}\phi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},\beta_k)}(x_k)\right.\\
&\qquad\qquad\left.+\sum_{\substack{1\le i<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t+1} p_{i_1}\cdots p_{i_t}\psi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k)\right]
\end{aligned}$$

$$+ a_{\beta_1,\ldots,\beta_{k-1},0} - 2b_{\beta_1,\ldots,\beta_{k-1},0} - 2\sum_{\substack{1\le i<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^t p_{i_1}\cdots p_{i_t} b_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)}$$
$$- 2\sum_{\substack{1\le i<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t+1} p_{i_1}\cdots p_{i_t} a_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)}. \tag{3.29}$$

Further since

$$\begin{aligned} &a_{\beta_1,\ldots,\beta_{k-1},0} - 2b_{\beta_1,\ldots,\beta_{k-1},0} \\ &\quad - 2\sum_{\substack{1\le i<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^t p_{i_1}\cdots p_{i_t} b_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)} \\ &\quad - 2\sum_{\substack{1\le i<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t+1} p_{i_1}\cdots p_{i_t} a_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)} = 0 \end{aligned} \tag{3.30}$$

by (3.17), we have

$$\begin{aligned} &(1-2p_kx_k)\psi_{\beta_1,\ldots,\beta_k}(x_k) \\ &\quad = (2-2p_kx_k) \\ &\qquad \times \Bigg[\phi_{\beta_1,\ldots,\beta_k}(x_k) + \sum_{\substack{1\le i<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^t p_{i_1}\cdots p_{i_t} \phi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},\beta_k)}(x_k) \\ &\qquad\quad + \sum_{\substack{1\le i<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t+1} p_{i_1}\cdots p_{i_t} \psi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k)\Bigg]. \end{aligned} \tag{3.31}$$

Thus (3.21) holds for $j = k$. Assume that (3.21) holds from k to j and consider the case for $j-1$. Note that the last two terms of the right-hand side of (3.21) can be divided into three terms, respectively, as follows:

$$\begin{aligned} &\sum_{\substack{1\le i_1<\cdots<i_t\le j-1,\\ 1\le t\le j-1}} (-1)^t p_{i_1}\cdots p_{i_t} \phi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j) \\ &\quad = -p_{j-1}\phi_{\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k}(x_k,\ldots,x_j) \\ &\qquad - p_{j-1}\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^t p_{i_1}\cdots p_{i_t} \phi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j) \\ &\qquad + \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^t p_{i_1}\cdots p_{i_t} \phi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j), \end{aligned} \tag{3.32}$$

$$\begin{aligned}
&\sum_{\substack{1\le i_1<\cdots<i_t\le j-1,\\ 1\le t\le j-1}} (-1)^{t+1} p_{i_1}\cdots p_{i_t}\psi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j)\\
&\quad = p_{j-1}\psi_{\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k}(x_k,\ldots,x_j)\\
&\qquad - p_{j-1}\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t+1} p_{i_1}\cdots p_{i_t}\psi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j)\\
&\qquad + \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t+1} p_{i_1}\cdots p_{i_t}\phi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j).
\end{aligned} \tag{3.33}$$

Taking $\sum_{\beta_{j-1}=1}^{\infty} x_{j-1}^{\beta_{j-1}}$ of both sides of (3.21), we have

$$\begin{aligned}
&\mu(x_k,\ldots,x_j,x_{j-1})\psi_{\beta_1,\ldots,\beta_k}(x_{j-1},\ldots,x_k)\\
&\quad = (1+\mu(x_k,\ldots,x_j,x_{j-1}))\\
&\qquad \times \Bigg[\phi_{\beta_1,\ldots,\beta_k}(x_{j-1},\ldots,x_k) + \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t} p_{i_1}\cdots p_{i_t}\phi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j,x_{j-1})\\
&\qquad\quad + \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t+1} p_{i_1}\cdots p_{i_t}\psi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j,x_{j-1})\Bigg]\\
&\qquad + \mu(x_k,\ldots,x_j)\psi_{\beta_1,\ldots,\beta_{j-2},0,\beta_j,\ldots,\beta_k}(x_k,\ldots,x_j)\\
&\qquad - (1+\mu(x_k,\ldots,x_j))\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t} p_{i_1}\cdots p_{i_t}\phi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{j-2},0,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j)\\
&\qquad - (1+\mu(x_k,\ldots,x_j))\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t+1} p_{i_1}\cdots p_{i_t}\psi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{j-2},0,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j).
\end{aligned} \tag{3.34}$$

Note that

$$\begin{aligned}
&\mu(x_k,\ldots,x_j)\psi_{\beta_1,\ldots,\beta_{j-2},0,\beta_j,\ldots,\beta_k}(x_k,\ldots,x_j)\\
&\quad - (1+\mu(x_k,\ldots,x_j))\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t} p_{i_1}\cdots p_{i_t}\phi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{j-2},0,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j)\\
&\quad - (1+\mu(x_k,\ldots,x_j))\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t+1} p_{i_1}\cdots p_{i_t}\psi_{\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{j-2},0,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j) = 0
\end{aligned} \tag{3.35}$$

by induction hypothesis. Thus

$$\begin{aligned}
&\mu(x_k,\dots,x_j,x_{j-1})\psi_{\beta_1,\dots,\beta_k}(x_{j-1},\dots,x_k)\\
&\quad=(1+\mu(x_k,\dots,x_j,x_{j-1}))\\
&\qquad\times\Bigg[\phi_{\beta_1,\dots,\beta_k}(x_{j-1},\dots,x_k)+\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\1\le t\le j-2}}(-1)^t p_{i_1}\cdots p_{i_t}\phi_{\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_k)}(x_k,\dots,x_j,x_{j-1})\\
&\qquad\quad+\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\1\le t\le j-2}}(-1)^{t+1} p_{i_1}\cdots p_{i_t}\psi_{\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_k)}(x_k,\dots,x_j,x_{j-1})\Bigg].
\end{aligned}\tag{3.36}$$

Therefore, (3.21) holds for $j-1$. □

Equation (3.21) with $j=2$ gives us that

$$\begin{aligned}
&\mu(x_k,\dots,x_2)\psi_{\beta_1,\dots,\beta_k}(x_k,\dots,x_2)\\
&\quad=(1+\mu(x_k,\dots,x_2))\\
&\qquad\times\left[\phi_{\beta_1,\dots,\beta_k}(x_k,\dots,x_2)-p_1\phi_{\beta_1-1,\beta_2,\dots,\beta_k}(x_k,\dots,x_2)+p_1\psi_{\beta_1-1,\beta_2,\dots,\beta_k}(x_k,\dots,x_2)\right].
\end{aligned}\tag{3.37}$$

Taking $\sum_{\beta_1=1}^{\infty}x_1^{\beta_1}$ of both sides of (3.37), we get that

$$\begin{aligned}
&\mu(x_k,\dots,x_1)\psi_{\beta_1,\dots,\beta_k}(x_k,\dots,x_2,x_1)\\
&\quad=(1+\mu(x_k,\dots,x_1))\phi_{\beta_1,\dots,\beta_k}(x_k,\dots,x_2,x_1)\\
&\qquad+\mu(x_k,\dots,x_2)\psi_{0,\beta_2,\dots,\beta_k}(x_k,\dots,x_2)-(1+\mu(x_k,\dots,x_2))\phi_{0,\beta_2,\dots,\beta_k}(x_k,\dots,x_2).
\end{aligned}\tag{3.38}$$

Lemma 3.5. *If $k\ge 2$, then*

$$\mu(x_k,\dots,x_2)\psi_{0,\beta_2,\dots,\beta_k}(x_k,\dots,x_2)=(1+\mu(x_k,\dots,x_2))\phi_{0,\beta_2,\dots,\beta_k}(x_k,\dots,x_2).\tag{3.39}$$

Proof. If $k=2$, then since $\psi_{0,\beta_2}(x_2)=\psi_{\beta_2}(x_2)$ and $\phi_{0,\beta_2}(x_2)=\phi_{\beta_2}(x_2)$, the equation

$$\mu(x_2)\psi_{0,\beta_2}(x_2)=(1+\mu(x_2))\phi_{0,\beta_2}(x_2)\tag{3.40}$$

holds by (3.20). Assume now that (3.39) holds for k. Then by (3.38) we get that

$$\mu(x_k,\dots,x_1)\psi_{\beta_1,\dots,\beta_k}(x_k,\dots,x_2,x_1)=(1+\mu(x_k,\dots,x_1))\phi_{\beta_1,\dots,\beta_k}(x_k,\dots,x_2,x_1),\tag{3.41}$$

which implies that

$$\mu(x_{k+1},\ldots,x_2)\psi_{0,\beta_2,\ldots,\beta_{k+1}}(x_{k+1},\ldots,x_2) = (1+\mu(x_{k+1},\ldots,x_2))\phi_{0,\beta_2,\ldots,\beta_{k+1}}(x_{k+1},\ldots,x_2). \tag{3.42}$$

Thus (3.39) holds for $k+1$. □

By Lemmas 3.4 and 3.5 and (3.38), we have

$$\mu(x_k,\ldots,x_1)\psi_{\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1) = (1+\mu(x_k,\ldots,x_1))\phi_{\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1). \tag{3.43}$$

We now need to find the function $\phi_{\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1)$ explicitly.

Lemma 3.6. *If $p_1 = 2$, then*

$$\phi_{\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1) = \begin{cases} \dfrac{2}{\lambda(x_1)}, & \text{if } k=1, \\ \left[1+\dfrac{1}{\lambda(x_k,\ldots,x_2)}\right]\dfrac{1}{\lambda(x_k,\ldots,x_1)}, & \text{if } k \geq 2. \end{cases} \tag{3.44}$$

If $p_i \neq 2$ for $i = 1,2,\ldots,k$, then

$$\phi_{\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1) = \left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]\frac{1}{\lambda(x_k,\ldots,x_1)}. \tag{3.45}$$

Proof. We first assume that $p_1 = 2$. Then by Proposition 2.2,

$$b_{\beta_1,\beta_2,\ldots,\beta_k} = R\left(\mathbb{Z}_{p_1^{\beta_1+1}p_2^{\beta_2}p_3^{\beta_3}\cdots p_k^{\beta_k}}\right) \tag{3.46}$$

is the coefficient of $x_1^{\beta_1+1}x_2^{\beta_2}x_3^{\beta_3}\cdots x_k^{\beta_k}$ of

$$\frac{1}{\lambda(x_k,\ldots,x_1)}, \tag{3.47}$$

which implies that $b_{\beta_1,\beta_2,\ldots,\beta_k}$ is the coefficient of $x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3}\cdots x_k^{\beta_k}$ of

$$\begin{gathered} \frac{2}{\lambda(x_1)} \quad \text{if } k=1, \\ \left[1+\frac{1}{\lambda(x_k,\ldots,x_2)}\right]\frac{1}{\lambda(x_k,\ldots,x_1)} \quad \text{if } k\geq 2, \end{gathered} \tag{3.48}$$

and hence by the definition of ϕ we get that

$$\phi_{\beta_1,\dots,\beta_k}(x_k,\dots,x_1)=\begin{cases}\dfrac{2}{\lambda(x_1)}, & \text{if } k=1,\\[2ex] \left[1+\dfrac{1}{\lambda(x_k,\dots,x_2)}\right]\dfrac{1}{\lambda(x_k,\dots,x_1)}, & \text{if } k\geq 2.\end{cases} \tag{3.49}$$

Assume now that $p_i\neq 2$ for $i=1,2,\dots,k$. Since $b_{\beta_1,\dots,\beta_k}=R(\mathbb{Z}_{2p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}})$, by Proposition 2.2 $b_{\beta_1,\dots,\beta_k}$ is the coefficient of $x_1^1x_2^{\beta_1}x_3^{\beta_2}\cdots x_{k+1}^{\beta_k}$ of

$$\frac{1}{\lambda(x_{k+1},\dots,x_1)}. \tag{3.50}$$

Since

$$\begin{aligned}\frac{1}{\lambda(x_{k+1},\dots,x_1)}&=\frac{1}{\lambda(x_{k+1},\dots,x_2)-(1+\lambda(x_{k+1},\dots,x_2))x_1}\\&=\frac{1}{\lambda(x_{k+1},\dots,x_2)}\frac{1}{1-[1+(1/\lambda(x_{k+1},\dots,x_2))]x_1}\end{aligned} \tag{3.51}$$

by the definition, $b_{\beta_1,\dots,\beta_k}$ is the coefficient of $x_2^{\beta_1}x_3^{\beta_2}\cdots x_{k+1}^{\beta_k}$ of

$$\frac{1}{\lambda(x_{k+1},\dots,x_2)}\left[1+\frac{1}{\lambda(x_{k+1},\dots,x_2)}\right]. \tag{3.52}$$

By changing the variables $x_2,x_3,\dots,x_{k+1}$ by $x_1,x_2,\dots,x_k$, respectively, we get that $b_{\beta_1,\dots,\beta_k}$ is the coefficient of $x_1^{\beta_1}x_2^{\beta_2}\cdots x_k^{\beta_k}$ of

$$\frac{1}{\lambda(x_k,\dots,x_1)}\left[1+\frac{1}{\lambda(x_k,\dots,x_1)}\right]. \tag{3.53}$$

By the definition of $\phi_{\beta_1,\dots,\beta_k}(x_k,\dots,x_1)$, we have

$$\phi_{\beta_1,\dots,\beta_k}(x_k,\dots,x_1)=\frac{1}{\lambda(x_k,\dots,x_1)}\left[1+\frac{1}{\lambda(x_k,\dots,x_1)}\right]. \tag{3.54}$$

□

By Proposition 2.1, (3.43), and Lemma 3.6, we have the following theorem.

Theorem 3.7. *Let*

$$n:=p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}, \tag{3.55}$$

be a positive integer such that $p_1, \ldots, p_k$ are distinct prime numbers and $\beta_1, \ldots, \beta_k$ are positive integers. Let

$$B_{4n} := \left\langle a, b \mid a^{2n} = e, b^2 = a^n, bab^{-1} = a^{-1} \right\rangle \tag{3.56}$$

be the dicyclic group of order $4n$. Let $R(B_{4n})$ be the number of rooted chains of subgroups in the lattice of subgroups of B_{4n}.

(1) *If $p_1 = 2$, then $R(B_{4n})$ is the coefficient of $x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ of*

$$\psi_{\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) = \begin{cases} \left[1 + \dfrac{1}{\mu(x_1)}\right] \dfrac{2}{\lambda(x_1)}, & \text{if } k = 1, \\ \left[1 + \dfrac{1}{\mu(x_k, \ldots, x_1)}\right] \left[1 + \dfrac{1}{\lambda(x_k, \ldots, x_2)}\right] \dfrac{1}{\lambda(x_k, \ldots, x_1)}, & \text{if } k \geq 2. \end{cases} \tag{3.57}$$

(2) *If $p_i \neq 2$ for $i = 1, 2, \ldots, k$, then $R(B_{4n})$ is the coefficient of $x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ of*

$$\psi_{\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) = \left[1 + \frac{1}{\mu(x_k, \ldots, x_1)}\right] \frac{1}{\lambda(x_k, \ldots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \ldots, x_1)}\right]. \tag{3.58}$$

Furthermore, the number $C(B_{4n})$ of chains of subgroups in the lattice of subgroups of B_{4n} is the coefficient of $x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ of

$$2\psi_{\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) - \prod_{i=1}^{k} \frac{1}{1 - x_i}. \tag{3.59}$$

We now want to find the coefficient of $x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ of $\psi_{\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1)$ explicitly. Since

$$\frac{1}{\mu(x_k, \ldots, x_1)} = \frac{1}{\mu(x_k, \ldots, x_2)} \frac{1}{1 - \left[1 + (1/\mu(x_k, \ldots, x_2))\right] p_1 x_1}, \tag{3.60}$$

by the definition, the coefficient of $x_1^{\beta_1}$ of $1/\mu(x_k, \ldots, x_1)$ is

$$\begin{aligned} &\frac{1}{\mu(x_k, \ldots, x_2)} \left[1 + \frac{1}{\mu(x_k, \ldots, x_2)}\right]^{\beta_1} p_1^{\beta_1} \\ &= p_1^{\beta_1} \sum_{i_1=0}^{\beta_1} \binom{\beta_1}{i_1} \left[\frac{1}{\mu(x_k, \ldots, x_2)}\right]^{i_1+1} \\ &= p_1^{\beta_1} \sum_{i_1=0}^{\beta_1} \binom{\beta_1}{i_1} \left[\frac{1}{\mu(x_k, \ldots, x_3)}\right]^{i_1+1} \left[\frac{1}{1 - \left[1 + (1/\mu(x_k, \ldots, x_3))\right] p_2 x_2}\right]^{i_1+1}. \end{aligned} \tag{3.61}$$

Thus the coefficient of $x_1^{\beta_1}x_2^{\beta_2}$ of $1/\mu(x_k,\dots,x_1)$ is

$$\begin{aligned}
&p_1^{\beta_1}p_2^{\beta_2}\sum_{i_1=0}^{\beta_1}\binom{\beta_1}{i_1}\binom{i_1+\beta_2}{\beta_2}\left[\frac{1}{\mu(x_k,\dots,x_3)}\right]^{i_1+1}\left[1+\frac{1}{\mu(x_k,\dots,x_3)}\right]^{\beta_2}\\
&\quad=p_1^{\beta_1}p_2^{\beta_2}\sum_{i_1=0}^{\beta_1}\sum_{i_2=0}^{\beta_2}\binom{\beta_1}{i_1}\binom{\beta_2}{i_2}\binom{i_1+\beta_2}{\beta_2}\left[\frac{1}{\mu(x_k,\dots,x_3)}\right]^{i_1+i_2+1}.
\end{aligned}\tag{3.62}$$

Continuing this process, one can see that the coefficient of $x_1^{\beta_1}x_2^{\beta_2}\cdots x_k^{\beta_k}$ of $1/\mu(x_k,\dots,x_1)$ is

$$2^{\beta_k}p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}\sum_{i_1=0}^{\beta_1}\sum_{i_2=0}^{\beta_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\prod_{r=1}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+\sum_{m=1}^{r}i_m}{\beta_{r+1}}.\tag{3.63}$$

Similarly one can see that the coefficient of $x_1^{\beta_1}x_2^{\beta_2}\cdots x_k^{\beta_k}$ of $1/\lambda(x_k,\dots,x_1)$ is

$$2^{\beta_k}\sum_{i_1=0}^{\beta_1}\sum_{i_2=0}^{\beta_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\prod_{r=1}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+\sum_{m=1}^{r}i_m}{\beta_{r+1}},\tag{3.64}$$

the coefficient of $x_1^{\beta_1}x_2^{\beta_2}\cdots x_k^{\beta_k}$ of $[1+(1/\lambda(x_k,\dots,x_2))](1/\lambda(x_k,\dots,x_1))$ is

$$2^{\beta_k}\sum_{i_1=0}^{\beta_1+1}\sum_{i_2=0}^{\beta_2}\sum_{i_3=0}^{\beta_3}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\binom{\beta_1+1}{i_1}\binom{\beta_2+i_1}{\beta_2}\prod_{r=2}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+\sum_{m=1}^{r}i_m}{\beta_{r+1}}\tag{3.65}$$

and the coefficient of $x_1^{\beta_1}x_2^{\beta_2}\cdots x_k^{\beta_k}$ of $1/\lambda(x_k,\dots,x_1)^2$ is

$$(\beta_1+1)2^{\beta_k}\sum_{i_1=0}^{\beta_1}\sum_{i_2=0}^{\beta_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\prod_{r=1}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+\sum_{m=1}^{r}i_m+1}{\beta_{r+1}}.\tag{3.66}$$

Therefore, one can have the following.

Corollary 3.8. *Let n and B_{4n} be the positive integer and the dicyclic group, respectively, defined in Theorem 3.7. Let $R(B_{4n})$ be the number of rooted chains of subgroups in the lattice of subgroups of B_{4n}.*

(1) *If* $p_1 = 2$, *then*

$$R(B_{4n}) = 2^{\beta_k}\sum_{i_1=0}^{\beta_1+1}\sum_{i_2=0}^{\beta_2}\sum_{i_3=0}^{\beta_3}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\binom{\beta_1+1}{i_1}\binom{\beta_2+i_1}{\beta_2}\prod_{r=2}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+\sum_{m=1}^{r} i_m}{\beta_{r+1}}$$
$$+\,2^{\beta_k}\sum_{j_1=0}^{\beta_1}\sum_{j_2=0}^{\beta_2}\cdots\sum_{j_k=0}^{\beta_k}\left[\left[p_1^{j_1}p_2^{j_2}\cdots p_k^{j_k}\sum_{i_1=0}^{j_1}\sum_{i_2=0}^{j_2}\cdots\sum_{i_{k-1}=0}^{j_{k-1}}\prod_{r=1}^{k-1}\binom{j_r}{i_r}\binom{j_{r+1}+\sum_{m=1}^{r} i_m}{j_{r+1}}\right]\right.$$
$$\times\left[\sum_{i_1=0}^{\beta_1-j_1+1}\sum_{i_2=0}^{\beta_2-j_2}\sum_{i_3=0}^{\beta_3-j_3}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}-j_{k-1}}\binom{\beta_1-j_1+1}{i_1}\binom{\beta_2-j_2+i_1}{\beta_2-j_2}\right.$$
$$\left.\left.\times\prod_{r=2}^{k-1}\binom{\beta_r-j_r}{i_r}\binom{\beta_{r+1}-j_{r+1}+\sum_{m=1}^{r} i_m}{\beta_{r+1}-j_{r+1}}\right]\right], \tag{3.67}$$

where if $k = 1$, *then* $R(B_{4\cdot 2^{\beta_1}}) = 2^{2\beta_1+2}$ *and if* $k = 2$, *then*

$$R\left(B_{4\cdot 2^{\beta_1}p_2^{\beta_2}}\right) = 2^{\beta_2}\sum_{i_1=0}^{\beta_1+1}\binom{\beta_1+1}{i_1}\binom{\beta_2+i_1}{\beta_2}$$
$$+\,2^{\beta_2}\sum_{j_1=0}^{\beta_1}\sum_{j_2=0}^{\beta_2}\left[\left[2^{j_1}p_2^{j_2}\sum_{i_1=0}^{j_1}\binom{j_1}{i_1}\binom{j_2+i_1}{j_2}\right]\right. \tag{3.68}$$
$$\left.\times\left[\sum_{i_1=0}^{\beta_1-j_1+1}\binom{\beta_1-j_1+1}{i_1}\binom{\beta_2-j_2+i_1}{\beta_2-j_2}\right]\right].$$

(2) *If* $p_i \neq 2$ *for* $i = 1, 2, \ldots, k$, *then*

$$R(B_{4n}) = 2^{\beta_k}\sum_{i_1=0}^{\beta_1}\sum_{i_2=0}^{\beta_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\prod_{r=1}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+\sum_{m=1}^{r} i_m}{\beta_{r+1}}$$
$$+\,(\beta_1+1)2^{\beta_k}\sum_{i_1=0}^{\beta_1}\sum_{i_2=0}^{\beta_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\prod_{r=1}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+1+\sum_{m=1}^{r} i_m}{\beta_{r+1}}$$
$$+2^{\beta_k}\sum_{j_1=0}^{\beta_1}\sum_{j_2=0}^{\beta_2}\cdots\sum_{j_k=0}^{\beta_k}\left[\left[p_1^{j_1}p_2^{j_2}\cdots p_k^{j_k}\sum_{i_1=0}^{j_1}\sum_{i_2=0}^{j_2}\cdots\sum_{i_{k-1}=0}^{j_{k-1}}\prod_{r=1}^{k-1}\binom{j_r}{i_r}\binom{j_{r+1}+\sum_{m=1}^{r} i_m}{j_{r+1}}\right]\right.$$
$$\left.\times\left[\sum_{i_1=0}^{\beta_1-j_1}\sum_{i_2=0}^{\beta_2-j_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}-j_{k-1}}\prod_{r=1}^{k-1}\binom{\beta_r-j_r}{i_r}\binom{\beta_{r+1}-j_{r+1}+\sum_{m=1}^{r} i_m}{\beta_{r+1}-j_{r+1}}\right]\right]$$

$$
\begin{aligned}
&+2^{\beta_k}\sum_{j_1=0}^{\beta_1}\sum_{j_2=0}^{\beta_2}\cdots\sum_{j_k=0}^{\beta_k}\left[\left[p_1^{j_1}p_2^{j_2}\cdots p_k^{j_k}\sum_{i_1=0}^{j_1}\sum_{i_2=0}^{j_2}\cdots\sum_{i_{k-1}=0}^{j_{k-1}}\prod_{r=1}^{k-1}\binom{j_r}{i_r}\binom{j_{r+1}+\sum_{m=1}^{r}i_m}{j_{r+1}}\right]\right.\\
&\qquad\times\left[(\beta_1-j_1+1)\sum_{i_1=0}^{\beta_1-j_1}\sum_{i_2=0}^{\beta_2-j_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}-j_{k-1}}\prod_{r=1}^{k-1}\binom{\beta_r-j_r}{i_r}\right.\\
&\qquad\left.\left.\times\binom{\beta_{r+1}-j_{r+1}+1+\sum_{m=1}^{r}i_m}{\beta_{r+1}-j_{r+1}}\right]\right],
\end{aligned}
\tag{3.69}
$$

where if $k=1$, *then*

$$
\begin{aligned}
R\left(B_{4p_1^{\beta_1}}\right)&=2^{\beta_1}+(\beta_1+1)2^{\beta_1}+2^{\beta_1}\sum_{j_1=0}^{\beta_1}p_1^{j_1}+2^{\beta_1}\sum_{j_1=0}^{\beta_1}p_1^{j_1}(\beta_1-j_1+1)\\
&=2^{\beta_1}\left[\beta_1+2+\frac{p_1^{\beta_1+1}-1}{p_1-1}+\frac{p_1^{\beta_1+2}-(\beta_1+2)p_1+\beta_1+1}{(p_1-1)^2}\right].
\end{aligned}
\tag{3.70}
$$

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