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Research Article

Restricted p-Isometry Properties of Partially Sparse Signal Recovery

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By generalizing the restricted p-isometry property to the partially sparse signal recovery problem, we give a sufficient condition for exactly recovering partially sparse signal via the partial l_p minimization (truncated l_p minimization) problem with $p \in (0, 1]$. Based on this, we establish a simpler sufficient condition which can show how the p-RIP bounds vary corresponding to different ps.

1. Introduction

The partially sparse signal recovery (PSSR) is the problem of recovering a partially sparse signal from a certain number of linear measurements when the part of the signal is known to be sparse, which was coined by Bandeira et al. [1, 2]. This type of problems has many applications in signal and image processing, derivative-free optimizations, and so on; see, for example, [1–4]. Clearly, PSSR includes *sparse signal recovery (SSR)* as a special case. The latter is the well-known NP-hard problem in the compressed sensing (CS), which is also called *cardinality minimization problem (CMP, or l_0-norm minimization problems)*; see, for example, [5–8]. In particular, Candés and Tao [8] introduced a restricted isometry property (RIP) of a sensing matrix which guarantees to recover a sparse solution of SSR by minimizing its convex relaxation (ℓ_1-norm minimization). However, there are some problems which cannot be reformulated as an SSR, but a PSSR. As we know, PSSR happens naturally in sparse Hessian recovery; see, for example, [2], where Bandeira et al. employed partially sparse recovery approach for building sparse quadratic interpolation models of functions with sparse Hessian. They have successfully applied the ℓ_1-norm minimization of PSSR in interpolation-based trust-region methods for derivative-free optimization. Vaswani and Lu [3] successfully applied modified CS (partially sparse recovery) in image reconstruction, where the sufficient RIP condition is weaker than the RIP for SSR. Moreover, Bandeira et al. [1] considered the RIP and null space properties (NSP) for PSSR and extended recovery results under noisy measurements to the partially sparse case, where partial NSP is a necessary and sufficient condition for PSSR. In [4], Jacques also established the partial RIP condition for PSSR with noise via its convex relaxation problem.

Note that in the CS context, the SSR problem can also be relaxed to a l_p-norm minimization (truncated l_p-minimization) problem with $0 < p < 1$; see, for example, [9–19]. It is well known that Chartrand [20] firstly show that fewer measurements are required for exact reconstruction if we replace l_1-norm with l_p-norm ($0 < p < 1$), and Chartrand and Staneva [10] established p-RIP conditions for exact SSR via l_p-minimization. In particular, the numerical experiments in magnetic resonance imaging (MRI) showed that this approach works very efficiently; see [9] for details. Wang et al. [19] studied the performance of l_p-minimization for strong recovery and *weak recovery* where we need to recover all the sparse vectors on one support with one sign pattern. Moreover, Saab et al. [16] provided a sufficient condition for SSR via l_p-minimization and provided a lower bound of the support size up to which l_p-minimization can recover all such sparse vectors, and Foucart and Lai [14] improved this bound by considering a generalized version of RIP condition. While SSR and l_p-minimization have been the focus point of some recent research, there are fewer research related to PSSR and the partially l_p-minimization. One may naturally wonder whether we can generalize the p-RIP conditions introduced by [10] from the SSR to the PSSR case. This paper will deal with this issue. We will give a different p-RIP recovery condition for PSSR via its nonconvex relaxation. Furthermore, based on the recent work by Oymak et al. [21], we also extend our result to the matrix setting.

In the next section, we give the PSSR model and review some preliminaries on p-RIP conditions. In Section 3, we establish the exact partially p-RIP recovery conditions for PSSR via its nonconvex l_p-minimization. In Section 4, we give a sufficient condition for partially low-rank matrix recovery via the partially Schatten-p minimization problem.

2. Preliminaries

In this section, we will review some basic concepts and results on the p-RIP recovery conditions for SSR and introduce the p-RIP definition for PSSR. We begin with defining the mathematical model of the PSSR problem as follows:

$$\min \|x\|_0, \quad \text{s.t. } A_1 x + A_2 y = b, \tag{1}$$

where the l_0-norm $\|x\|_0$ is defined as $\|x\|_0 := |\{i : x_i \neq 0\}|$ (which is not really a norm since it is not positive homogeneous). For any positive number s, we say x is *s-sparse* if $\|x\|_0 \leq s$. $A = [A_1, A_2] \in \mathbb{R}^{M\times N}$ is a sensing matrix with $A_1 \in \mathbb{R}^{M\times(N-r)}$, $A_2 \in \mathbb{R}^{M\times r}$, and $b \in \mathbb{R}^M$. It means that the unknown vector consists of two parts (x, y), where $x \in \mathbb{R}^{N-r}$ is sparse and $y \in \mathbb{R}^r$ is possibly dense. When $A = A_1$, the previous problem reduces to the following *l_0-norm minimization problem (sparse signal recovery, SSR)*:

$$\min \|x\|_0, \quad \text{s.t. } Ax = b. \tag{2}$$

The previous PSSR problem (1) is an NP-hard problem, since its special case SSR (2) is well-known NP-hard problem in the compressed sensing (CS). As we mentioned in Section 1, one popular and powerful approach is to solve it via *ℓ_1-norm minimization* (its convex relaxation), where the l_0-norm is replaced by the l_1-norm in SSR (2). Moreover, we can also use a nonconvex approach for exact reconstruction with fewer measurements than the convex relaxation; see, for example, [9, 10]. That is the *l_p-norm minimization* problem with $0 < p < 1$, where we replace the l_0-norm with the l_p-norm in (2) as follows:

$$\min \|x\|_p^p, \quad \text{s.t. } Ax = b. \tag{3}$$

Note that $\|\cdot\|_p$ is not a norm when $p \in (0, 1)$, but it is much close to l_0-norm. Moreover, the numerical experiments in MRI showed that the approach via l_p-minimization works very efficiently; see [9] for details. In particular, Chartrand and Staneva [10] introduced the concept of restricted isometry constant via l_p-norm.

Definition 1 (p-RIC [10]). Given a matrix $A \in \mathbb{R}^{M\times N}$, where $M < N$, s is a positive number and $0 < p < 1$, then we say that δ_s is the restricted p-isometry constant (or p-RIC) of order s of the matrix A if δ_s is the smallest number, such that

$$\left(1-\delta_s\right)\|x\|_2^p \leq \|Ax\|_p^p \leq \left(1+\delta_s\right)\|x\|_2^p, \tag{4}$$

for all s-sparse vectors x.

In the same paper, Chartrand and Staneva gave the following sufficient condition for exact SSR via l_p-minimization.

Theorem 2 (see [10]). *Let $A \in \mathbb{R}^{M\times N}$, $x \in \mathbb{R}^N$, and $k = \|x\|_0$ be the size of the support of x, $0 < p < 1$, $a_1 > 1$, and $a_2 = a_1^{2/(2-p)}$, rounded up, so that $a_2 k$ is an integer ($a_2 = \lceil a_1^{2/(2-p)} k\rceil / k$). If A satisfies*

$$\delta_{a_2 k} + a_1 \delta_{(a_2+1)k} < a_1 - 1, \tag{5}$$

then x is the unique minimizer of problem (2).

Inspired by the previous analysis, it is natural to give the partially l_p-norm minimization problem for PSSR (1) as follows:

$$\min \|x\|_p^p, \quad \text{s.t. } A_1 x + A_2 y = b. \tag{6}$$

In order to establish the link between the PSSR (1) and its partially l_p-norm minimization problem, we need to give a partially p-RIC definition. Here we borrow the idea from Bandeira et al. [1]. Assume that A_2 is full column rank. For $A = [A_1, A_2]$ as mentioned above, let

$$B := I - A_2\left(A_2^T A_2\right)^{-1} A_2^T, \tag{7}$$

which is the matrix of the orthogonal projection from $\mathbb{R}^N$ to $\mathfrak{R}(A_2)^\perp$.

Definition 3 (Partially p-RIC). Let $A = [A_1, A_2] \in \mathbb{R}^{M\times N}$, where $A_1 \in \mathbb{R}^{M\times(N-r)}$, and $A_2 \in \mathbb{R}^{M\times r}$ is full column rank. We say that δ_{s-r}^r is the Partially Restricted Isometry Constant (Partially p-RIC) of order $s-r$ of the matrix A if δ_{s-r}^r is the p-RIC of order $s-r$ of the matrix BA_1; that is, δ_{s-r}^r is the smallest number, such that

$$\left(1-\delta_{s-r}^r\right)\|x\|_2^p \leq \left\|BA_1 x\right\|_p^p \leq \left(1+\delta_{s-r}^r\right)\|x\|_2^p, \tag{8}$$

for all $(s-r)$-sparse vectors x, where B is given by (7).

3. Main Results

We will give our main results which state sufficient p-RIP recovery conditions on the exact PSSR via the nonconvex l_p-norm minimization. We begin with the following useful lemma.

Lemma 4. *For $0 < p \leq 1$, let $c_1 = a_1^{1-p/2}$ and $c_2 = a_2^{1-p/2}$ with $a_1 > 0$ and $a_2 > 0$. If $(c_1-1)/c_2 > |a_1 - a_2|/a_2$, then $a_1 > 1$ and $a_2 > 1$.*

Proof. In order to prove the lemma, we consider the following two cases.

Case 1 ($a_1 \geq a_2$). In this case, from the fact $(c_1-1)/c_2 > |a_1 - a_2|/a_2$, we have

$$\frac{a_1}{a_2} - \left(\frac{a_1}{a_2}\right)^{1-p/2} < 1 - \left(\frac{1}{a_2}\right)^{1-p/2}. \tag{9}$$

If $0 < a_2 \le 1$, from the previous inequality we easily obtain

$$0 < \frac{a_1}{a_2} - \left(\frac{a_1}{a_2}\right)^{1-p/2} < 1 - \left(\frac{1}{a_2}\right)^{1-p/2} \le 0, \quad (10)$$

which is a contradiction. Hence $a_1 \ge a_2 > 1$.

Case 2 ($a_1 < a_2$). Similarly, in this case, from $(c_1 - 1)/c_2 > |a_1 - a_2|/a_2$, we obtain

$$0 < 1 - \frac{a_1}{a_2} < \frac{c_1 - 1}{c_2}. \quad (11)$$

If $0 < a_1 \le 1$, then

$$c_1 = a_1^{1-p/2} \le 1. \quad (12)$$

Combining the previous inequalities we obtain

$$0 < 1 - \frac{a_1}{a_2} < \frac{c_1 - 1}{c_2} \le 0, \quad (13)$$

which is a contradiction. Hence $1 < a_1 < a_2$.

Therefore, taking into account the previous two cases, we completed the proof. □

We below propose a general recovery condition for PSSR via its p-norm minimization.

Theorem 5. *Let $A = [A_1, A_2] \in \mathbb{R}^{M\times N}$ with $A_1 \in \mathbb{R}^{M\times(N-r)}$ and $A_2 \in \mathbb{R}^{M\times r}$. Suppose that A_2 is full column rank, and let $A_1 x + A_2 y = b$ with $\|x\|_0 = k$. For $0 < p \le 1$, $a_1 > 0$, and $a_2 > 0$, let $c_1 = a_1^{1-p/2}$, $c_2 = a_2^{1-p/2}$ with $(c_1-1)/c_2 > |a_1-a_2|/a_2$. If*

$$\begin{aligned} &a_2 c_1 \delta_{(a_2+1)k} + \left(|a_1 - a_2| c_2 + a_2\right)\delta_{a_1 k} \\ &\quad < a_2 c_1 - |a_1 - a_2| c_2 - a_2, \end{aligned} \quad (14)$$

then (x, y) is the unique minimizer of problem (6).

Proof. Note that (x, y) is a feasible solution to optimization problem (6). We remain to show that the solution set is a singleton $\{(x, y)\}$. This proof generally modifies that of [10], but under different assumptions. (Specifically, we use a different way to arrange the elements of T_0^C in the following.) Let (u, v) be an arbitrary solution to problem (6). we will show that $u = x$ and $y = v$. We will prove $u = x$ firstly. Taking $h = u - x$, we will show that $h = 0$. Let $\Phi = BA_1$. For $T \in \{1, \dots, N - r\}$, Φ_T denotes the matrix equaling Φ in those columns whose indices belong to T and otherwise zero. Similarly, we define the vector h_T. Let T_0 be the support of x. Then, the supports of x and $h_{T_0^C}$ are disjoint since $T_0 \cap T_0^C = \emptyset$. From direct calculation, we obtain

$$\begin{aligned} \|x\|_p^p \ge \|u\|_p^p &= \|x + h\|_p^p \\ &= \left\|x + h_{T_0} + h_{T_0^C}\right\|_p^p \\ &= \left\|x + h_{T_0}\right\|_p^p + \left\|h_{T_0^C}\right\|_p^p \\ &\ge \|x\|_p^p - \left\|h_{T_0}\right\|_p^p + \left\|h_{T_0^C}\right\|_p^p, \end{aligned} \quad (15)$$

where the first inequality holds because (u, v) solves (6), and the last one holds by the triangle inequality for $\|\cdot\|_p^p$. Then we have

$$\left\|h_{T_0^C}\right\|_p^p \le \left\|h_{T_0}\right\|_p^p. \quad (16)$$

Now we arrange the elements of T_0^C in order of decreasing magnitude of $|h|$ and partition into $T_0^C = T_1 \bigcup T_2 \bigcup \cdots \bigcup T_J$, where T_1 has $a_2 k$ elements and T_j ($j \ge 2$) each has $a_1 k$ elements (except possibly T_J). Set $T_{01} = T_0 \bigcup T_1$. Note that

$$BA_2 = \left[I - A_2\left(A_2^T A_2\right)^{-1} A_2^T\right] A_2 = 0. \quad (17)$$

Direct calculations yield

$$\begin{aligned} 0 &= \left\|B\left(A_1 x + A_2 y - A_1 u - A_2 v\right)\right\|_p^p \\ &= \left\|BA_1 x - BA_1 u\right\|_p^p = \|\Phi x - \Phi u\|_p^p \\ &= \|\Phi h\|_p^p = \left\|\Phi h_{T_{01}} + \sum_{j\ge 2} \Phi h_{T_j}\right\|_p^p \\ &\ge \left\|\Phi h_{T_{01}}\right\|_p^p - \left\|\sum_{j\ge 2} \Phi h_{T_j}\right\|_p^p \\ &\ge \left\|\Phi h_{T_{01}}\right\|_p^p - \sum_{j\ge 2} \left\|\Phi h_{T_j}\right\|_p^p \\ &\ge \left(1 - \delta_{(a_2+1)k}\right)\left\|h_{T_{01}}\right\|_2^p - \left(1 + \delta_{a_1 k}\right) \\ &\quad \times \sum_{j\ge 2} \left\|h_{T_j}\right\|_2^p. \end{aligned} \quad (18)$$

Now we discuss the relation between l_2-norm and l_p-norm. For each $t \in T_j$ and $s \in T_{j-1}$, it holds $|h_t| \le |h_s|$. So, we have for $j = 2$,

$$\begin{aligned} &\left|h_t\right|^p \le \frac{\left\|h_{T_1}\right\|_p^p}{a_2 k} \\ &\Longrightarrow \left|h_t\right|^2 \le \frac{\left\|h_{T_1}\right\|_p^2}{\left(a_2 k\right)^{2/p}} \\ &\Longrightarrow \frac{\left\|h_{T_2}\right\|_2^2}{a_1 k} \le \frac{\left\|h_{T_1}\right\|_p^2}{\left(a_2 k\right)^{2/p}} \\ &\Longrightarrow \left\|h_{T_2}\right\|_2^p \le \frac{a^{p/2}}{a_2 k^{1-p/2}} \left\|h_{T_1}\right\|_p^p. \end{aligned} \quad (19)$$

Similarly, we obtain that for $j \geq 3$,

$$\left\|h_{T_j}\right\|_2^p \leq \frac{1}{(a_1 k)^{1-p/2}} \left\|h_{T_{j-1}}\right\|_p^p. \tag{20}$$

Applying the Holder's inequality, we obtain

$$\begin{aligned}\left\|h_{T_0}\right\|_p^p &= \sum_{t\in T_0} |h_t|^p \cdot 1 \\ &\leq \left(\sum_{t\in T_0} |h_t|^2\right)^{p/2} \left(\sum_{t\in T_0} 1\right)^{1-p/2} \\ &= \left\|h_{T_0}\right\|_2^p \cdot k^{1-p/2}.\end{aligned} \tag{21}$$

Similarly, we have

$$\left\|h_{T_1}\right\|_p^p \leq \left\|h_{T_1}\right\|_2^p \cdot (a_2 k)^{1-p/2}. \tag{22}$$

Therefore,

$$\begin{aligned}&\sum_{j\geq 2} \left\|h_{T_j}\right\|_2^p \\ &\leq \frac{a_1^{p/2}}{a_2 k^{1-p/2}} \left\|h_{T_1}\right\|_p^p \\ &\quad + \frac{1}{(a_1 k)^{1-p/2}} \sum_{j\geq 2} \left\|h_{T_j}\right\|_p^p \quad (\text{By } (19) \text{ and } (20)) \\ &= \frac{1}{(a_1 k)^{1-p/2}} \sum_{j\geq 1} \left\|h_{T_j}\right\|_p^p + \frac{(a_1 - a_2) a_1^{p/2}}{a_1 a_2 k^{1-p/2}} \left\|h_{T_1}\right\|_p^p \\ &= \frac{1}{(a_1 k)^{1-p/2}} \left\|h_{T_0^C}\right\|_p^p + \frac{(a_1 - a_2) a_1^{p/2}}{a_1 a_2 k^{1-p/2}} \left\|h_{T_1}\right\|_p^p \\ &\leq \frac{1}{(a_1 k)^{1-p/2}} \left\|h_{T_0}\right\|_p^p + \frac{(a_1 - a_2) a_1^{p/2}}{a_1 a_2 k^{1-p/2}} \left\|h_{T_1}\right\|_p^p \quad (\text{By } (16)) \\ &\leq \frac{k^{1-p/2}}{(a_1 k)^{1-p/2}} \left\|h_{T_0}\right\|_2^p \\ &\quad + \frac{(a_1 - a_2)(a_2 k)^{1-p/2}}{a_2 (a_1 k)^{1-p/2}} \left\|h_{T_1}\right\|_2^p \quad (\text{By } (21) \text{ and } (22)) \\ &= \frac{1}{c_1} \left\|h_{T_0}\right\|_2^p + \frac{(a_1 - a_2) c_2}{a_2 c_1} \left\|h_{T_1}\right\|_2^p \\ &\leq \frac{1}{c_1} \left\|h_{T_0}\right\|_2^p + \frac{|a_1 - a_2| c_2}{a_2 c_1} \left\|h_{T_1}\right\|_2^p \\ &\leq \frac{1}{c_1} \left\|h_{T_{01}}\right\|_2^p + \frac{|a_1 - a_2| c_2}{a_2 c_1} \left\|h_{T_{01}}\right\|_2^p.\end{aligned} \tag{23}$$

Thus by (18) and (23), we have

$$\begin{aligned}0 &\geq \left(1 - \delta_{(a_2+1)k}\right) \left\|h_{T_{01}}\right\|_2^p \\ &\quad - \left(1 + \delta_{a_1 k}\right) \sum_{j\geq 2} \left\|h_{T_j}\right\|_2^p \quad (\text{By } (18)) \\ &\geq \left(1 - \delta_{(a_2+1)k}\right) \left\|h_{T_{01}}\right\|_2^p - \frac{1 + \delta_{a_1 k}}{c_1} \left\|h_{T_{01}}\right\|_2^p \\ &\quad - \frac{|a_1 - a_2| c_2 \left(1 + \delta_{a_1 k}\right)}{a_2 c_1} \left\|h_{T_{01}}\right\|_2^p \quad (\text{By } (23)) \\ &= \left[1 - \delta_{(a_2+1)k} - \frac{1 + \delta_{a_1 k}}{c_1} - \frac{|a_1 - a_2| c_2 \left(1 + \delta_{a_1 k}\right)}{a_2 c_1}\right] \\ &\quad \times \left\|h_{T_{01}}\right\|_2^p.\end{aligned} \tag{24}$$

Clearly, the assumption ensures that the scalar factor is positive, and hence we obtain $h_{T_{01}} = 0$. That means $h_{T_0} = 0$. Using $\|h_{T_0^C}\|_p^p \leq \|h_{T_0}\|_p^p$, we obtain $h_{T_0^C} = 0$. Therefore, $h = 0$, which means $x = u$.

Now we remain to show that $y = v$. It is obvious that $A_1 x + A_2 y = A_1 u + A_2 v$. Since $x = u$, we have $A_2(y - v) = 0$. Then $y = v$ because A_2 is full column rank. □

Theorem 5 states a different sufficient condition for the exactly PSSR via its nonconvex relaxation from the existing conditions for SSR.

Theorem 6. *Let $A = [A_1, A_2] \in \mathbb{R}^{M\times N}$ with $A_1 \in \mathbb{R}^{M\times(N-r)}$ and $A_2 \in \mathbb{R}^{M\times r}$. Suppose that A_2 is full column rank, and let $A_1 x + A_2 y = b$ with $\|x\|_0 = k$. For $0 < p \leq 1$ and $a > 1$, if*

$$\delta_{(a+1)k} < \frac{a^{1-p/2} - 1}{a^{1-p/2} + 1}, \tag{25}$$

then (x, y) is the unique minimizer of problem (6). Specifically, for all $0 < p \leq 1$, if

$$\delta_{(a+1)k} < \frac{\sqrt{a} - 1}{\sqrt{a} + 1}, \tag{26}$$

then (x, y) is the unique minimizer of problem (6).

Proof. Applying Theorem 5, here we only need to show that if (25) holds, we can find a_1 and a_2, such that (14) holds. We consider the three cases in the following.

Case i ($a_1 \geq a_2 + 1$). In this case, we easily obtain $a_1 - a_2 \geq 1$ and $\delta_{a_1 k} \geq \delta_{(a_2+1)k}$. Therefore the following condition can guarantee the inequality (14):

$$a_2 c_1 \delta_{a_1 k} + \left[(a_1 - a_2) c_2 + a_2\right] \delta_{a_1 k} < a_2 \left(c_1 + c_2 - 1\right) - a_1 c_2. \tag{27}$$

Simplifying the previous inequality, we obtain

$$\delta_{a_1 k} < \frac{a_2 c_1 - (a_1 - a_2) c_2 - a_2}{a_2 c_1 + (a_1 - a_2) c_2 + a_2} = \frac{a_1^{1-p/2} - (a_1 - a_2) a_2^{-p/2} - 1}{a_1^{1-p/2} + (a_1 - a_2) a_2^{-p/2} + 1}. \quad (28)$$

In this case, employing $a_2^{-p/2} > 0$, we easily get that $a_1 = a_2 + 1$ gives the maximum value of the right of the inequality (the strongest result) which satisfies the condition (14). That is,

$$\delta_{(a_2+1)k} < \frac{(a_2+1)^{1-p/2} - a_2^{-p/2} - 1}{(a_2+1)^{1-p/2} + a_2^{-p/2} + 1}. \quad (29)$$

Case ii ($a_2 \le a_1 < a_2 + 1$). In this case, we can get that $0 \le a_1 - a_2 < 1$ and $\delta_{a_1 k} < \delta_{(a_2+1)k}$. Similarly, the following condition can guarantee the inequality (14):

$$a_2 c_1 \delta_{(a_2+1)k} + [(a_1 - a_2) c_2 + a_2] \delta_{(a_2+1)k} < a_2 (c_1 + c_2 - 1) - a_1 c_2. \quad (30)$$

Simplifying the previous inequality, we obtain

$$\delta_{(a_2+1)k} < \frac{a_1^{1-p/2} - (a_1 - a_2) a_2^{-p/2} - 1}{a_1^{1-p/2} + (a_1 - a_2) a_2^{-p/2} + 1}. \quad (31)$$

In this case, employing $a_2^{-p/2} > 0$, we get that $a_1 = a_2$ give the maximum value of the right of the inequality; that is,

$$\delta_{(a_2+1)k} < \frac{a_2^{1-p/2} - 1}{a_2^{1-p/2} + 1}. \quad (32)$$

Case iii ($a_1 < a_2$). In this case, it is clear that $0 < a_2 - a_1$, $a_1 < a_2 + 1$, and $\delta_{a_1 k} < \delta_{(a_2+1)k}$. So the following condition can guarantee the inequality (14):

$$a_2 c_1 \delta_{(a_2+1)k} + [(a_2 - a_1) c_2 + a_2] \delta_{(a_2+1)k} < a_2 (c_1 - c_2 - 1) + a_1 c_2. \quad (33)$$

Simplifying the previous inequality, we obtain

$$\delta_{(a_2+1)k} < \frac{a_1^{1-p/2} - (a_2 - a_1) a_2^{-p/2} - 1}{a_1^{1-p/2} + (a_2 - a_1) a_2^{-p/2} + 1}. \quad (34)$$

In this case, employing $a_2^{-p/2} > 0$, we chose $a_2 - a_1 \to 0$ to give the maximum value of the right of the inequality. That is,

$$\delta_{(a_2+1)k} < \frac{a_2^{1-p/2} - 1}{a_2^{1-p/2} + 1}. \quad (35)$$

It is easy to see that $((a_2+1)^{1-p/2} - a_2^{-p/2} - 1)/((a_2+1)^{1-p/2} + a_2^{-p/2} + 1) < (a_2^{1-p/2} - 1)/(a_1^{1-p/2} + 1)$. In fact, $(a_2+1)^{1-p/2} + a_2^{-p/2} + 1 > a_2^{1-p/2} + 1$. On the other hand, $(a_2+1)^{1-p/2} - a_2^{-p/2} - 1 - (a_2^{1-p/2} - 1) = (a_2+1)[(a_2+1)^{-p/2} - a_2^{-p/2}] < 0$, which means $(a_2+1)^{1-p/2} - a_2^{-p/2} - 1 < a_2^{1-p/2} - 1$.

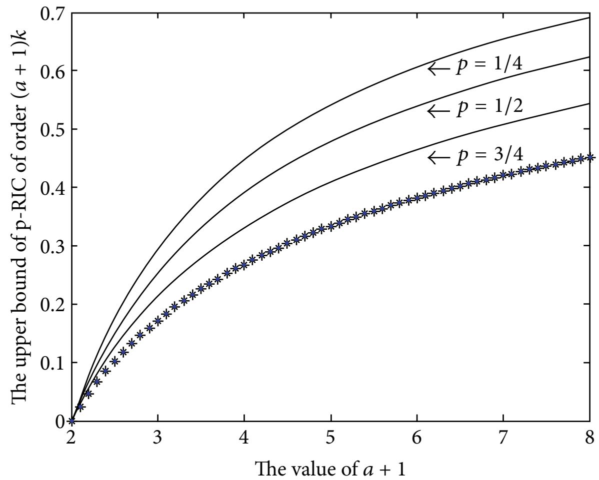

FIGURE 1: The upper bound of p-RIC of order $(a+1)k$ with particular values of p.

Therefore, combining the previous three cases, we obtain that one can choose $a_1 = a_2$ to get the weakest sufficient condition. It is easy to see that $a_1 = a_2$ satisfying the assumptions of Theorem 5.

After the previous discussion, using condition (25) and choosing $a_1 = a_2 = a$, we can derive condition (14).

Specifically, we consider the following function:

$$f(p) = \frac{a^{1-p/2} - 1}{a^{1-p/2} + 1} = 1 - \frac{2}{a^{1-p/2} + 1}. \quad (36)$$

Clearly,

$$f'(p) = -\frac{a^{1-p/2} \ln a}{(a^{1-p/2} + 1)^2} < 0, \quad (37)$$

and hence $f(p)$ is a decreasing function of p. Thus, for all $0 < p \le 1$, condition

$$\delta_{(a+1)k} < \frac{\sqrt{a} - 1}{\sqrt{a} + 1} \quad (38)$$

can guarantee condition (14).

The proof is completed. □

Applying Theorem 6, we understand how the p-RIP bounds related to p as in Figure 1. From Figure 1, it is easy to give a stronger result (i.e., weaker sufficient condition) for smaller p. Moreover, by taking different values of p with $p = 1/4, 1/2, 3/4$, we obtain some interesting p-RIP bounds as in Table 1.

4. Final Remark

In this paper, we studied the restricted p-isometry property to the partially sparse signal recovery problem and proposed a sufficient p-RIP condition for exactly recovering partially sparse signal via the partially l_p-minimization problem with $p \in (0, 1]$. It is worth generalizing the p-RIP condition

Table 1: Bounds comparison on different values of p.

$p=\frac{1}{4}$	0.1756	0.2943	0.3807	0.4468	0.4991	0.5417
$p=\frac{1}{2}$	0.1509	0.2542	0.3307	0.3902	0.4380	0.4776
$p=\frac{3}{4}$	0.1260	0.2133	0.2788	0.3304	0.3726	0.4080
$p=1$	0.1010	0.1716	0.2251	0.2679	0.3033	0.3333

from the vector case to the matrix case. Note that the well-known low-rank matrix recovery (LMR) problem has many applications and appeared in the literature of a diverse set of fields including matrix completion, quantum state tomography, face recognition, magnetic resonance imaging (MRI), computer vision, and system identification and control; see, for example, [21, 22] for more details and the reference therein. In particular, Oymak et al. [21] showed that several sufficient RIP recovery conditions for k sparse vector are also sufficient for recovery of matrices of rank up to $2k$ via Schatten p-norm minimization. According to our approach to extend the p-RIP bound from SSP to partially SSR, we can obtain some different restricted p-isometry properties for LMR problem by using the idea in [21].

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References

[1] A. Bandeira, K. Scheinberg, and L. N. Vicente, "On partially sparse recovery," Department of Mathematics, University of Coimbra, 2011.

[2] A. S. Bandeira, K. Scheinberg, and L. N. Vicente, "Computation of sparse low degree interpolating polynomials and their application to derivative-free optimization," *Mathematical Programming*, vol. 134, no. 1, pp. 223–257, 2012.

[3] N. Vaswani and W. Lu, "Modified-CS: modifying compressive sensing for problems with partially known support," *IEEE Transactions on Signal Processing*, vol. 58, no. 9, pp. 4595–4607, 2010.

[4] L. Jacques, "A short note on compressed sensing with partially known signal support," *Signal Processing*, vol. 90, no. 12, pp. 3308–3312, 2010.

[5] D. L. Donoho, "Compressed sensing," *IEEE Transactions on Information Theory*, vol. 52, no. 4, pp. 1289–1306, 2006.

[6] A. M. Bruckstein, D. L. Donoho, and M. Elad, "From sparse solutions of systems of equations to sparse modeling of signals and images," *SIAM Review*, vol. 51, no. 1, pp. 34–81, 2009.

[7] E. J. Candés, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on Information Theory*, vol. 52, no. 2, pp. 489–509, 2006.

[8] E. J. Candés and T. Tao, "Decoding by linear programming," *IEEE Transactions on Information Theory*, vol. 51, no. 12, pp. 4203–4215, 2005.

[9] R. Chartrand, "Fast algorithms for nonconvex compressive sensing: MRI reconstruction from very few data," in *Proceedings of the IEEE International Symposium on Biomedical Imaging (ISBI '09)*, pp. 262–265, 2009.

[10] R. Chartrand and V. Staneva, "Restricted isometry properties and nonconvex compressive sensing," *Inverse Problems*, vol. 24, no. 3, pp. 1–14, 2008.

[11] X. Chen, F. Xu, and Y. Ye, "Lower bound theory of nonzero entries in solutions of ℓ_2-ℓ_p minimization," *SIAM Journal on Scientific Computing*, vol. 32, no. 5, pp. 2832–2852, 2010.

[12] X. Chen and W. Zhou, "Convergence of reweighted ℓ_1 minimization algorithms and unique solution of truncated lp minimization," Tech. Rep., 2010.

[13] M. E. Davies and R. Gribonval, "Restricted isometry constants where ℓ_p sparse recovery can fail for $0 < p \leq 1$," *IEEE Transactions on Information Theory*, vol. 55, no. 5, pp. 2203–2214, 2009.

[14] S. Foucart and M.-J. Lai, "Sparsest solutions of underdetermined linear systems via ℓ_q-minimization for $0 < p \leq 1$," *Applied and Computational Harmonic Analysis*, vol. 26, no. 3, pp. 395–407, 2009.

[15] D. Ge, X. Jiang, and Y. Ye, "A note on the complexity of ℓ_p minimization," *Mathematical Programming*, vol. 129, no. 2, pp. 285–299, 2011.

[16] R. Saab, R. Chartrand, and O. Yilmaz, "Stable sparse approximations via nonconvex optimization," in *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing*, pp. 3885–3888, 2008.

[17] Y. Shen and S. Li, "Restricted p-isometry property and its application for nonconvex compressive sensing," *Advances in Computational Mathematics*, vol. 37, no. 3, pp. 441–452, 2012.

[18] R. Saab and O. Yılmaz, "Sparse recovery by non-convex optimization–instance optimality," *Applied and Computational Harmonic Analysis*, vol. 29, no. 1, pp. 30–48, 2010.

[19] M. Wang, W. Xu, and A. Tang, "On the performance of sparse recovery via ℓ_p-minimization $(0 \leq p \leq 1)$," *IEEE Transactions on Information Theory*, vol. 57, no. 11, pp. 7255–7278, 2011.

[20] R. Chartrand, "Exact reconstructions of sparse signals via nonconvex minimization," *IEEE Signal Processing Letters*, vol. 14, no. 10, pp. 707–710, 2007.

[21] S. Oymak, K. Mohan, M. Fazel, and B. Hassibi, "A simplified approach to recovery conditions for low rank matrices," in *Proceedings of the IEEE International Symposium on Information Theory Proceedings (ISIT '11)*, pp. 2318–2322, 2011.

[22] B. Recht, M. Fazel, and P. A. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization," *SIAM Review*, vol. 52, no. 3, pp. 471–501, 2010.