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Research Article

Local Stability of Period Two Cycles of Second Order Rational Difference Equation

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We consider the second order rational difference equation $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n + Cx_{n-1})$, $n = 0, 1, 2, \ldots$, where the parameters $\alpha, \beta, \gamma, A, B, C$ are positive real numbers, and the initial conditions x_{-1}, x_0 are nonnegative real numbers. We give a necessary and sufficient condition for the equation to have a prime period-two solution. We show that the period-two solution of the equation is locally asymptotically stable. In particular, we solve Conjecture 5.201.2 proposed by Camouzis and Ladas in their book (2008) which appeared previously in Conjecture 11.4.3 in Kulenović and Ladas monograph (2002).

1. Introduction

Difference equations proved to be effective in modelling and analysing discrete dynamical systems that arise in signal processing, populations dynamics, health sciences, economics, and so forth. They also arise naturally in studying iterative numerical schemes. Furthermore, they appear when solving differential equations using series solution methods or studying them qualitatively using, for example, Poincaré maps. For an introduction to the general theory of difference equations, we refer the readers to Agarwal [1], Elaydi [2], and Kelley and Peterson [3].

Rational difference equations; particularly bilinear ones, that is,

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} \alpha_i x_{n-i}}{\beta + \sum_{j=0}^{\ell} \beta_j x_{n-j}}, \quad n = 0, 1, 2, 3 \ldots \tag{1.1}$$

attracted the attention of many researchers recently. For example, see the articles [4–24], monographs Kocić and Ladas [25], Kulenović and Ladas [26], and Camouzis and Ladas [27], and the references cited therein. We believe that behavior of solutions of rational difference equations provides prototypes towards the development of the basic theory of the global behavior of solutions of nonlinear difference equations of order greater than one [26, page 1].

Our aim in this paper is to study the second order bilinear rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where the parameters $\alpha, \beta, \gamma, A, B, C$ are positive real numbers, and the initial conditions x_{-1}, x_0 are nonnegative real numbers. Our concentration is on the periodic character of the positive solutions of (1.2). Indeed, our interest in (1.2) was stimulated by the following conjecture proposed by Camouzis and Ladas in [27, Conjecture 5.201.2].

Conjecture 1.1. *Show that the period-two solution of* (1.2) *is locally asymptotically stable.*

It is worth mentioning that the aforementioned conjecture appeared previously in Conjecture 11.4.3 in the Kulenović and Ladas monograph [26].

To this end and using transformations similar to the ones used by [22, 27], let

$$x_n = \frac{\gamma}{C} y_n \tag{1.3}$$

then (1.2) reduces to

$$y_{n+1} = \frac{r + py_n + y_{n-1}}{z + qy_n + y_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where

$$r = \frac{\alpha C}{\gamma^2}, \qquad p = \frac{\beta}{\gamma}, \qquad z = \frac{A}{\gamma}, \qquad q = \frac{B}{C} \tag{1.5}$$

are positive real numbers, and the initial conditions y_{-1}, y_0 are nonnegative real numbers.

That being said, the remainder of this paper is organized as follows. In the next section, we present some definitions and results that are needed in the sections to follow. Next, using elementary mathematics and nontrivial combinations of ideas, we establish our main results in Section 3. Our main results provide positive confirmation of Conjecture 1.1. Finally, we conclude in Section 4 with suggestions for future research.

2. Preliminaries

For the sake of self-containment and convenience, we recall the following definitions and results from [26].

Let I be a nondegenerate interval of real numbers and let $f : I \times I \rightarrow I$ be a continuously differentiable function. Then for every set of initial conditions $x_0, x_{-1} \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

A constant sequence, $x_n = \overline{x}$ for all n where $\overline{x} \in I$, is called an *equilibrium solution* of (2.1) if

$$\overline{x} = f(\overline{x}, \overline{x}). \tag{2.2}$$

Definition 2.1. Let $\overline{x}$ be an equilibrium solution of (2.1).

(i) $\overline{x}$ is called *locally stable* if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_0, x_{-1} \in I$, with $|x_0 - \overline{x}| + |x_{-1} - \overline{x}| < \delta$, we have

$$|x_n - \overline{x}| < \epsilon, \quad \forall n \geq -1. \tag{2.3}$$

(ii) $\overline{x}$ is called *locally asymptotically stable* if it is locally stable, and if there exists $\gamma > 0$, such that for all $x_0, x_{-1} \in I$, with $|x_0 - \overline{x}| + |x_{-1} - \overline{x}| < \gamma$, we have

$$\lim_{n \to \infty} x_n = \overline{x}. \tag{2.4}$$

(iii) $\overline{x}$ is called a *global attractor* if for every $x_0, x_{-1} \in I$, we have

$$\lim_{n \to \infty} x_n = \overline{x}. \tag{2.5}$$

(iv) $\overline{x}$ is called *globally asymptotically stable* if it is locally stable and a global attractor.

(v) $\overline{x}$ is called *unstable* if it is not stable.

(vi) $\overline{x}$ is called a *source*, or a *repeller*, if there exists $r > 0$ such that for all $x_0, x_{-1} \in I$, with $0 < |x_0 - \overline{x}| + |x_{-1} - \overline{x}| < r$, there exists $N \geq 1$ such that

$$|x_N - \overline{x}| \geq r. \tag{2.6}$$

Clearly a source is an unstable equilibrium.

Definition 2.2. Let

$$a = \frac{\partial f}{\partial u}(\overline{x}, \overline{x}), \qquad b = \frac{\partial f}{\partial v}(\overline{x}, \overline{x}) \tag{2.7}$$

denote the partial derivatives of $f(u,v)$ evaluated at the equilibrium $\overline{x}$ of (2.1). Then the equation

$$y_{n+1} = ay_n + by_{n-1}, \quad n = 0,1,\dots \tag{2.8}$$

is called the *linearized equation associated with* (2.1) about the equilibrium solution $\overline{x}$.

Definition 2.3. A solution $\{x_n\}$ of (2.1) is said to be *periodic with period P* if

$$x_{n+P} = x_n, \quad \forall n \geq -1. \tag{2.9}$$

A solution $\{x_n\}$ of (2.1) is said to be periodic with prime period P, or a *P-cycle* if it is periodic with period P and P is the least positive integer for which (2.9) holds.

Theorem 2.4 (Linearized stability). *(a) If both roots of the quadratic equation*

$$\lambda^2 - a\lambda - b = 0. \tag{2.10}$$

lie in the open unit disk $|\lambda| < 1$, *then the equilibrium* $\overline{x}$ *of* (2.1) *is locally asymptotically stable.*

(b) *If at least one of the roots of* (2.10) *has absolute value greater than one, then the equilibrium* $\overline{x}$ *of* (2.1) *is unstable.*

(c) *A necessary and sufficient condition for both roots of* (2.10) *to lie in the open unit disk* $|\lambda| < 1$ *is*

$$|a| < 1 - b < 2. \tag{2.11}$$

In this case, the locally asymptotically stable equilibrium $\overline{x}$ *is also called a sink.*

(d) *A necessary and sufficient condition for both roots of* (2.10) *to have absolute value greater than one is*

$$|b| > 1, \qquad |a| < |1 - b|. \tag{2.12}$$

In this case, $\overline{x}$ *is a repeller.*

(e) *A necessary and sufficient condition for one root of* (2.10) *to have absolute value greater than one and for the other to have absolute value less than one is*

$$a^2 + 4b > 0, \qquad |a| > |1 - b|. \tag{2.13}$$

In this case, the unstable equilibrium $\overline{x}$ *is called a saddle point.*

(f) *A necessary and sufficient condition for a root of* (2.10) *to have absolute value equal to one is*

$$|a| = |1 - b| \tag{2.14}$$

or

$$b = -1, \qquad |a| \leq 2. \tag{2.15}$$

In this case, the equilibrium $\overline{x}$ is called a nonhyperbolic point.

3. Main Result

In this section, we give a necessary and sufficient condition for (1.4) to have a prime period-two solution. We show that the period-two solution of (1.4) is locally asymptotically stable.

Equation (1.4) has a unique positive equilibrium given by

$$\overline{y} = \frac{1 + p - z + \sqrt{(1 + p - z)^2 + 4r(q + 1)}}{2(q + 1)}. \tag{3.1}$$

Furthermore, the linearized equation associated with (1.4) about the equilibrium solution is given by

$$z_{n+1} = \frac{p - q\overline{y}}{z + (q + 1)\overline{y}} z_n + \frac{1 - \overline{y}}{z + (q + 1)\overline{y}} z_{n-1}. \tag{3.2}$$

Therefore, its characteristic equation is

$$\lambda^2 - \frac{p - q\overline{y}}{z + (q + 1)\overline{y}} \lambda - \frac{1 - \overline{y}}{z + (q + 1)\overline{y}} = 0. \tag{3.3}$$

Lemma 3.1. *(a) When*

$$p + z \geq 1, \tag{3.4}$$

Equation (1.4) *has no nonnegative prime period-two solution.*

(b) *When*

$$p + z < 1, \tag{3.5}$$

Equation (1.4) *has prime period-two solution,*

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots; \tag{3.6}$$

if and only if

$$r < \frac{(1 - p - z)\left[q(1 - p - z) - (1 + 3p - z)\right]}{4}, \tag{3.7}$$

where Φ *and* Ψ *are the positive and distinct solutions of the quadratic equation*

$$t^2 - (1 - z - p)t + \frac{p(1 - z - p) + r}{q - 1} = 0, \quad q > 1. \tag{3.8}$$

Proof. (a) Suppose $p + z \geq 1$ and assume, for the sake of contradiction, there exist distinct positive real numbers Φ and Ψ such that

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots \tag{3.9}$$

is a prime period-two solution of (1.4). Then, Φ and Ψ satisfy

$$\Phi = \frac{r + p\Psi + \Phi}{z + q\Psi + \Phi}, \qquad \Psi = \frac{r + p\Phi + \Psi}{z + q\Phi + \Psi}. \tag{3.10}$$

Furthermore,

$$\Phi(z + q\Psi + \Phi) = r + p\Psi + \Phi, \tag{3.11}$$

$$\Psi(z + q\Phi + \Psi) = r + p\Phi + \Psi. \tag{3.12}$$

Subtracting (3.11) from (3.12), we have

$$(\Phi + \Psi) = (1 - z - p). \tag{3.13}$$

But, $p + z \geq 1$ implies that $\Phi + \Psi \leq 0$ which contradicts the hypothesis that Φ and Ψ are distinct positive real numbers.

(b) Now, suppose $p + z < 1$, and let Φ and Ψ be two positive real numbers such that

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots \tag{3.14}$$

is a prime period-two solution of (1.4). Thus, Φ and Ψ satisfy

$$\Phi = \frac{r + p\Psi + \Phi}{z + q\Psi + \Phi}, \tag{3.15}$$

$$\Psi = \frac{r + p\Phi + \Psi}{z + q\Phi + \Psi}. \tag{3.16}$$

Moreover,

$$\Phi(z+q\Psi+\Phi)=r+p\Psi+\Phi, \tag{3.17}$$

$$\Psi(z+q\Phi+\Psi)=r+p\Phi+\Psi. \tag{3.18}$$

Subtracting (3.17) from (3.18), we have

$$(\Phi+\Psi)=(1-z-p). \tag{3.19}$$

Furthermore, adding (3.17) to (3.18), we have

$$\Phi\Psi=\frac{p(1-z-p)+r}{q-1}, \quad q>1. \tag{3.20}$$

Hence, Φ and $\Psi>0$ satisfy the quadratic equation

$$t^2-(1-z-p)t+\frac{p(1-z-p)+r}{q-1}=0, \quad q>1. \tag{3.21}$$

In other words, Φ and Ψ are given by

$$t=\frac{(1-z-p)\pm\sqrt{(1-z-p)^2-4((p(1-z-p)+r)/(q-1))}}{2}, \quad q>1. \tag{3.22}$$

□

Theorem 3.2. *Suppose* (1.4) *has a prime period-two solution. Then, the period-two solution is locally asymptotically stable.*

Proof. To start off, we first vectorize (1.4) by introducing the following change of variables:

$$u_n=y_{n-1}, \qquad v_n=y_n, \quad \text{for } n=0,1,2,\ldots, \tag{3.23}$$

and write (1.4) in the following equivalent form:

$$\begin{pmatrix}u_{n+1}\\ v_{n+1}\end{pmatrix}=T\begin{pmatrix}u_n\\ v_n\end{pmatrix}; \quad n=0,1,2,\ldots, \tag{3.24}$$

where

$$T\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}v\\ \dfrac{r+pv+u}{z+qv+u}\end{pmatrix}. \tag{3.25}$$

Now Φ and Ψ generate a period-two solution of (1.4) if and only if

$$\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \tag{3.26}$$

is a fixed point of T^2, the second iterate of T. Furthermore,

$$T^2\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}g(u,v)\\ h(u,v)\end{pmatrix}, \tag{3.27}$$

where

$$g(u,v) = \frac{r+pv+u}{z+qv+u}, \qquad h(u,v) = \frac{r+pg(u,v)+v}{z+qg(u,v)+v}. \tag{3.28}$$

The prime period-two solution of (1.4) is asymptotically stable if the eigenvalues of the Jacobian matrix J_{T^2}, evaluated at $\left(\begin{smallmatrix}\Phi\\ \Psi\end{smallmatrix}\right)$ lie inside the unit disk. But,

$$J_{T^2}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} = \begin{pmatrix} \dfrac{1-\Phi}{z+q\Psi+\Phi} & \dfrac{p-q\Phi}{z+q\Psi+\Phi} \\ \dfrac{(p-q\Psi)(1-\Phi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)} & \dfrac{1-\Psi}{z+q\Phi+\Psi} + \dfrac{(p-q\Phi)(p-q\Psi)}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)} \end{pmatrix} \tag{3.29}$$

and, hence, its characteristic equation is

$$\lambda^2 - \eta\lambda + \mu = 0, \tag{3.30}$$

where

$$\begin{gathered} \eta = \frac{1-\Phi}{z+q\Psi+\Phi} + \frac{1-\Psi}{z+q\Phi+\Psi} + \frac{(p-q\Phi)(p-q\Psi)}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)}, \\ \mu = \frac{(1-\Phi)(1-\Psi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)}. \end{gathered} \tag{3.31}$$

By Theorem 2.4(c), the eigenvalues of

$$J_{T^2}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \tag{3.32}$$

lie inside the unit disk if and only if

$$|\eta| < 1+\mu < 2 \quad \text{or,} \quad \text{equivalently, } |\eta| < 1+\mu, \quad \mu < 1. \tag{3.33}$$

To this end, without loss of generality, assume that $0 < \Phi < \Psi$. Then, by (3.15),

$$1 = \frac{(r/\Phi) + p(\Psi/\Phi) + 1}{z + q\Psi + \Phi} > \frac{1}{z + q\Psi + \Phi}. \tag{3.34}$$

Hence,

$$z + q\Psi + \Phi > 1. \tag{3.35}$$

Similarly, we observe that

$$z + q\Phi + \Psi > 1. \tag{3.36}$$

Furthermore, since $p + z < 1$, (3.19) implies the sum of Φ and Ψ is less than 1 and, a fortiori, each is less than 1. Indeed, we have

$$0 < \Phi < \min\left\{\Psi, \frac{1}{2}\right\} < 1. \tag{3.37}$$

With that in mind, it is clear that

$$\mu = \frac{(1-\Phi)(1-\Psi)}{(z + q\Psi + \Phi)(z + q\Phi + \Psi)} < 1. \tag{3.38}$$

In addition, with the understanding that

$$\begin{gathered}
\eta = \underbrace{\frac{1-\Phi}{z + q\Psi + \Phi} + \frac{1-\Psi}{z + q\Phi + \Psi}}_{L} + \underbrace{\frac{(p - q\Phi)(p - q\Psi)}{(z + q\Psi + \Phi)(z + q\Phi + \Psi)}}_{Q}, \\
1 + \mu - L = \left(1 - \frac{1-\Phi}{z + q\Psi + \Phi}\right)\left(1 - \frac{1-\Psi}{z + q\Phi + \Psi}\right) = \frac{(1 - z - 2\Phi - q\Psi)(1 - z - 2\Psi - q\Phi)}{(z + q\Psi + \Phi)(z + q\Phi + \Psi)}, \\
-1 - \mu - L = -\left(1 + \frac{1-\Phi}{z + q\Psi + \Phi}\right)\left(1 + \frac{1-\Psi}{z + q\Phi + \Psi}\right) = -\frac{(1 + z + q\Psi)(1 + z + q\Phi)}{(z + q\Psi + \Phi)(z + q\Phi + \Psi)},
\end{gathered} \tag{3.39}$$

we have

$$\begin{aligned}
|\eta| < 1 + \mu \Longleftrightarrow & -1 - \mu - L < Q < 1 + \mu - L \\
\Longleftrightarrow & (p - q\Phi)(p - q\Psi) < (1 - z - 2\Phi - q\Psi)(1 - z - 2\Psi - q\Phi), \\
& -(1 + z + q\Psi)(1 + z + q\Phi) < (p - q\Phi)(p - q\Psi).
\end{aligned} \tag{3.40}$$

The second inequality follows immediately since $p + z < 1$ and

$$\left|(p - q\Phi)(p - q\Psi)\right| < (p + q\Phi)(p + q\Psi) < (1 + z + q\Phi)(1 + z + q\Psi), \tag{3.41}$$

However, the proof of the first inequality is a bit tricky.

By (3.19),

$$(p - q\Phi)(p - q\Psi) = \Bigg(\underbrace{1 - z - \Phi - q\Psi}_{a} - \Psi\Bigg)\Bigg(\underbrace{1 - z - \Psi - q\Phi}_{b} - \Phi\Bigg). \tag{3.42}$$

Therefore,

$$(p - q\Phi)(p - q\Psi) < (1 - z - 2\Phi - q\Psi)(1 - z - 2\Psi - q\Phi) \tag{3.43}$$

if and only if

$$(a - \Psi)(b - \Phi) < (a - \Phi)(b - \Psi), \tag{3.44}$$

which is true if and only if

$$(b - a)\Phi < (b - a)\Psi. \tag{3.45}$$

The latter inequality holds true because $\Phi < \Psi$ and

$$b - a = (q - 1)(\Psi - \Phi) > 0. \tag{3.46}$$

This completes the proof. □

4. Conclusion

Without dispute, the existence of a periodic solution or an equilibrium solution does not imply its local stability. As such, it is natural to ask about the class of difference equations which afford the ELAS property, that is, the existence of a periodic solution implies its local asymptotic stability. In particular, one may want to investigate the class of bilinear difference equations (1.1) and completely characterize those equations enjoying the ELAS property.

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