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Research Article

Discrete Symmetries Analysis and Exact Solutions of the Inviscid Burgers Equation

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We discuss the Lie point symmetries and discrete symmetries of the inviscid Burgers equation. By employing the Lie group method of infinitesimal transformations, symmetry reductions and similarity solutions of the governing equation are given. Based on discrete symmetries analysis, two groups of discrete symmetries are obtained, which lead to new exact solutions of the inviscid Burgers equation.

1. Introduction

Burgers equation is one of the basic partial differential equations of fluid mechanics. It occurs in various fields of applied mathematics, such as modeling of gas dynamics and traffic flow.

For a given velocity u and viscosity coefficient ν, the general form of Burgers equation is:

$$u_t(x,t) + g(u)u_x(x,t) = \nu u_{xx}(x,t), \tag{1.1}$$

where $g(u)$ is a smooth function of u. If $\nu = 0$, Burgers equation reduces to the inviscid Burgers equation:

$$\text{IBE} : u_t(x,t) + g(u)u_x(x,t) = 0, \tag{1.2}$$

which is a prototype for equations for which the solution can develop discontinuities (shock waves). There are many methods to solve (1.2). In [1], the authors discussed the matrix exponential representations of solutions to similar equation of (1.2). Here we can use the method of Lie symmetries and discrete symmetries analysis to solve (1.2).

The classical Lie symmetries of the partial differential equations (PDEs) which can be obtained through the Lie group method of infinitesimal transformations were originally developed by Lie [2]. We can use the basic prolongation method and the infinitesimal criterion of invariance to find some particular Lie point symmetries group of the nonlinear partial differential equations. The Lie groups of transformations admitted by a given system of differential equations can be used (1) to lower the order or eventually reduce the equation to quadrature, in the case of ordinary differential equations; (2) to determine particular solutions, called invariant solutions, or generate new solutions, once a special solution is known, in the case of ordinary differential equations or PDEs.

In the past decades, much attention has been paid to the symmetry method and a series of achievements have been obtained [3–9]. Particularly, In [9], a five-dimensional symmetry algebra consisting of Lie point symmetries is firstly computed for the nonlinear Schrödinger equation. But it seems that very few research on discrete symmetries is available up to now. In fact, discrete symmetries also play an important role in solving PDEs. For instance, to understand how a system changes its stability, to simplify the numerical computation of solutions of PDEs and to create new exact solutions from known solutions. Discrete symmetries are usually easy to guess but difficult to get in a systematic way. They can be obtained from the continuous Lie point symmetries. In [10, 11], Hydon studied the application of the method in differential equations.

For the Burgers equation, many researches have been carried on [12–18]. In these papers, Ouhadan and El Kinani used Lie symmetry method for obtaining exact solutions of inviscid Burgers equation in some cases [14]. However, the analysis presented there was not complete. Nadjafikhah extended the study to include other cases of interest [15, 16]. But in all their work, they only used the Lie symmetry method, and the discrete symmetries approach has never been considered. In this work, we obtain the analytical solutions of the inviscid Burgers equation by using Lie group method. Also by applying discrete symmetries, we introduce new groups of analytical solutions of our problem.

2. Lie Symmetries and Lie Algebra

In this section, we recall the general procedure for determining symmetries for any system of PDEs [19–21]. To begin, let us consider the general case of a nonlinear system of PDEs,

$$\Delta_\nu\left(x, u^{(n)}\right) = 0, \quad \nu = 1, \ldots, l, \tag{2.1}$$

that involve p independent variables $x = (x^1, \ldots, x^p)$, q dependent variables $u = (u^1, \ldots, u^q)$, and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n.

We consider a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of (2.1):

$$\begin{aligned} \hat{x}^i &= x^i + \epsilon \xi^i(x,u) + O\left(\epsilon^2\right), \quad i = 1,\dots,p, \\ \hat{u}^j &= u^j + \epsilon \eta^j(x,u) + O\left(\epsilon^2\right), \quad j = 1,\dots,q, \end{aligned} \tag{2.2}$$

where ϵ is the parameter of the transformation and ξ^i, η^j are the infinitesimals of the transformations of the independent and dependent variables, respectively. The infinitesimal generator X associated with the above group of transformations can be written as

$$X = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \eta^j(x,u) \frac{\partial}{\partial u^j}. \tag{2.3}$$

A symmetry of differential equation is a transformation which maps solutions of the equation to other solutions.

The invariance of system (2.1) under the infinitesimal transformations leads to the invariance condition

$$pr^{(n)} X\left[\Delta_\nu\left(x,u^{(n)}\right)\right]|_{\Delta_\nu(x,u^{(n)})=0} = 0, \quad \nu = 1,\dots,l, \tag{2.4}$$

where $pr^{(n)}$ is the nth-order prolongation of the infinitesimal generator given by

$$pr^{(n)} X = X + \sum_{\alpha=1}^{q} \sum_{J} \varphi_\alpha^J\left(x,u^{(n)}\right) \frac{\partial}{\partial u_J^\alpha}, \tag{2.5}$$

where $J = (j_1,\dots,j_k)$, and $1 \le j_k \le p, 1 \le k \le n$,

$$\varphi_\alpha^J\left(x,u^{(n)}\right) = D_J\left(\varphi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha, \tag{2.6}$$

where

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}, \qquad u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i}. \tag{2.7}$$

For (1.2), following the general Lie's algorithm [19, 20], we consider the one-parameter Lie group of infinitesimal transformation in (x, t, u) given by

$$\begin{aligned}
\widehat{x} &= x + \xi(x,t,u)\epsilon + O\left(\epsilon^2\right), \\
\widehat{t} &= t + \tau(x,t,u)\epsilon + O\left(\epsilon^2\right), \\
\widehat{u} &= u + \eta(x,t,u)\epsilon + O\left(\epsilon^2\right),
\end{aligned} \tag{2.8}$$

where ϵ is the group parameter. The infinitesimal generator of the symmetry algebra takes the form

$$\mathrm{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}, \tag{2.9}$$

where ξ, τ, η are the same as those in (1.2). And the first prolongation is

$$pr^{(1)}\mathrm{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \eta_x\frac{\partial}{\partial u_x} + \eta_t\frac{\partial}{\partial u_t}. \tag{2.10}$$

Equation (1.2) can be written as

$$\triangle = u_t(x,t) + g(u)u_x(x,t) = 0. \tag{2.11}$$

The invariance of (1.2) under the infinitesimal transformations (2.8) needs

$$pr^{(1)}\mathrm{X}[\triangle]|_{\triangle=0} = 0. \tag{2.12}$$

From (2.11) and (2.12), we get

$$\eta_t + g(u)\eta_x + \eta g'(u)u_x = 0, \tag{2.13}$$

where

$$\begin{aligned}
\eta_x &= \frac{\partial}{\partial x}(\eta - \xi u_x - \tau u_t) + \xi\frac{\partial}{\partial x}(u_x) + \tau\frac{\partial}{\partial t}(u_x), \\
\eta_t &= \frac{\partial}{\partial t}(\eta - \xi u_x - \tau u_t) + \xi\frac{\partial}{\partial x}(u_t) + \tau\frac{\partial}{\partial t}(u_t).
\end{aligned} \tag{2.14}$$

Conditions on the infinitesimals ξ, τ, and η are determined by equating coefficients of like derivatives of monomials in u_x and u_t and higher derivatives by zero. This will produce a series of PDEs; by analyzing these equations, we can get that ξ, τ, and η have the following form:

$$\begin{aligned}
\xi &= C_8 x^2 + C_7 xt + C_3 x + C_4 t + C_1, \\
\tau &= C_8 xt + C_7 t^2 - C_5 x + (C_3 - C_6) t + C_2, \\
\eta &= \frac{C_7 x + C_4 + (C_8 x - C_7 t + C_6) g(u) + (-C_8 t + C_5) g(u)^2}{g'(u)}.
\end{aligned} \tag{2.15}$$

Here we omit the redundant computational process for simplification. Associated with this Lie group, we have an 8-dimensional Lie algebra that can be represented by the generators

$$\begin{gathered}
X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \qquad X_4 = t\frac{\partial}{\partial x} + \frac{1}{g'(u)}\frac{\partial}{\partial u}, \\
X_5 = -x\frac{\partial}{\partial t} + \frac{g^2(u)}{g'(u)}\frac{\partial}{\partial u}, \qquad X_6 = -t\frac{\partial}{\partial t} + \frac{g(u)}{g'(u)}\frac{\partial}{\partial u}, \\
X_7 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + \frac{x - tg(u)}{g'(u)}\frac{\partial}{\partial u}, \qquad X_8 = x^2\frac{\partial}{\partial x} + xt\frac{\partial}{\partial t} + \frac{g(u)(x - tg(u))}{g'(u)}\frac{\partial}{\partial u}.
\end{gathered} \tag{2.16}$$

3. Discrete Symmetries

In this section, we will derive the discrete symmetries of (1.2), which has a 6-dimensional Lie subalgebra $\mathcal{L} : \{X_1, X_2, X_3, X_4, X_5, X_6\}$. we Will calculate the discrete symmetry following the method presented in [11].

By the commutator relation

$$[X_i, X_j] = c_{ij}^k X_k, \quad i < j, i, j = 1, 2, \ldots, 6, \tag{3.1}$$

we can get the nonzero commutators in the following form:

$$\begin{gathered}
[X_1, X_3] = X_1, \qquad [X_1, X_5] = -X_2, \qquad [X_2, X_3] = X_2, \qquad [X_2, X_4] = X_1, \\
[X_2, X_6] = -X_2, \qquad [X_4, X_5] = 2X_6 + X_3, \qquad [X_4, X_6] = X_4, \qquad [X_5, X_6] = -X_5.
\end{gathered} \tag{3.2}$$

Then the nonzero structure constants are

$$\begin{gathered}
C_{13}^1 = 1, \quad C_{15}^2 = -1, \quad C_{23}^2 = 1, \quad C_{24}^1 = 1, \quad C_{26}^2 = -1, \quad C_{45}^6 = 2, \quad C_{45}^3 = 1, \\
C_{46}^4 = 1, \quad C_{56}^5 = -1, \quad C_{31}^1 = -1, \quad C_{51}^2 = 1, \quad C_{32}^2 = -1, \quad C_{42}^1 = -1, \quad C_{62}^2 = 1, \\
C_{54}^6 = -2, \quad C_{54}^3 = -1, \quad C_{64}^4 = -1, \quad C_{65}^5 = 1.
\end{gathered} \tag{3.3}$$

The matrices $C(j)$ are

$$C(1)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\quad C(2)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&-1&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\end{pmatrix},$$
$$C(3)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\quad C(4)=\begin{pmatrix}0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-1&0&0&-2\\0&0&0&-1&0&0\end{pmatrix},\tag{3.4}$$
$$C(5)=\begin{pmatrix}0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&2\\0&0&0&0&0&0\\0&0&0&0&1&0\end{pmatrix},\quad C(6)=\begin{pmatrix}0&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\end{pmatrix}.$$

The next step is to calculate the matrices $A(\epsilon,j)$. Exponentiating the matrices $\epsilon C(j)$, we obtain

$$A(1,\epsilon)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\-\epsilon&0&1&0&0&0\\0&0&0&1&0&0\\0&\epsilon&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\quad A(2,\epsilon)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&-\epsilon&1&0&0&0\\-\epsilon&0&0&1&0&0\\0&0&0&0&1&0\\0&\epsilon&0&0&0&1\end{pmatrix},\tag{3.5}$$

$$A(3,\epsilon)=\begin{pmatrix}e^{\epsilon}&0&0&0&0&0\\0&e^{\epsilon}&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\quad A(4,\epsilon)=\begin{pmatrix}1&0&0&0&0&0\\\epsilon&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&-\epsilon&\epsilon^2&1&-2\epsilon\\0&0&0&-\epsilon&0&1\end{pmatrix},$$
$$A(5,\epsilon)=\begin{pmatrix}1&-\epsilon&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&\epsilon&1&\epsilon^2&2\epsilon\\0&0&0&0&1&0\\0&0&0&0&\epsilon&1\end{pmatrix},\quad A(6,\epsilon)=\begin{pmatrix}1&0&0&0&0&0\\0&e^{-\epsilon}&0&0&0&0\\0&0&1&0&0&0\\0&0&0&e^{\epsilon}&0&0\\0&0&0&0&e^{-\epsilon}&0\\0&0&0&0&0&1\end{pmatrix}.\tag{3.6}$$

From the nonlinear constants

$$c_{lm}^{n} b_i^l b_j^m = c_{ij}^{k} b_k^n, \quad 1 \le i < j \le 6, 1 \le n \le 6, \tag{3.7}$$

and using the adjoint matrices $A(i, \epsilon)$, $i = 1, \ldots, 6$, the matrix B can be simplified as the following two nonsingular forms:

$$B_1 = \begin{pmatrix} \frac{\alpha}{\beta} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -\frac{\alpha}{\beta} & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta^{-1} & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}, \quad \alpha, \beta \in R. \tag{3.8}$$

The determining equations for the discrete symmetries are given by the system:

$$\begin{pmatrix} X_1\hat{x} & X_1\hat{t} & X_1\hat{u} \\ X_2\hat{x} & X_2\hat{t} & X_2\hat{u} \\ X_3\hat{x} & X_3\hat{t} & X_3\hat{u} \\ X_4\hat{x} & X_4\hat{t} & X_4\hat{u} \\ X_5\hat{x} & X_5\hat{t} & X_5\hat{u} \\ X_6\hat{x} & X_6\hat{t} & X_6\hat{u} \end{pmatrix} = B \begin{pmatrix} \hat{X}_1\hat{x} & \hat{X}_1\hat{t} & \hat{X}_1\hat{u} \\ \hat{X}_2\hat{x} & \hat{X}_2\hat{t} & \hat{X}_2\hat{u} \\ \hat{X}_3\hat{x} & \hat{X}_3\hat{t} & \hat{X}_3\hat{u} \\ \hat{X}_4\hat{x} & \hat{X}_4\hat{t} & \hat{X}_4\hat{u} \\ \hat{X}_5\hat{x} & \hat{X}_5\hat{t} & \hat{X}_5\hat{u} \\ \hat{X}_6\hat{x} & \hat{X}_6\hat{t} & \hat{X}_6\hat{u} \end{pmatrix} = B \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hat{x} & \hat{t} & 0 \\ \hat{t} & 0 & \frac{1}{g'(\hat{u})} \\ 0 & -\hat{x} & \frac{g(\hat{u})^2}{g'(\hat{u})} \\ 0 & -\hat{t} & \frac{g(\hat{u})}{g'(\hat{u})} \end{pmatrix}. \tag{3.9}$$

First we consider $B = B_1$. From (3.9), we obtain the following system

$$\begin{pmatrix} X_1\hat{x} & X_1\hat{t} & X_1\hat{u} \\ X_2\hat{x} & X_2\hat{t} & X_2\hat{u} \\ X_3\hat{x} & X_3\hat{t} & X_3\hat{u} \\ X_4\hat{x} & X_4\hat{t} & X_4\hat{u} \\ X_5\hat{x} & X_5\hat{t} & X_5\hat{u} \\ X_6\hat{x} & X_6\hat{t} & X_6\hat{u} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\beta} & 0 & 0 \\ 0 & \alpha & 0 \\ \hat{x} & \hat{t} & 0 \\ \frac{\hat{t}}{\beta} & 0 & \frac{1}{\beta g'(\hat{u})} \\ 0 & -\beta\hat{x} & \frac{\beta g(\hat{u})^2}{g'(\hat{u})} \\ 0 & -\hat{t} & \frac{g(\hat{u})}{g'(\hat{u})} \end{pmatrix}. \tag{3.10}$$

The solution to system (3.10) is

$$\hat{x} = \frac{\alpha}{\beta} x, \qquad \hat{t} = \alpha t, \qquad \hat{u} = g^{-1}\left(\frac{1}{\beta} g(u)\right), \tag{3.11}$$

where α, β are arbitrary constants. We can prove that (3.11) satisfies the invariant condition

$$\hat{u}_{\hat{t}} + g(\hat{u})\hat{u}_{\hat{x}} = 0, \quad \text{when } u_t + g(u)u_x = 0. \tag{3.12}$$

Therefore, the first group of discrete symmetries is

$$\Gamma_1 : (x, t, u) \longmapsto \left(\frac{\alpha}{\beta}x, \alpha t, g^{-1}\left(\frac{1}{\beta}g(u)\right)\right). \tag{3.13}$$

Then we consider $B = B_2$. From (3.9), we obtain the following system:

$$\begin{pmatrix} X_1\hat{x} & X_1\hat{t} & X_1\hat{u} \\ X_2\hat{x} & X_2\hat{t} & X_2\hat{u} \\ X_3\hat{x} & X_3\hat{t} & X_3\hat{u} \\ X_4\hat{x} & X_4\hat{t} & X_4\hat{u} \\ X_5\hat{x} & X_5\hat{t} & X_5\hat{u} \\ X_6\hat{x} & X_6\hat{t} & X_6\hat{u} \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{\alpha}{\beta} & 0 \\ \alpha & 0 & 0 \\ \hat{x} & \hat{t} & 0 \\ 0 & -\dfrac{\hat{x}}{\beta} & \dfrac{g(\hat{u})^2}{\beta g'(\hat{u})} \\ \beta\,\hat{t} & 0 & \dfrac{\beta}{g'(\hat{u})} \\ -\hat{x} & 0 & -\dfrac{g(\hat{u})}{g'(\hat{u})} \end{pmatrix}. \tag{3.14}$$

The solution to system (3.14) is

$$\hat{x} = \alpha t, \qquad \hat{t} = -\frac{\alpha}{\beta}x, \qquad \hat{u} = g^{-1}\left(-\frac{\beta}{g(u)}\right), \tag{3.15}$$

where α, β are arbitrary constants. We can prove that (3.15) also satisfies the invariant condition

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} = 0, \quad \text{when } u_t + uu_x = 0. \tag{3.16}$$

Therefore, the first group of discrete symmetries is

$$\Gamma_2 : (x, t, u) \longmapsto \left(\alpha t, -\frac{\alpha}{\beta}x, g^{-1}\left(-\frac{\beta}{g(u)}\right)\right). \tag{3.17}$$

Here, we have obtained two groups of discrete symmetries of (1.2). Using them, we can simplify the numerical computation in solving (1.2), create new exact solutions from known solutions and so on. In later chapters, we will introduce how they generate new solutions.

3.1. Lie Symmetries of (1.2)

To obtain the group transformation which is generated by the infinitesimal generators (2.16), we need to solve the system of first-order ordinary differential equations:

$$\frac{d\hat{x}}{d\epsilon} = \xi\left(\hat{x}, \hat{t}, \hat{u}\right), \qquad \hat{x}(0) = x,$$
$$\frac{d\hat{t}}{d\epsilon} = \tau\left(\hat{x}, \hat{t}, \hat{u}\right), \qquad \hat{t}(0) = t, \tag{3.18}$$
$$\frac{d\hat{u}}{d\epsilon} = \eta\left(\hat{x}, \hat{t}, \hat{u}\right), \qquad \hat{u}(0) = u.$$

Then, we can get the one-parameter groups G_i generated by X_i for $i = 1, \ldots, 8$:

$$G_1 : (x, t, u) \longmapsto (x + \epsilon, t, u),$$
$$G_2 : (x, t, u) \longmapsto (x, t + \epsilon, u),$$
$$G_3 : (x, t, u) \longmapsto (xe^{\epsilon}, te^{\epsilon}, u),$$
$$G_4 : (x, t, u) \longmapsto \left(x + \epsilon t, t, g^{-1}(\epsilon + g(u))\right),$$
$$G_5 : (x, t, u) \longmapsto \left(x, t - x\epsilon, g^{-1}\left(\frac{g(u)}{1 - \epsilon g(u)}\right)\right), \tag{3.19}$$
$$G_6 : (x, t, u) \longmapsto \left(x, te^{-\epsilon}, g^{-1}(g(u)e^{\epsilon})\right),$$
$$G_7 : (x, t, u) \longmapsto \left(\frac{x}{1 - \epsilon t}, \frac{t}{1 - \epsilon t}, g^{-1}(g(u) + (x - tg(u))\epsilon)\right),$$
$$G_8 : (x, t, u) \longmapsto \left(\frac{x}{1 - \epsilon x}, \frac{t}{1 - \epsilon x}, g^{-1}\left(\frac{g(u)}{1 - (x - tg(u))\epsilon}\right)\right).$$

In addition, we have two groups of discrete symmetries

$$\Gamma_1 : (x, t, u) \longmapsto \left(\frac{\alpha}{\beta}x, \alpha t, g^{-1}\left(\frac{1}{\beta}g(u)\right)\right),$$
$$\Gamma_2 : (x, t, u) \longmapsto \left(\alpha t, -\frac{\alpha}{\beta}x, g^{-1}\left(-\frac{\beta}{g(u)}\right)\right). \tag{3.20}$$

If $u = f(x,t)$ is a solution of (1.2), so are the functions

$$\begin{aligned}
&G_1 \cdot f(x,t) = f(x+\epsilon,t),\\
&G_2 \cdot f(x,t) = f(x,t+\epsilon),\\
&G_3 \cdot f(x,t) = f(xe^{\epsilon},te^{\epsilon}),\\
&G_4 \cdot f(x,t) = g^{-1}\big(g\big(f(x+\epsilon t,t)\big)-\epsilon\big),\\
&G_5 \cdot f(x,t) = g^{-1}\left(\frac{g\big(f(x,t-x\epsilon)\big)}{1+\epsilon g\big(f(x,t-x\epsilon)\big)}\right),\\
&G_6 \cdot f(x,t) = g^{-1}\big(e^{-\epsilon}g\big(f(x,te^{-\epsilon})\big)\big),\\
&G_7 \cdot f(x,t) = g^{-1}\left(\frac{g\big(f(x/(1-\epsilon t),t/(1-\epsilon t))\big)-x\epsilon}{1-t\epsilon}\right),\\
&G_8 \cdot f(x,t) = g^{-1}\left(\frac{(1-\epsilon x)g\big(f(x/(1-\epsilon x),t/(1-\epsilon x))\big)}{1-\epsilon t g\big(f(x/(1-\epsilon x),t/(1-\epsilon x))\big)}\right),\\
&\Gamma_1 \cdot f(x,t) = g^{-1}\left(\beta g\left(f\left(\frac{\alpha}{\beta}x,\alpha t\right)\right)\right),\\
&\Gamma_2 \cdot f(x,t) = g^{-1}\left(-\frac{\beta}{g\big(f(\alpha t,-(\alpha/\beta)x)\big)}\right).
\end{aligned} \tag{3.21}$$

To illustrate how this technique may be of great interest, we let $u(x,t) = 1$ is a constant solution of (1.2). We conclude trivial solutions $\Gamma_1 \cdot 1$, $\Gamma_2 \cdot 1$, $G_i \cdot 1$, $i = 1,\ldots,6$, and nontrivial solutions for (1.2):

$$\begin{aligned}
&G_7 \cdot 1 = g^{-1}\left(\frac{g(1)-\epsilon x}{1-\epsilon t}\right),\\
&G_8 \cdot 1 = g^{-1}\left(\frac{(1-\epsilon x)g(1)}{1-\epsilon t g(1)}\right).
\end{aligned} \tag{3.22}$$

Now, by applying Γ_2, we get the following solutions:

$$\begin{aligned}
&\Gamma_2 \cdot G_7 \cdot 1 = g^{-1}\left(\frac{\alpha\epsilon x+\beta}{\alpha\epsilon t-g(1)}\right),\\
&\Gamma_2 \cdot G_8 \cdot 1 = g^{-1}\left(\frac{\alpha\epsilon g(1)x+\beta}{\alpha\epsilon g(1)t-g(1)}\right).
\end{aligned} \tag{3.23}$$

If we let $g(u) = u$, (1.2) becomes

$$u_t(x,t) + uu_x(x,t) = 0. \tag{3.24}$$

From (3.23), we get that

$$u = \frac{\alpha\epsilon x + \beta}{\alpha\epsilon t - 1} = \frac{ax + b}{at - 1} \tag{3.25}$$

is a solution to (3.24), where a, b are arbitrary constants.

If we let $g(u) = (1 - u)/(1 + u)$, (1.2) becomes

$$u_t(x,t) + \frac{1-u}{1+u} u_x(x,t) = 0. \tag{3.26}$$

From (3.23), we get

$$u = \frac{a(t-x) - b}{a(t+x) + b} \tag{3.27}$$

is a solution to (3.26), where a, b are arbitrary constants.

4. Reduction and Invariant Solutions of (1.2)

The first advantage of symmetry group method is to construct new solutions from known solutions. The second is when a nonlinear system of differential equations admits infinite symmetries, so it is possible to transform it to a linear system. In this section, symmetry group method will be applied to the inviscid Burgers equation to be connected directly to some order differential equations. To do this, particular linear combinations of infinitesimals are considered and their corresponding invariants are determined. Using discrete symmetries, a series of new interesting results are obtained.

4.1. Reduction with $X_2 + X_4$

As a first example, we perform a reduction of (1.2) using the generator

$$X_2 + X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{1}{g'(u)}\frac{\partial}{\partial u}. \tag{4.1}$$

Having determined the infinitesimals, the symmetry variables can be found by solving the characteristic equation

$$\frac{dx}{t} = \frac{dt}{1} = g'(u)du. \tag{4.2}$$

The similarity transformation is

$$\xi = \frac{1}{2}t^2 - x, \qquad u = g^{-1}(t - f(\xi)). \tag{4.3}$$

The similarity representation is

$$\frac{1-f(\xi)f'(\xi)}{g'(u)}=0. \tag{4.4}$$

The similarity solution is

$$f(\xi)=\pm\sqrt{2\xi+C_1}. \tag{4.5}$$

In the end, we obtain that

$$u=g^{-1}\left(t\pm\sqrt{t^2-2x+C_1}\right) \tag{4.6}$$

is a solution of (1.2). Now, by applying Γ_1, Γ_2 to (4.6), we conclude the other two solutions

$$\begin{aligned}\Gamma_1 : u&=g^{-1}\left(\alpha\beta t\pm\beta\sqrt{\alpha^2t^2-\frac{2\alpha}{\beta}x+C_1}\right),\\ \Gamma_2 : u&=g^{-1}\left(\frac{\beta}{(\alpha/\beta)x\pm\sqrt{((\alpha/\beta)x)^2-2\alpha t+C_1}}\right).\end{aligned} \tag{4.7}$$

If we let $g(u)=u$ in (1.2), from (4.6), (4.7), we can get the following solutions:

$$\begin{aligned}&u_{1,2}=t\pm\sqrt{t^2-2x+C_1}, \qquad u_{3,4}=\alpha\beta t\pm\beta\sqrt{\alpha^2t^2-\frac{2\alpha}{\beta}x+C_1},\\ &u_{5,6}=\frac{\beta}{(\alpha/\beta)x\pm\sqrt{((\alpha/\beta)x)^2-2\alpha t+C_1}},\end{aligned} \tag{4.8}$$

where α, β, C_1 are arbitrary constants.

If we let $g(u)=(1-u)/(1+u)$ in (1.2), from (4.6), (4.7), the following solutions can be obtained:

$$\begin{aligned}&u_{1,2}=\frac{1-t\pm\sqrt{t^2-2x+C_1}}{1+t\pm\sqrt{t^2-2x+C_1}}, \qquad u_{3,4}=\frac{1-\alpha\beta t\pm\beta\sqrt{\alpha^2t^2-2\alpha x/\beta+C_1}}{1+\alpha\beta t\pm\beta\sqrt{\alpha^2t^2-2\alpha x/\beta+C_1}},\\ &u_{5,6}=\frac{\alpha x\pm\beta\sqrt{-((-\alpha^2x^2+2\alpha\beta^2t-C_1\beta^2)/\beta^2)}-\beta^2}{\alpha x\pm\beta\sqrt{-((-\alpha^2x^2+2\alpha\beta^2t-C_1\beta^2)/\beta^2)}+\beta^2},\end{aligned} \tag{4.9}$$

where α, β, C_1 are arbitrary constants.

4.2. Reduction with $X_2 + X_5$

Similarly, the generator

$$X_2 + X_5 = (1-x)\frac{\partial}{\partial t} + \frac{g^2(u)}{g'(u)}\frac{\partial}{\partial u} \tag{4.10}$$

with similarity transformation

$$\xi = x, \qquad u = g^{-1}\left(\frac{1-x}{(1-x)f(\xi)-t}\right) \tag{4.11}$$

leads to the solution

$$u = g^{-1}\left(\frac{x-1}{t+C_1}\right). \tag{4.12}$$

By applying Γ_1, Γ_2 to (4.12), we obtain the invariant solution

$$u = g^{-1}\left(\frac{\alpha x-\beta}{\alpha t+C_1}\right). \tag{4.13}$$

If $g(u) = u/(1+u^2)$ in (1.2), then we have the solutions

$$u_{1,2} = \frac{\alpha t + C_1 \pm \sqrt{(\alpha t + C_1)^2 - 4(\alpha x - \beta)^2}}{2(\alpha x - \beta)}. \tag{4.14}$$

4.3. Reduction with $X_1 + X_6$

In the last, we discuss the generator

$$X_1 + X_6 = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + \frac{g(u)}{g'(u)}\frac{\partial}{\partial u}. \tag{4.15}$$

The similarity transformation is

$$\xi = x + \ln(t), \qquad u = g^{-1}\left(\frac{1}{t}f(\xi)\right), \tag{4.16}$$

which leads to the solution

$$u = g^{-1}\left(\frac{1}{t}\mathrm{LW}\left(te^{x+C_1}\right)\right), \tag{4.17}$$

where LW is the Lambert W-function, which satisfies

$$\mathrm{LW}(x)e^{\mathrm{LW}(x)} = x. \tag{4.18}$$

By applying Γ_1, Γ_2 to this solution, we conclude other two solutions:

$$u_1 = g^{-1}\left(\frac{\beta}{\alpha t}\mathrm{LW}\left(\alpha t e^{(\alpha/\beta)x+C_1}\right)\right), \qquad u_2 = g^{-1}\left(\frac{\alpha x}{\mathrm{LW}\left(-(\alpha/\beta)x e^{\alpha t+C_1}\right)}\right). \tag{4.19}$$

Comparing (4.17) and (4.19), we can see (4.17) is a special case of (4.19).

In (1.2), if we let $g(u) = (1-u)/(1+u)$, (4.19) lead to the following solutions

$$u_1 = \frac{\alpha t - \beta \mathrm{LW}\left(\alpha t e^{(\alpha x + C_1\beta)/\beta}\right)}{\alpha t + \beta \mathrm{LW}\left(\alpha t e^{(\alpha x + C_1\beta)/\beta}\right)}, \qquad u_2 = \frac{\mathrm{LW}\left(-\alpha x e^{\alpha t+C_1}/\beta\right) - \alpha x}{\mathrm{LW}\left(-\alpha x e^{\alpha t+C_1}/\beta\right) + \alpha x}. \tag{4.20}$$

If we let $g(u) = u/(1+u^2)$, we can get the following solutions:

$$\begin{gathered} u_{1,2} = \frac{\alpha t \pm \sqrt{\alpha^2 t^2 - 4\beta^2 \mathrm{LW}\left(\alpha t e^{(\alpha x/\beta)+C_1}\right)^2}}{2\beta \mathrm{LW}\left(\alpha t e^{(\alpha x/\beta)+C_1}\right)}, \\ u_{3,4} = \frac{\mathrm{LW}\left(-\alpha x e^{\alpha t+C_1}/\beta\right) \pm \sqrt{\mathrm{LW}\left(-\alpha x e^{\alpha t+C_1}/\beta\right)^2 - 4\alpha^2 x^2}}{2\alpha x}. \end{gathered} \tag{4.21}$$

5. Summary and Discussion

In this paper, we obtain eight infinitesimal generators for (1.2) by means of the Lie symmetry method. Considering it is 6-dimensional Lie subalgebra, we get two groups of discrete symmetries following the method presented by Hydon. Using the symmetry group, the similarity variable, similarity transformations, and the reduced equations are given. Solving the reduced equations, from the similarity transformations, we get the solutions of the inviscid Burgers equation. In additional, by applying discrete symmetries, we conclude other new solutions. If given $g(u)$ different forms in (1.2), many different types of solutions can be obtained directly. In forthcoming days, we will further discuss the problem. It is also interesting for us to see how the discrete symmetries will be under the 8-dimensional Lie algebra case.

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