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Research Article

Perturbation of Stochastic Boussinesq Equations with Multiplicative White Noise

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This paper studies the Boussinesq equations perturbed by multiplicative white noise and shows the existence and uniqueness of the global solution. It also gets some regularity results for the unique solution.

1. Introduction

The Boussinesq equation is a mathematics model of thermohydraulics, which consists of equations of fluid and temperature in the Boussinesq approximation. The deterministic case has been studied systematically by many authors (e.g., see [1–3]). However, in many practical circumstances, small irregularity has to be taken into account. Thus, it is necessary to add to the equation a random force, which is in general a space-time white noise, as considered recently by many authors for other equations (see [4–11]). The random attractors of boussinesq equations with multiplicative noise have been investigated by [12]. In this paper, We will study the perturbation of stochastic boussinesq equations with multiplicative white noise.

We will consider the following stochastic two-dimension a Boussinesq equations perturbed by a multiplicative white noise of Stratonovich form:

$$\begin{gathered} dv + \left[(v \cdot \nabla) v - \nu \Delta v + \nabla p \right] dt \\ = e_2 \left(T - T_1 \right) dt + \sigma v \circ dw \left(t \right), \\ dT + \left[(v \cdot \nabla) T - \lambda \Delta T \right] dt = 0, \\ \operatorname{div} v = 0. \end{gathered} \tag{1}$$

The domain occupied by the fluid is $D = (0,1) \times (0,1)$, and e_1, e_2 is the canonical basis of $\mathbb{R}^2$. The unknown $v = (v_1, v_2)$, T, and p stand for the velocity vector, temperature, and pressure, respectively. T_1 is the temperature at the top, $x_2 = 1$, while $T_0 = T_1 + 1$ is the temperature at the boundary below, $x_2 = 0$. The constant numbers $\lambda > 0$, $\beta > 0$, and $\sigma > 0$ are related to the usual Prandtl, Grashof, and Rayleigh numbers.

$W(t)$ is two-sided Wiener processes on the probability space $(\Omega, \mathbb{F}, P)$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathbb{F}$ is the Borel sigma-algebra induced by the compact-open topology of Ω, and P is a Wiener measure.

We supplement (1) with the following boundary condition:

$$\begin{gathered} v = 0 \quad \text{at } x_2 = 0,\ x_2 = 1, \\ T = T_0 \quad \text{at } x_2 = 0, \qquad T = T_1 = T_0 - 1 \quad \text{at } x_2 = 1, \\ \psi\big|_{x_1=0} = \psi\big|_{x_1=1} \quad \text{for } \psi = v,\ T,\ p,\ \frac{\partial v}{\partial x_1},\ \frac{\partial T}{\partial x_1}. \end{gathered} \tag{2}$$

When an initial-valued problem is considered, we supplement these equations with

$$v(x,0) = v_0(x), \quad T(x,0) = T_0(x) \quad \text{for } x \in D. \tag{3}$$

The existence of a compact random attractor and its Hausdorff, fractal dimension estimates have been investigated by [12]. We will solve pathwise (1)–(3). By using the Faedo-Galerkin approximation and a priori estimates, we prove the existence and uniqueness of the global solution and show that the solution continuously depends on the initial value. We also get some regularity results of the solutions.

2. Mathematical Setting and Basic Estimates

Let

$$\eta := T - T_0 + x_2, \tag{4}$$

and change p to $p - x_2 + x_2^2/2$; then (1) can be rewritten as

$$\begin{gathered} dv + \left[(v\cdot\nabla)v - \beta\Delta v + \nabla p\right]dt = e_2\eta dt + \sigma v \circ dW, \\ d\eta + \left[(v\cdot\nabla)\eta\right]dt = v_2 dt, \\ \operatorname{div} v = 0. \end{gathered} \tag{5}$$

Let the process be

$$\alpha(t) := e^{-\sigma W(t)}. \tag{6}$$

Then $d\alpha = -\sigma\alpha \circ dW$, and if we let

$$\xi := \alpha v, \tag{7}$$

we get the new equations (no stochastic differential appears here)

$$\frac{d\xi}{dt} + \alpha^{-1}(\xi\cdot\nabla)\xi - \beta\Delta\xi + \alpha\nabla p = \alpha e_2\eta, \tag{8}$$

$$\frac{d\eta}{dt} + \alpha^{-1}(\xi\cdot\nabla)\eta - \lambda\Delta\eta = \alpha^{-1}\xi_2, \tag{9}$$

$$\operatorname{div}\xi = 0, \tag{10}$$

with the boundary conditions

$$\begin{gathered} \xi = 0 \quad \text{at } x_2 = 0,\ x_2 = 1, \\ \eta = 0 \quad \text{at } x_2 = 0,\ x_2 = 1, \\ \psi|_{x_1=0} = \psi|_{x_1=1} \quad \text{for } \psi = \xi, \eta, p, \frac{\partial\xi}{\partial x_1}, \frac{\partial\eta}{\partial x_1} \end{gathered} \tag{11}$$

and the initial value conditions

$$\xi(0) = \xi_0, \qquad \eta(0) = \eta_0. \tag{12}$$

To solve (8)–(12), we consider the Hilbert space $H = H_1 \times H_2$ with the scalar products $(\cdot,\cdot)$ and norms $|\cdot|$, where $H_2 = L^2(D)$ and

$$H_1 = \left\{\xi \in L^2(D) : \operatorname{div}\xi = 0,\ \xi_i|_{x_i=0} = \xi_i|_{x_i=1},\ i = 1,2\right\}. \tag{13}$$

We also consider the subspace $V = V_1 \times V_2$ of H, where V_2 is the space of functions in $H^1(D)$ vanishing at $x_2 = 0$ and $x_2 = 1$ and periodic in the direction of x_1. V_2 is a Hilbert space for the scalar product and the norm

$$((\eta_1,\eta_2)) = \int_D \operatorname{grad}\eta_1 \operatorname{grad}\eta_2\, dx, \quad \|\eta\| = ((\eta,\eta))^{1/2}, \tag{14}$$

and $V_1 = \{\xi \in V_2^2 : \operatorname{div}\xi = 0\}$. We also denote by $((\cdot,\cdot))$ and $\|\cdot\|$ the canonical scalar product and norm in V_1 and V.

The bilinear form

$$\begin{gathered} \mu(u_1,u_2) = \beta((\xi_1,\xi_2)) + \lambda((\eta_1,\eta_2)), \\ \forall\{\xi_i,\eta_i\} \in V,\ i = 1,2, \end{gathered} \tag{15}$$

determines a linear isomorphism A from $D(A)$ into H and from V into the dual space V', defined by

$$(Au_1,u_2) = \mu(u_1,u_2), \quad \forall u_i = \{\xi_i,\eta_i\} \in V,\ i = 1,2, \tag{16}$$

with $D(A) = D(A_1) \times D(A_2)$, where

$$\begin{gathered} D(A_1) = \left\{\xi \in V_1 \cap H^2(D)^2 : \left.\frac{\partial\xi}{\partial x_1}\right|_{x_1=0} = \left.\frac{\partial\xi}{\partial x_1}\right|_{x_1=1}\right\} \\ D(A_2) = \left\{\eta \in V_2 \cap H^2(D) : \left.\frac{\partial\eta}{\partial x_1}\right|_{x_1=0} = \left.\frac{\partial\eta}{\partial x_1}\right|_{x_1=1}\right\}. \end{gathered} \tag{17}$$

Four spaces $D(A)$, V, H, and V' satisfy

$$D(A) \subset V \subset H \subset V' \tag{18}$$

and all embedding injections are densely continuous. It is well known that $A : D(A) \to H$ is self-adjoint and positive and A^{-1} is a compact self-adjoint in H.

We also consider the trilinear forms γ on V defined by

$$\begin{gathered} \gamma(u_1,u_2,u_3) = ((\xi_1\cdot\nabla)\xi_2,\xi_3) + ((\xi_1\cdot\nabla)\eta_2,\eta_3), \\ \forall u_i = \{\xi_i,\eta_i\} \in V,\ i = 1,2,3. \end{gathered} \tag{19}$$

The trilinear form γ is continuous on V or even on $H^1(D)^2 \times H^1(D)$. We associate with the form γ the bilinear continuous operator B which map $V \times V$ into V' and $D(A) \times D(A)$ into H, defined by

$$\begin{gathered} (B(u_1,u_2),u_3) = \gamma(u_1,u_2,u_3), \\ \forall u_i = \{\xi_i,\eta_i\} \in V,\ i = 1,2,3. \end{gathered} \tag{20}$$

Finally, we define the continuous operators $R(t)$ in H

$$R(t) : u = \{\xi,\eta\} \longrightarrow Ru = \left\{\alpha(t)e_2\eta, \alpha^{-1}(t)\xi_2\right\}. \tag{21}$$

Now, we can set (8) in the operator form. If $u = \{\xi,\eta\}$ is the solution of (8) and $\psi = \{f,g\}$ is a test function in V, we multiply (8) by f and (9) by g, integrate over D, and add the resulting equation. The pressure term disappears and after simplification we find

$$\begin{gathered} \frac{d}{dt}(u,\psi) + \alpha^{-1}(u,u,\psi) + \mu(u,\psi) + (R(t)u,\psi) = 0, \\ \forall\psi \in V, \end{gathered} \tag{22}$$

which can be reinterpreted as

$$\frac{du}{dt} + Au + \alpha^{-1}(t)B(u,u) + R(t)u = 0. \tag{23}$$

Note that this equation differs from the determined case, and in determined case, the family $R(t)$ of operator is independent of the time t. Initial condition (12) can be reinterpreted as

$$u(0) = u_0 := \{\xi_0, \eta_0\}. \tag{24}$$

To solve (23)-(24), we also need some Sobolev norm estimates on the bilinear B and the operators R and A.

Lemma 1. *The bilinear operators* $B : V \times V \to V'$ *and* $D(A) \times D(A) \to H$ *are continuous and satisfy*

(i) $(B(u,v),v) = 0$, *for all* $u, v \in V$,

(ii) $|(B(u,v),w)| \le c_1 |u|^{\theta_1} \|u\|^{1-\theta_1} \|v\| \|w\|^{\theta_1} |w|^{1-\theta_1}$, *for all* $u, v, w \in V$,

(iii) $|B(u,v)| + |B(v,u)| \le c_2 \|u\| \|v\|^{1-\theta_2} |Av|^{\theta_2}$, *for all* $u \in V$, $v \in D(A)$,

(iv) $|B(u,v)| \le c_3 |u|^{\theta_3} \|u\|^{1-\theta_3} |Av|^{\theta_3} \|v\|^{1-\theta_3}$, *for all* $u \in V$, $v \in D(A)$,

where c_1, c_2, c_3 *are appropriate constants and* $\theta_i \in [0,1)$, $i = 1, 2, 3$.

Proof. The proof is the same as the deterministic case (see [10]). □

Lemma 2. *The linear continuous operators* $R : V \to V'$ *and* $D(A) \to H$ *satisfy*

$$|R(t)u| \le c_4 (\alpha(t)) \ + \alpha^{-1}(t) \|u\|, \quad \forall u \in V, \ \forall t \ge 0, \tag{25}$$

$$|(R(t)u, u)| \le c_4 \left(\alpha(t) + \alpha^{-1}(t)\right) \|u\| \, |u|, \quad \forall u \in V, \ \forall t \ge 0. \tag{26}$$

Proof. By (21), we have

$$\begin{aligned} |Ru| &= \left|\alpha e_2 \eta\right| + \left|\alpha^{-1} \xi_2\right| \\ &\le \alpha |\eta| + \alpha^{-1} |\xi| \le \left(\alpha + \alpha^{-1}\right)(|\xi| + |\eta|) \end{aligned} \tag{27}$$

which implies by the Poincare inequality

$$|u| \le c_4 \|u\|, \quad \text{for } u \in V \tag{28}$$

that (25) holds true. Since $|(Ru,u)| \le |Ru||u|$, it follows from (25) that (26) holds true. □

Lemma 3. *The bilinear form* μ *on* $V \times V$ *satisfies*

$$c_5 \|u\|^2 \le \mu(u,u) \le c_6 \|u\|^2, \quad \textit{for } u \in V. \tag{29}$$

Proof. By (15), we have

$$\begin{aligned} \mu(u,u) &= \beta \|\xi\|^2 + \lambda \|\eta\|^2 \le (\beta + \lambda)(\|\xi\| + \|\eta\|)^2 \\ &= (\beta + \lambda)\|u\|^2, \\ \mu(u,u) &\ge \min\{\beta, \lambda\}\left(\|\xi\|^2 + \|\eta\|^2\right) \\ &\ge \frac{1}{2} \min\{\beta, \lambda\}\left(\|\xi\|^2 + \|\eta\|^2\right), \end{aligned} \tag{30}$$

which imply (28). □

3. Existence and Uniqueness

In this section, we will prove the existence and uniqueness of the global solution of (23)-(24), equivalently (8)–(12) or (1)–(3). We are working almost surely for $\omega \in \Omega$.

Theorem 4. *Assume that* $u_0 \in H$, *then there exists a unique solution of* (23)-(24), *such that*

$$u \in C([0,\infty), H) \cap L^2_{\mathrm{loc}}(0,\infty; V), \tag{31}$$

and the mapping $u_0 \mapsto u(t)$ *is continuous from* H *into* $D(A)$, *for all* $t > 0$.

Proof. Since $A^{-1} : H \to D(A)$ is a self-adjoint compact operator in H, it follows from a classical spectral theorem that there exists a sequence $\lambda_j : 0 < \lambda_1 \le \lambda_2 \le \cdots, \lambda_j \to \infty$ and a family of elements $w_j \in D(A)$ which is completely orthogonal in H such that

$$A w_j = \lambda_j w_j, \quad \forall j. \tag{32}$$

For each m, we look for an approximate solution u_m of the following form:

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i \tag{33}$$

satisfying

$$\begin{aligned} &\frac{d}{dt}\left(u_m, w_j\right) + \mu\left(u_m, w_j\right) \\ &\quad + \alpha^{-1}\gamma\left(u_m, u_m, w_j\right) + \left(R u_m, w_j\right) = 0, \quad j = 1, 2, \ldots, m \end{aligned} \tag{34}$$

and initial condition

$$u_m(0) = P_m u_0, \tag{35}$$

where P_m is the projector in H (or V) on the space spanned by $w_1, w_2, \ldots, w_m$. Since A and P_m commute, the above equation is also equivalent to

$$\frac{du_m}{dt} + A u_m + \alpha^{-1} P_m B(u_m, u_m) + P_m(R u_m) = 0, \tag{36}$$

where

$$P_m(R u_m) = \sum_{j=1}^{m} g_{jm}(t) P_m R w_j, \tag{37}$$

in view of the linearity of P_m, R.

The existence of u_m on any finite interval $[0, T_m)$ follows from standard results of the existence of solutions of ordinary differential equations that $T_m = +\infty$ is a consequence of these results and of the following priori estimates:

$$\begin{aligned} &u_m \text{ remains bounded in } L^\infty(0,T;H) \cap L^2(0,T;V), \\ &\qquad \forall T > 0. \end{aligned} \tag{38}$$

Indeed, multiplying (34) by g_{jm}, summing these relations for $j = 1, 2, \ldots, m$, and noting that $\alpha^{-1}\mu(u_m, u_m, u_m) = 0$ (by Lemma 1), we find

$$\frac{1}{2}\frac{d}{dt}|u_m|^2 + \mu(u_m, u_m) + (Ru_m, u_m) = 0, \tag{39}$$

which implies by Lemma 2, (29), and the Young inequality that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u_m|^2 &+ c_5\|u_m\|^2 \\ &\leq |(Ru_m, u_m)| \\ &\leq c_4 \sup_{0\leq t\leq T}\left(\alpha(t) + \alpha^{-1}(t)\right)\|u_m\| \cdot |u_m| \\ &\leq \frac{c_5}{2}\|u_m\|^2 + c_4'|u_m|^2, \end{aligned} \tag{40}$$

that is,

$$\frac{d}{dt}|u_m|^2 + c_5\|u_m\|^2 \leq c'|u_m|^2, \tag{41}$$

where c_5 is defined in (29) and c' is a appropriate constant. Using the classical Gronwall lemma we find

$$|u_m(t)|^2 \leq |u_m(0)|^2 e^{c'T} \leq |u_0|^2 e^{c'T} = M_1, \quad \forall 0 < t \leq T. \tag{42}$$

Integrating (41) for t from 0 to T and using above estimates we have

$$\begin{aligned} &\int_0^T \|u_m\|^2 dt \\ &\quad\leq \frac{1}{c_5}\left(|u_m(0)|^2 - |u_m(T)|^2\right) + \frac{1}{c_5}\int_0^T |u_m(t)|^2 dt \\ &\quad\leq M_2, \end{aligned} \tag{43}$$

where M_1 and M_2 are independent of m. Thus, we have proved (38).

We also claim that

$$\frac{du_m}{dt} \text{ remains bounded in } L^2\left(0, T; V'\right). \tag{44}$$

Indeed, it follows from Lemma 1 that $|B(u_m, u_m)|V' \leq c|u_m|\|u_m\|$ with appropriate constant c, which, together with (38), implies that $B(u_m, u_m)$ and thus $P_m B(u_m, u_m)$ remain bounded in $L^2(0, T; V')$. Since both operators $R : V \to V'$ and $A : V \to V'$ are continuous (Lemmas 2 and 3), it follows from (38) that Au_m, Ru_m and thus $P_m Ru_m$ remain bounded in $L^2(0, T; V')$. Therefore, by (36),

$$\frac{du_m}{dt} = -Au_m - \alpha^{-1}P_m B(u_m, u_m) - P_m(Ru_m) \tag{45}$$

remains bounded in $L^2(0, T; V')$, which proved (44).

By weak compactness, it follows from (38) and (44) that there exists a $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$, for all $T > 0$ subsequence still denoted by u_m, such that

$$\begin{aligned} u_m &\longrightarrow u \quad \text{in } L^2(0, T; V) \text{ weakly}, \\ u_m &\longrightarrow u \quad \text{in } L^\infty(0, T; H) \text{ weakly star}, \\ \frac{du_m}{dt} &\longrightarrow \frac{du}{dt} \quad \text{in } L^2\left(0, T; V'\right) \text{ weakly}. \end{aligned} \tag{46}$$

We pass to the limit in (34) and find that

$$\frac{d}{dt}(u, \varphi) + \mu(u, \varphi) + \alpha^{-1}\gamma(u, u, \varphi) + (Ru, \varphi), \quad \forall \varphi \in V, \tag{47}$$

which implies that u satisfies (23). In particular, $u' = (du/dt) \in L^2(0, T; V') \in L^1(0, T; V')$. This implies by [10, Lemma II.3.1] that u is almost everywhere equal to a continuous function from $[0, T]$ into V'. Therefore initial condition (24) follows by a passage to the limit in (35). $u \in C([0, T], H)$ follows from [10, Lemma II.3.2] and the facts that $H \subset V \subset V'$ and $u' \in L^2(0, T; V')$. Furthermore, if we show that uniqueness, then the fact that $u \in C([0, T], H)$, for all $T > 0$, implies that $u \in C([0, \infty), H)$.

To prove the uniqueness and continuous dependence of $u(t)$ on u_0 (in H), we let u be a solution of (23)-(24) such that $u \in C([0, \infty), H) \cap L^2_{\text{loc}}(0, \infty; V)$. Similar to (39), u must satisfy the energy equality

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \mu(u, u) + (Ru, u) = 0. \tag{48}$$

By using Lemmas 1–3 and Gronwall lemma, we get the following similar estimates:

$$|u(t)|^2 \leq |u(0)|^2 e^{ct}, \tag{49}$$

which has proved the continuous dependence. For the uniqueness, we let u_1, u_2 be two solutions of (23)-(24) and $u = u_1 - u_2$. Then u is also a solution with $u(0) = 0$. Thus, (49) implies that $|u(t)|^2 \leq 0$, that is, $u_1 = u_2$. ☐

4. Regularity Results

In this section, we will consider further regularity results for the unique solution. The main result is that $u \in D(A)$, and thus $u \in H^2(D)^2 \times H^2(D)$ provided the initial function $u_0 \in V$. More precisely, we have the following.

Theorem 5. *Assume that $u_0 \in V$, and let u be the unique solution of* (23)-(24). *Then,*

$$u \in C([0, \infty), V) \cap L^2_{\text{loc}}(0, \infty; D(A)). \tag{50}$$

Proof. Let u_m be the approximate solution (33) in the proof of Theorem 4. We first claim that

$$u_m \text{ remains bounded in } L^\infty(0, T; V) \cap L^2(0, T; D(A)), \quad \forall T > 0. \tag{51}$$

Indeed, multiplying (34) by $\lambda_j g_{jm}$, summing these relations for $j = 1, 2, \dots, m$, and using (32), we find

$$\begin{aligned}&\left(\frac{du_m}{dt}, Au_m\right) + \mu\left(u_m, Au_m\right) + \alpha^{-1}\gamma\left(u_m, u_m, Au_m\right)\\ &\quad + \left(Ru_m, Au_m\right) = 0.\end{aligned} \tag{52}$$

By Lemma 1(iv) and the Young inequality, we find

$$\begin{aligned}&\left|\alpha^{-1}\gamma\left(u_m, u_m, Au_m\right)\right|\\ &\quad = \left|\alpha^{-1}\left(B\left(u_m, u_m\right), Au_m\right)\right|\\ &\quad \leq \sup_{0\leq s\leq T} \alpha^{-1}(s)\cdot c_3\left|u_m\right|^{\theta_3}\left\|u_m\right\|^{2(1-\theta_3)}\left|Au_m\right|^{(1+\theta_3)}\\ &\quad \leq \frac{1}{4}\left|Au_m\right|^2 + c_3'(T)\left\|u_m\right\|^4|u|^{2\theta_3/(1-\theta_3)}.\end{aligned} \tag{53}$$

For $0 \leq t \leq T$, by Lemma 2, (25), and the Young inequality, we have

$$\begin{aligned}&\left|\left(Ru_m, Au_m\right)\right|\\ &\quad \leq \sup_{0\leq s\leq T}\left(\alpha(s) + \alpha^{-1}(s)\right)c_4\left\|u_m\right\|\left|Au_m\right|\\ &\quad \leq \frac{1}{4}\left|Au_m\right|^2 + c_4'\left\|u_m\right\|^2.\end{aligned} \tag{54}$$

Noting also that

$$\begin{aligned}\left(\frac{du_m}{dt}, Au_m\right) &= \left(Au_m, u_m'\right)\\ &= \mu\left(u_m, u_m'\right) = \frac{1}{2}\frac{d}{dt}\mu\left(u_m, u_m\right)\end{aligned} \tag{55}$$

and $\mu(u_m, Au_m) = |Au_m|^2$, we find from (52) and all the above estimates that

$$\begin{aligned}&\frac{d}{dt}\mu\left(u_m, u_m\right) + \left|Au_m\right|^2\\ &\quad \leq 2c_4'\left\|u_m\right\|^2 + 2c_3'\left\|u_m\right\|^4\left|u_m\right|^{2\theta_3/(1-\theta_3)}.\end{aligned} \tag{56}$$

By (38), u_m is bounded in $L^\infty(0, T; H)$. This, together with Lemma 3, implies that (56) can be rewritten as

$$\begin{aligned}&\frac{d}{dt}\mu\left(u_m, u_m\right) + \left|Au_m\right|^2\\ &\quad \leq 2c\left(1 + \left\|u_m\right\|^2\right)\mu\left(u_m, u_m\right), \quad 0 \leq t \leq T\end{aligned} \tag{57}$$

for some appropriate constant $c > 0$. By Gronwall lemma, it follows from (57) and (38) that

$$\begin{aligned}&\mu\left(u_m(t), u_m(t)\right)\\ &\quad \leq \left\|u_m(0)\right\|^2\exp\left(\int_0^t c\left(1 + \left\|u_m(s)\right\|^2\right)ds\right)\\ &\quad \leq M_1(T), \quad 0 \leq t \leq T,\end{aligned} \tag{58}$$

which implies by Lemma 3 again that u_m remains bounded in $L^\infty(0, T; V)$. Integrating in (57) from $t = 0$ to $t = T$, we have

$$\int_0^T\left|Au_m(t)\right|^2 ds \leq 2M_1 + \int_0^T M_1\left(1 + \left\|u_m(t)\right\|^2\right)ds \leq M_2. \tag{59}$$

which proved the second argument of (51), and thus (51) holds.

Taking the limit in (51) (by weak compactness), we then find that u is in $L^\infty(0, T; V) \cap L^2(0, T; D(A))$. We need also to prove that u is continuous from $[0, T]$ into V. This is proved as follows.

Since $u \in C([0, T], H) \cap L^\infty(0, T; V)$ and $V \subset H$ with densely continuous injection, it follows from [10, Lemma II.3.3] that $u : [0, T] \to V$ is weakly continuous; that is, $t \mapsto ((u(t), v))$ is continuous for every $v \in V$. Similarly $t \mapsto \mu((u(t), v))$ is continuous for every $v \in V$. Thus, by taking the limit in (52) and applying [10, Lemma II.3.2], we obtain an equality similar to (52) for u:

$$\frac{d}{dt}\mu(u, u) + 2|Au|^2 + 2\alpha^{-1}\lambda(u, u, Au) + 2(Ru, u) = 0, \tag{60}$$

which holds in the distribution sense on $(0, T)$. Since $u \in L^\infty(0, T; V) \cap L^2(0, T; D(A))$, it follows from Lemma 1 and Lemma 2 that

$$|Au|^2 + \alpha^{-1}\gamma(u, u, Au) + (Ru, u) \in L^1(0, T; \mathbb{R}) \tag{61}$$

and thus $(d/dt)\mu(u(t), u(t)) \in L^1(0, T; \mathbb{R})$, which implies by [10, Lemma II.3.1] that the function $t \mapsto \mu(u(t), u(t))$ is continuous. Therefore, since $\mu(\varphi, \varphi)^{1/2}$ is a norm on V equivalent to $\|\varphi\|$ (by Lemma 3), it follows that $u : [0, T] \to V$ is continuous for the norm topology. □

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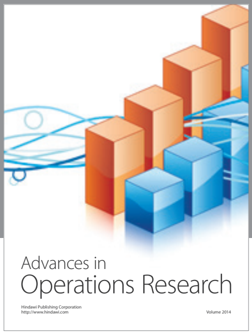

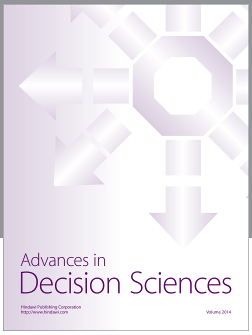

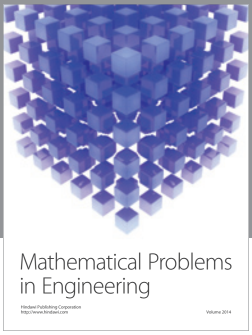

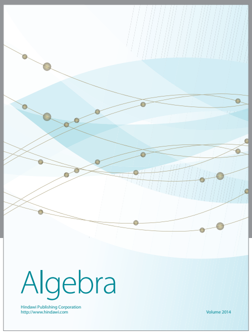

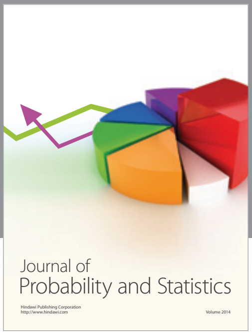

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