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Research Article

Characterizations of Strongly Paracompact Spaces

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Characterizations of strongly compact spaces are given based on the existence of a star-countable open refinement for every increasing open cover. It is proved that a countably paracompact normal space (a perfectly normal space or a monotonically normal space) is strongly paracompact if and only if every increasing open cover of the space has a star-countable open refinement. Moreover, it is shown that a space is linearly D provided that every increasing open cover of the space has a point-countable open refinement.

1. Introduction

The strongly paracompact property has been an interesting covering property in general topology. It is a natural generalization of compact spaces. It retains enough structure to enjoy many of the properties of compact spaces, yet sufficiently general to include a much wider class of spaces. On one hand, the strongly paracompact property is special since it is different in many aspects with other covering properties. For example, it is not implied even by metrizability; it is not preserved under finite-to-one closed mappings; it has no F_σ-heredity. On the other hand, the property is general since every regular Lindelöf space is strongly paracompact.

Unlike paracompactness, the strongly paracompact property has not many characterizations. The definition of the property is based on the existence of star-finite open refinement of every open cover. It is difficult to discover strongly paracompact spaces with only such a definition. So it has been an interesting subject to characterize the class in easier ways. In [1], Smirnov characterized the class in the way that a regular space is strongly paracompact if and only if every open cover of the space has a star-countable open refinement. Recently, Qu showed us another characterization in [2] that a regular space is strongly paracompact if and only if every increasing open cover of the space has a star-finite open refinement. The Tychonoff linearly Lindelöf nonparacompact space constructed in [3] helps us to know that we cannot obtain the conclusion only by weakening the condition "star-finite" in Qu's result to "star-countable." Then, it is natural to consider what more conditions we need to characterize the strongly paracompact space in the way that every increasing open cover of the space has a star-countable open refinement.

In Section 2, we mainly deal with this problem and first obtain that a countably paracompact normal space is strongly paracompact if and only if every increasing open cover of the space has a star-countable open refinement. Moreover, we also obtain a characterization of linearly D-spaces introduced in [4], that is, a space is linearly D provided that every increasing open cover of the space has a point-countable open refinement. It helps us to know that a monotonically normal space is strongly paracompact if and only if every increasing open cover of the space has a star-countable open refinement.

Throughout the paper, all spaces are assumed to be regular T_1-spaces.

2. Definitions

Note that throughout the paper, we denote by $(\mathcal{F})_A$ the family $\{F \in \mathcal{F} : F \cap A \neq \emptyset\}$ and by $St(A;\mathcal{F})$ the set $\bigcup\{F \in \mathcal{F} : F \cap A \neq \emptyset\}$ for any set A and any family $\mathcal{F}$ of a space X. In particular, if $A = \{x\}$, then we use the symbols $(\mathcal{F})x$ and $St(x;\mathcal{F})$ instead of $(\mathcal{F})_{\{x\}}$ and $St(\{x\};\mathcal{F})$.

To make it easier to read, we recall some definitions.

A family $\mathcal{A}$ of subsets of a space is *star-finite (star-countable)* if $(\mathcal{A})_A$ is finite (countable) for every $A \in \mathcal{A}$.

A space X is *strongly paracompact* if every open cover of X has a star-finite open refinement.

A family $\mathcal{A}$ of subsets of a space X is *locally finite* if each $x \in X$ has a neighborhood meeting only finitely many $A \in \mathcal{A}$.

A space X is *paracompact* if every open cover of X has a locally finite open refinement.

A space X is *countably paracompact* if every countable open cover of X has a locally finite open refinement.

A space X is *perfectly normal* if each pear of disjoint closed sets A and B in X, there is a continuous function $f : X \to \mathbb{I}$ such that $A = f^{-1}(0)$ and $A = f^{-1}(1)$. Here, the space $\mathbb{I}$ is the open interval $(0,1)$ of reals equipped with usual metric topology.

A subset B of a space X is *discrete* if each $x \in X$ has a neighborhood meeting at most one element in B.

The *extent* of a space X is the smallest infinite cardinal number τ such that $|F| \leq \tau$ for every discrete subset F of X.

A space X is *linearly Lindelöf* if every increasing open cover of X has a countable subcover. In the paper, we call a family $\mathcal{O}$ of subsets of X is *increasing* if the family is well ordered by proper inclusion.

A space X is *linearly D* provided that every increasing open cover $\mathcal{U}$ of X without a countable subcover has a closed and discrete $\mathcal{U}$-big set. Here, a set A is $\mathcal{U}$-big if $A \not\subseteq U$ for every $U \in \mathcal{U}$. Note that in T_1-spaces, every discrete subset is closed. So in the proof of Theorem 3.6, we only need to prove that the increasing open cover $\mathcal{U}$ has a discrete $\mathcal{U}$-big set.

A space X is *monotonically normal* if to each pair (H,K) of disjoint closed subsets of X, one can assign an open set $D(H,K)$ such that

(i) $H \subset D(H,K) \subset \overline{D(H,K)} \subset X \setminus K$;

(ii) if $H \subset H'$ and $K \supset K'$, then $D(H,K) \subset D(H',K')$.

For terminologies without definitions that appear in the paper, we refer the readers to [5, 6].

3. Main Results

Theorem 3.1. *A countably paracompact normal space is strongly paracompact if and only if every increasing open cover of the space has a star-countable open refinement.*

In order to prove Theorem 3.1, we need the following results.

Lemma 3.2 (see [2]). *A space is strongly paracompact if and only if every increasing open cover of the space has a star-finite open refinement.*

Lemma 3.3 (see [7]). *Every countable open cover of a countably paracompact normal space has a star-finite open refinement.*

Proof of Theorem 3.1. Necessity. By the definition of a strongly paracompact space, it is trivial to know that every increasing open cover of the space has a star-countable open refinement.

Sufficiency. Assume that X is a countably paracompact normal space and every increasing open cover of X has a star-countable open refinement. To prove that X is strongly paracompact, let $\mathcal{O}$ be an increasing open cover of X and suppose that $\mathcal{U}$ is a star-countable open refinement of $\mathcal{O}$. With the help of Lemma 3.2, we prove that the cover $\mathcal{O}$ has a star-finite open refinement.

Firstly, we present the family $\mathcal{U}$ in the following way.

Claim. The family $\mathcal{U}$ can be presented as $\mathcal{U} = \{\mathcal{B}_\alpha : \alpha \in \Lambda\}$, where each $\mathcal{B}_\alpha$ is a countable family and $(\bigcup \mathcal{B}_\alpha) \bigcap (\bigcup \mathcal{B}_\beta) = \emptyset$ for $\alpha \neq \beta$.

Proof of claim. For all $A, B \in \mathcal{U}$, we call the finite subfamily $\{C_1, C_2, \ldots, C_n\}$ a chain from A to B, if $A = C_1$, $B = C_n$, and $C_i \cap C_{i+1} \neq \emptyset$ for $1 \leq i < n$. For every $A \in \mathcal{U}$, denote

$$\mathcal{B}(A) = \{B \in \mathcal{U} : \text{there is a chain from } A \text{ to } B\}. \tag{3.1}$$

It is easy to know that $\mathcal{B}(A)$ is countable, and, for any $A_1, A_2 \in \mathcal{U}$, $(\bigcup \mathcal{B}(A_1)) \bigcap (\bigcup \mathcal{B}(A_2)) \neq \emptyset$ if and only if $\mathcal{B}(A_1) = \mathcal{B}(A_2)$. We complete the proof of the claim.

For every $\alpha \in \Lambda$, let $Z_\alpha = \cup \mathcal{B}_\alpha$. By the above claim, we know that the family $\{Z_\alpha : \alpha \in \Lambda\}$ is an open and closed disjoint family of X. Since X is countably paracompact, the closed subspace Z_α of X is countably paracompact for every $\alpha \in \Lambda$. Moreover, it follows from the above claim that the family $\mathcal{B}_\alpha$ is a countable open cover of Z_α. By Lemma 3.3, we find a star-finite open family $\mathcal{W}_\alpha$ of the subspace Z_α refining $\mathcal{B}_\alpha$. Since each Z_α is open in X and since $X = \bigcup_{\alpha\in\Lambda} Z_\alpha$, it follows that the family $\bigcup_{\alpha\in\Lambda} \mathcal{W}_\alpha$ is an open cover of X. The family $\bigcup_{\alpha\in\Lambda} \mathcal{W}_\alpha$ is also star-finite since $\{Z_\alpha : \alpha \in \Lambda\}$ is a disjoint family of X. On the other hand, it is easy to see that $\bigcup_{\alpha\in\Lambda} \mathcal{W}_\alpha$ is a refinement of $\mathcal{O}$ since $\mathcal{U}$ refines $\mathcal{O}$.

By Lemma 3.2, the space X is strongly paracompact. □

Remark 3.4. It is well known that space Γ constructed in [3] is not strongly paracompact, while every increasing open cover of the space has a star-countable refinement since it is linearly Lindelöf. It helps us to know that in Theorem 3.1 we cannot get the conclusion if we remove the countably paracompact property.

Corollary 3.5. *Every perfectly normal space is strongly paracompact if and only if every increasing open cover of the space has a star-countable open refinement.*

Proof. Necessity. It is trivial by the definition of strongly paracompact spaces.

Sufficiency. It is known that every perfectly normal space is countably paracompact and normal (see [5]). Then it follows from Theorem 3.1 that a perfectly normal space is strongly paracompact if every increasing open cover of the space has a star-countable open refinement. □

Motivated by Theorem 3.1, we obtain a characterization of linearly D-spaces in the way that every open cover of the space has a point-countable open refinement, which will help us to obtain a new characterization of strongly paracompact spaces in monotonically normal spaces.

Theorem 3.6. *A space X is linearly D provided that every increasing open cover of X has a point-countable open refinement.*

Proof. Assume that $\mathcal{U}$ is an increasing open cover of X without a countable subcover, and $\mathcal{V}$ is a point-countable open refinement of $\mathcal{U}$.

In order to prove easily, well order X as $\{x_\alpha : \alpha < \Gamma\}$ and let $y_0 = x_0$. Since $\mathcal{V}$ is point countable, the family $(\mathcal{V})_{y_0}$ is countable. For every $V \in (\mathcal{V})_{y_0}$, let U_V be the first set of $\mathcal{U}$ such that $U_V \supset V$ and let $\mathcal{U}_0 = \{U_V \in \mathcal{U} : V \in (\mathcal{V})_{y_0}\}$. The family $\mathcal{U}_0$ cannot cover X since $\mathcal{U}_0$ is a countable family and $\mathcal{U}$ has no countable subcover according to our assumption above. We then take the first point of X which is not contained in $\cup\mathcal{U}_0$ and denote it by y_1. For every $V \in (\mathcal{V})_{y_1}$, let U_V be the first set of $\mathcal{U}$ such that $V \subset U_V$. The family $\mathcal{U}_1 = \{U_V \in \mathcal{U} : V \in (\mathcal{V})_{\{y_0,y_1\}}\}$ is still not a cover of X. Consequently, we are able to take the first point of X which is not contained in $\cup\mathcal{U}_1$ and denote it by y_2. Thus, we continue to define the family $\mathcal{U}_2 = \{U_V \in \mathcal{U} : V \in \mathcal{V}_{\{y_0,y_1,y_2\}}\}$, where each U_V is the first set of $\mathcal{U}$ such that $U_V \supset V$. Define y_α and $\mathcal{U}_\alpha$ successively in the same way. There must exist an ordinal $\Lambda \leq \Gamma$ such that the set $A = \{y_\alpha : \alpha < \Lambda\}$ satisfies that the family $\mathcal{U}' = \{U_V \in \mathcal{U} : V \in (\mathcal{V})_A\}$ covers X.

To prove that the set A is closed and discrete in X, it suffices to show that A is discrete since X is T_1. For every $x \in X$, if there exists some $V' \in (\mathcal{V})_x$ such that $V' \cap A \neq \emptyset$, let $\beta < \Lambda$ be the least such that $y_\beta \in V' \cap A$. Then $V' \in (\mathcal{V})_{y_\beta}$ and $y_\alpha \notin V'$ for every y_α with $\alpha < \beta$. On the other hand, for every $\alpha > \beta$, we know that $y_\alpha \notin \cup\mathcal{U}_\beta$ and $V' \in \mathcal{U}_\beta$, where $\mathcal{U}_\beta = \{U_V \in \mathcal{U} : V \in (\mathcal{V})_{\{y_\delta:\delta\leq\beta\}}\}$. It follows that $y_\alpha \notin V'$. Thus we have proved that such a neighborhood V' of x contains only one element of A. By the arbitrariness of x, we know that the set A is discrete.

To prove X is linearly D, it is enough to show that A is a $\mathcal{U}$-big set. To show this, pick an arbitrary $U \in \mathcal{U}$. Assume on the contrary that $U' \subset U$ for every $U' \in \mathcal{U}'$. Then $\cup\mathcal{U}' \subset U$. It is contradicted with the fact that $\mathcal{U}$ has no countable subcover. Therefore, there exists some $U' \in \mathcal{U}'$ such that $U \subset U'$. Then, we have $A \not\subset U$. Thus we know that A is a $\mathcal{U}$-big set.

We complete the proof of Theorem 3.6. □

Since a space of countable extent is linearly Lindelöf if and only if it is linearly D (see [4]), we have the following consequence of Theorem 3.6.

Corollary 3.7. *A space of countable extent is linearly Lindelöf if and only if every increasing open cover of the space has a point-countable open refinement.*

At last, we close the paper with another main result with the help of foregoing results and the following lemma.

Lemma 3.8 (see [4]). *Every monotonically normal linearly D-space is paracompact.*

Theorem 3.9. *A monotonically normal space X is strongly paracompact if and only if every increasing open cover of X has a star-countable open refinement.*

Proof. *Necessity.* It is trivial by the definition of a strongly paracompact space.

Sufficiency. Assume that X is a monotonically normal space and every increasing open cover of X has a star-countable open refinement. It follows from Theorem 3.6 and Lemma 3.8 that X is paracompact. By Theorem 3.1, we know that X is strongly paracompact. □

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