Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 943961, 9 pages
http://dx.doi.org/10.1155/2013/943961

Research Article

Solvability of Nonlocal Fractional Boundary Value Problems

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Received 28 January 2013; Revised 30 March 2013; Accepted 11 April 2013

Academic Editor: Ferhan M. Atici

This paper is devoted to introduce a new approach to investigate the existence of solutions for a three-point boundary value problem of fractional difference equations as fllows: $\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1), \Delta y(t+\nu-2))$, $y(\nu-2)=0$, and $[\Delta^{\alpha} y(t)]_{t=\nu+b-\alpha+1} = \gamma[\Delta^{\alpha} y(t)]_{t=\nu+\xi-\alpha}$. We present an existence result at resonance case. The proof relies on coincidence degree theory.

1. Introduction

In this paper, we consider a discrete fractional boundary value problem, for $t \in [0, b+1]_{\mathbb{N}_0}$, of the form

$$\Delta^{\nu} y(t) = f\left(t+\nu-1, y(t+\nu-1), \Delta y(t+\nu-2)\right), \quad (1)$$

subject to the conjugate boundary conditions

$$y(\nu-2) = 0, \quad [\Delta^{\alpha} y(t)]_{t=\nu+b-\alpha+1} = \gamma[\Delta^{\alpha} y(t)]_{t=\nu+\xi-\alpha}, \quad (2)$$

where $f(t+\nu-1,\cdot,\cdot) : [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $b \in \mathbb{N}^*$, $\nu \in (1,2]$, $\alpha \in (0,1)$, $\gamma > 0$, and $\xi \in [0,b]_{\mathbb{N}_0}$, and satisfies both $\nu-\alpha-1 \geq 0$ and $(\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}} = \gamma(\nu+\xi-\alpha)^{\underline{\nu-\alpha-1}}$.

The three-point boundary value problem (1), (2) happens to be at resonance in the sense that associated linear homogeneous boundary value problem

$$\begin{gathered} \Delta^{\nu} y(t) = 0, \quad t \in [0, b+1]_{\mathbb{N}_0}, \\ y(\nu-2) = 0, \quad [\Delta^{\alpha} y(t)]_{t=\nu+b-\alpha+1} = \gamma[\Delta^{\alpha} y(t)]_{t=\nu+\xi-\alpha} \end{gathered} \quad (3)$$

has nontrivial solution $y(t) = Ct^{\underline{\nu-1}}, C \in \mathbb{R}$.

The research into the boundary value problems for differential equation and fractional differential equation have been always very active subjects. Rich results has been obtained due to the various powerful devices such as coincidence degree theory and cone theory. For details, see [1–7] and the references therein. Discrete fractional calculus has generated interest in recent years. There are many literatures dealing with the discrete fractional difference equation subject to various boundary value conditions or initial value conditions. We refer to [8–18] and references therein. However, we note that these results were usually obtained by analytic techniques and various fixed point theorems. For example, in [13–16], authors investigated the existence to some boundary value problems by fixed point theorems on a cone. In [17], we given the existence of multiple solutions for a fractional difference boundary value problem with parameter by establishing the corresponding variational framework and using the mountain pass theorem, linking theorem, and Clark theorem in critical point theory. As we know, the coincidence degree theory has played an important role in dealing with the existence and multiple solutions for differential equations, which include the boundary value problems. To the best of our knowledge, it has not be used in discrete fractional boundary value problems. The aim of this paper is to establish the existence conditions for boundary value problem (1), (2). The proof relies on the coincidence degree theory.

Now, we will briefly recall some notations and an abstract existence result.

Let $\mathbb{Y}, \mathbb{Z}$ be real Banach space, let $\Phi : \mathrm{dom}(\Phi) \subset \mathbb{Y} \to \mathbb{Z}$ be a Fredholm map of index zero, and let $P : \mathbb{Y} \to \mathbb{Y}$, $Q : \mathbb{Z} \to \mathbb{Z}$ be continuous projectors such that $\mathrm{Im}(P) = \mathrm{Ker}(\Phi)$, $\mathrm{Ker}(Q) = \mathrm{Im}(\Phi)$, $\mathbb{Y} = \mathrm{Ker}(\Phi) \oplus \mathrm{Ker}(P)$, and $\mathbb{Z} = \mathrm{Im}(\Phi) \oplus \mathrm{Im}(Q)$. It follows that $\Phi|_{\mathrm{dom}(\Phi)\cap\mathrm{Ker}(P)} : \mathrm{dom}(\Phi) \cap \mathrm{Ker}(P) \to \mathrm{Im}(\Phi)$ is invertible. We denote the inverse of the map by K_p. If Ω is an open bounded subset of $\mathbb{Y}$ such that

$\mathrm{dom}(\Phi) \cap \Omega \neq \emptyset$, and the map $N : \mathbb{Y} \to \mathbb{Z}$ will be called L-compact on $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to \mathbb{Y}$ is compact.

We need the following known result for the sequel.

Theorem 1 (Mawhin continuation theorem, see [2]). *Let L be a Fredholm operator of index zero, and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\mathrm{dom}(L) \backslash \mathrm{Ker}(L)) \cap \partial\Omega] \times (0, 1)$;*

(ii) *$Nx \notin \mathrm{Im}(L)$ for every $x \in \mathrm{Ker}(L) \cap \partial\Omega$;*

(iii) *$\deg(JQN|_{\mathrm{Ker}(L)}, \Omega \cap \mathrm{Ker}(L), 0) \neq 0$, where $Q : \mathbb{Z} \to \mathbb{Z}$ is a projection as above with $\mathrm{Im}(L) = \mathrm{Ker}(Q)$, and $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$ is any isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{dom}(L) \cap \overline{\Omega}$.

2. Preliminaries

We first collect some basic lemmas for manipulating discrete fractional operators.

First, for any real number β, we let $\mathbb{N}_\beta = \{\beta, \beta+1, \beta+2, \ldots\}$. We define $t^{\underline{\alpha}} := \Gamma(t+1)/\Gamma(t+1-\alpha)$, for any t and α for which the right-hand side is defined. We also appeal to the convention that if $t+1-\alpha$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}} = 0$.

Definition 2 (see [13]). The νth fractional sum of f defined on $\mathbb{N}_a$, for $\nu > 0$, is defined to be

$$\Delta_a^{-\nu} f(t) = \Delta_a^{-\nu} f(t; a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s), \tag{4}$$

where $t \in \mathbb{N}_{a+\nu}$. We also define the νth fractional difference, where $\nu > 0$ and $0 \leq N-1 < \nu \leq N$ with $N \in \mathbb{N}$, to be $\Delta^\nu f(t) := \Delta^N \Delta^{-(N-\nu)} f(t)$ where $t \in \mathbb{N}_{a+N-\nu}$.

Lemma 3 (see [13]). *Let t and ν be any numbers for which $t^{\underline{\nu}}$ and $t^{\underline{\nu-1}}$ are defined. Then*

$$\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}. \tag{5}$$

Lemma 4 (see [13]). *Let $0 \leq N-1 < \nu \leq N$. Then*

$$\Delta^{-\nu}\Delta^{\nu} y(t) = y(t) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} + \cdots + C_N t^{\underline{\nu-N}}, \tag{6}$$

for some $C_i \in \mathbb{R}$, with $1 \leq i \leq N$.

Lemma 5 (see [11]). *For $\beta > 0$ and all $\mu \in \mathbb{R}$, for which the following is defined, we find that*

$$\Delta^\beta t^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\beta+1)} t^{\underline{\mu-\beta}}. \tag{7}$$

Lemma 6 (see [10]). *Let p be a positive integer, and let $\nu > p$. Then $\Delta^p \Delta^{-\nu} y(t) = \Delta^{-(\nu-p)} y(t)$.*

Lemma 7 (see [17]). *A real symmetric matrix A is positive definite if there exists a real nonsingular matrix M such that $A = M^\dagger M$, where $M^\dagger$ is the transpose.*

We define the Banach space $\mathbb{Y} = \mathscr{C}([\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}, \mathbb{R})$ with the norm $\|y\| = \max_{t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}} |y(t)|$ and Banach space $\mathbb{Z} = \mathscr{C}([0, b+1]_{\mathbb{N}_0}, \mathbb{R})$ with the norm $\|z\| = \max_{t \in [0,b+1]_{\mathbb{N}_0}} |y(t)|$. For $y \in \mathbb{Y}$, since

$$\Delta^\nu y(t) = \frac{1}{\Gamma(-\nu)} \sum_{s=\nu-2}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} y(s), \tag{8}$$

we can easily see that $\Delta^\nu y \in \mathscr{C}([0, b+1]_{\mathbb{N}_0}, \mathbb{R})$.

Define Φ to be the linear operator from $\mathrm{dom}(\Phi) \cap \mathbb{Y}$ to $\mathbb{Z}$ with

$$\begin{aligned} \mathrm{dom}(\Phi) = \Big\{ & y \in \mathbb{Y} \mid \Delta^\nu y \in \mathscr{C}\left([0, b+1]_{\mathbb{N}_0}, \mathbb{R}\right), \\ & y(\nu-2) = 0,\ \Delta^\alpha y(\nu+b-\alpha+1) \\ & = \gamma \Delta^\alpha y(\nu+\xi-\alpha) \Big\}, \\ \Phi y = {} & \Delta^\nu y, \quad y \in \mathrm{dom}(\Phi). \end{aligned} \tag{9}$$

We define $N : \mathbb{Y} \to \mathbb{Z}$ as

$$\begin{aligned} Ny(t) = {} & f\big(t+\nu-1, y(t+\nu-1), \\ & \Delta y(t+\nu-2)\big), \quad t \in [0, b+1]_{\mathbb{N}_0}. \end{aligned} \tag{10}$$

Then the boundary value problem (1), (2) can be written by

$$\Phi y = Ny. \tag{11}$$

Lemma 8. *$\Phi : \mathrm{dom}(\Phi) \cap \mathbb{Y} \to \mathbb{Z}$ is a Fredholm operator of index zero.*

Proof. By Lemma 4 and the condition $y(\nu-2) = 0$, we have $\mathrm{Ker}(\Phi) = \{C t^{\underline{\nu-1}} \mid C \in \mathbb{R}\}$.

Let $h(t+\nu-1) \in \mathbb{Z}$, $C_1 \in \mathbb{R}$, and

$$y(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} h(s+\nu-1) + C_1 t^{\underline{\nu-1}}, \tag{12}$$

then $\Phi y(t) = \Delta^\nu y(t) = h(t+\nu-1)$ and

$$\mathrm{Im}(\Phi) = \left\{ h(t+\nu-1) \in \mathbb{Z} \mid \sum_{s=0}^{b+1} G(s) h(s+\nu-1) = 0 \right\}, \tag{13}$$

where

$$G(s) := \begin{cases} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} \\ \quad -\dfrac{(\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}}}{(\nu+\xi-\alpha)^{\underline{\nu-\alpha-1}}} \\ \quad \times(\nu+\xi-\alpha-s-1)^{\underline{\nu-\alpha-1}}, & 0\le s\le \xi, \\ (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}}, & \xi+1\le s\le b+1. \end{cases} \tag{14}$$

In fact, if $g \in \mathrm{Im}(\Phi)$, then there exists $y \in \mathrm{dom}(\Phi)$ such that $g(t+\nu-1) = \Phi y(t) = \Delta^{\nu} y(t)$.

In view of Lemma 4, we have

$$y(t) = \Delta^{-\nu} g(t+\nu-1) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}}. \tag{15}$$

By $y(\nu-2)=0$, we get $C_2=0$. By Lemmas 5 and 6, we get

$$\begin{aligned} \Delta^{\alpha} y(t) &= \Delta^{\alpha}\Delta^{-\nu} g(t+\nu-1) + C_1 \Delta^{\alpha} t^{\underline{\nu-1}} \\ &= \Delta^{-(\nu-\alpha)} g(t+\nu-1) + C_1 \Delta^{\alpha} t^{\underline{\nu-1}} \\ &= \frac{1}{\Gamma(\nu-\alpha)} \sum_{s=0}^{t-\nu+\alpha} (t-s-1)^{\underline{\nu-\alpha-1}} g(s+\nu-1) \\ &\quad + C_1 \frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)} t^{\underline{\nu-\alpha-1}}, \end{aligned} \tag{16}$$

then

$$\begin{aligned} &\Delta^{\alpha} y(\nu+b-\alpha+1) \\ &\quad = \frac{1}{\Gamma(\nu-\alpha)} \sum_{s=0}^{b+1} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} g(s+\nu-1) \\ &\quad\quad + C_1 \frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)} (\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}}, \end{aligned} \tag{17}$$

$$\begin{aligned} &\Delta^{\alpha} y(\nu+\xi-\alpha) \\ &\quad = \frac{1}{\Gamma(\nu-\alpha)} \sum_{s=0}^{\xi} (\nu+\xi-\alpha-s-1)^{\underline{\nu-\alpha-1}} g(s+\nu-1) \\ &\quad\quad + C_1 \frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)} (\nu+\xi-\alpha)^{\underline{\nu-\alpha-1}}. \end{aligned} \tag{18}$$

By $[\Delta^{\alpha} y(t)]_{t=\nu+b-\alpha+1} = \gamma[\Delta^{\alpha} y(t)]_{t=\nu+\xi-\alpha}$ and $(\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}} = \gamma(\nu+\xi-\alpha)^{\underline{\nu-\alpha-1}}$, we have

$$\begin{aligned} &\sum_{s=0}^{b+1} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} g(s+\nu-1) - \frac{(\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}}}{(\nu+\xi-\alpha)^{\underline{\nu-\alpha-1}}} \\ &\quad \times \sum_{s=0}^{\xi} (\nu+\xi-\alpha-s-1)^{\underline{\nu-\alpha-1}} g(s+\nu-1) = 0. \end{aligned} \tag{19}$$

Therefore,

$$\sum_{s=0}^{b+1} G(s)\, g(s+\nu-1) = 0, \tag{20}$$

that is,

$$g(t+\nu-1) \in \left\{ h(t+\nu-1) \in \mathbb{Z} \mid \sum_{s=0}^{b+1} G(s)\, h(s+\nu-1) = 0 \right\}. \tag{21}$$

Next, we will show that $G(s) > 0$, for all $s \in [0, b+1]_{\mathbb{N}_0}$.

Obviously $G(s) > 0$ for $\xi+1 \le s \le b+1$. Suppose that $0 \le s \le \xi$, then

$$\begin{aligned} G(s) &= (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} - \frac{(\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}}}{(\nu+\xi-\alpha)^{\underline{\nu-\alpha-1}}} \\ &\quad \times (\nu+\xi-\alpha-s-1)^{\underline{\nu-\alpha-1}} \\ &= (\nu+\xi-\alpha-s-1)^{\underline{\nu-\alpha-1}} \\ &\quad \times \left[\frac{(\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}}}{(\nu+\xi-\alpha-s-1)^{\underline{\nu-\alpha-1}}} - \frac{(\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}}}{(\nu+\xi-\alpha)^{\underline{\nu-\alpha-1}}} \right] \\ &= (\nu+\xi-\alpha-s-1)^{\underline{\nu-\alpha-1}} \\ &\quad \times \left[\frac{\Gamma(\nu+b-\alpha-s+1)\Gamma(\xi-s+1)}{\Gamma(b-s+2)\Gamma(\nu+\xi-\alpha-s)} \right. \\ &\quad \left. - \frac{\Gamma(\nu+b-\alpha+2)\Gamma(\xi+2)}{\Gamma(b+3)\Gamma(\nu+\xi-\alpha+1)} \right] \\ &= (\nu+\xi-\alpha-s-1)^{\underline{\nu-\alpha-1}} \\ &\quad \times \left[\frac{(\nu+b-\alpha-s)(\nu+b-\alpha-s-1)\cdots(\nu+\xi-\alpha-s)}{(b-s+1)(b-s)\cdots(\xi-s+1)} \right. \\ &\quad \left. - \frac{(\nu+b-\alpha+1)(\nu+b-\alpha)\cdots(\nu+\xi-\alpha+1)}{(b+2)(b+1)\cdots(\xi+2)} \right]. \end{aligned} \tag{22}$$

Since $\nu-\alpha-1 \ge 0$, we may imply that

$$\begin{aligned} &\frac{\nu+b-\alpha-s-i}{b-s+1-i} - \frac{\nu+b-\alpha+1-i}{b+2-i} \\ &\quad = \frac{(\nu+b-\alpha-i)(b+1-i)+(\nu+b-\alpha-i)-s(b+2-i)}{(b-s+1-i)(b+2-i)} \\ &\quad\quad - \frac{(\nu+b-\alpha-i)(b+1-i)+(b+1-i)-s(\nu+b-\alpha+1-i)}{(b-s+1-i)(b+2-i)} \\ &\quad = \frac{(\nu-\alpha-1)(s+1)}{(b-s+1-i)(b+2-i)} > 0, \end{aligned} \tag{23}$$

for $i = 0, 1, 2, \ldots, b-\xi$, then $G(s) > 0$.

Let $h(t+\nu-1) \in \mathbb{Z}$, consider continuous linear mapping $Q : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by

$$Qh = \delta_0 \sum_{s=0}^{b+1} G(s) h(s+\nu-1), \tag{24}$$

where $\delta_0 = (1/\sum_{s=0}^{b+1} G(s)) > 0$.

Note that

$$Q^2 h = \delta_0 \sum_{s=0}^{b+1} G(s) Qh(s+\nu-1) = Qh. \tag{25}$$

That is, the map Q is idempotent. In fact, Q is a continuous linear projector. Note that $h \in \operatorname{Im}(\Phi)$ implies $Qh = 0$. Conversely, if $Qh = 0$, then $h \in \operatorname{Im}(\Phi)$. Therefore, $\operatorname{Im}(\Phi) = \operatorname{Ker}(Q)$.

Take $h \in \mathbb{Z}$ in the form $h = (h-Qh)+Qh$, so that $h-Qh \in \operatorname{Im}(\Phi)$ and $Qh \in \operatorname{Im}(Q)$. Thus, $\mathbb{Z} = \operatorname{Im}(\Phi) + \operatorname{Im}(Q)$. Let $h \in \operatorname{Im}(\Phi) \cap \operatorname{Im}(Q)$, and assume that $h(s+\nu-1) = b \neq 0$. Then, since $h \in \operatorname{Im}(\Phi)$, we have $b = 0$, which is a contradiction. Hence, $\operatorname{Im}(Q) \cap \operatorname{Im}(\Phi) = \{\theta\}$, thus $\mathbb{Z} = \operatorname{Im}(\Phi) \oplus \operatorname{Im}(Q)$.

Now, $\dim\operatorname{Ker}(\Phi) = 1 = \operatorname{codim}\operatorname{Im}(\Phi)$, and so Φ is a Fredholm operator of index zero. The proof is accomplished. □

Define a mapping $P : \mathbb{Y} \rightarrow \mathbb{Y}$ by

$$Py(t) = \frac{1}{\Gamma(\nu)} \Delta y(\nu-2) t^{\underline{\nu-1}}, \quad t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}, \tag{26}$$

we see that

$$\operatorname{Ker}(P) = \{y \in \mathbb{Y} \mid \Delta y(\nu-2) = 0\}. \tag{27}$$

Next, we will show that $\operatorname{Ker}(P) \oplus \operatorname{Ker}(\Phi) = \mathbb{Y}$.

In fact, for $\forall u \in \operatorname{Ker}(P) \cap \operatorname{Ker}(\Phi)$, there exists $y \in \mathbb{Y}$ such that $Py(t) = (1/\Gamma(\nu))\Delta y(\nu-2)t^{\underline{\nu-1}} = u(t)$ and $\Delta u(\nu-2) = 0$, and then

$$\Delta u(t)|_{t=\nu-2} = \frac{1}{\Gamma(\nu)} (\nu-1) \Delta y(\nu-2)(\nu-2)^{\underline{\nu-2}} = 0. \tag{28}$$

That is, $\Delta y(\nu-2) = 0$, and hence $\operatorname{Ker}(P) \cap \operatorname{Ker}(\Phi) = \{\theta\}$. Next, if $y \in \mathbb{Y}$, then there exists a $\varphi(t) \in C([0,b+1]_{\mathbb{N}_0}, \mathbb{R})$ such that $\Delta^\nu y(t) = \varphi(t)$; therefore $y(t) = (1/\Gamma(\nu))\sum_{s=0}^{t-2+\nu}(t-s-1)^{\underline{\nu-1}}\varphi(s) + c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}}$. Since $\Phi(c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}}) = 0$ and $\Delta(1/\Gamma(\nu))\sum_{s=0}^{t-2+\nu}(t-s-1)^{\underline{\nu-1}}\varphi(s)|_{t=\nu-2} = 0$, we see that $\operatorname{Ker}(P) \oplus \operatorname{Ker}(\Phi) = \mathbb{Y}$.

Note that the projectors P and Q are exact. Define $K_p : \operatorname{Im}(\Phi) \rightarrow \operatorname{dom}(\Phi) \cap \operatorname{Ker}(P)$ by

$$K_p y(t) = \Delta^{-\nu} y(t). \tag{29}$$

If $y \in \operatorname{Im}(\Phi)$, then

$$(\Phi K_p) y(t) = \Delta^{\nu}\Delta^{-\nu} y(t) = y(t). \tag{30}$$

Also, if $y \in \operatorname{dom}(\Phi) \cap \operatorname{Ker}(P)$, then

$$(K_p \Phi) y(t) = \Delta^{-\nu}\Delta^{\nu} y(t) = y(t) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}}. \tag{31}$$

We can easily see that $K_p\Phi(y) \in \operatorname{dom}(\Phi)\cap\operatorname{Ker}(P)$; then $C_2 = 0$ and $\Delta(y(t)+C_1 t^{\nu-1})|_{t=\nu-2} = \Delta y(\nu-2)+C_1(\nu-1)(\nu-2)^{\underline{\nu-2}} = 0$; in view of $y \in \operatorname{Ker}(P)$, we have $\Delta y(\nu-2) = 0$, then $C_1 = 0$, thus

$$(K_p \Phi) y(t) = y(t), \tag{32}$$

hence, $K_p = (\Phi|_{\operatorname{dom}(\Phi)\cap\operatorname{Ker}(P)})^{-1}$.

Lemma 9. *$K_p(I-Q)N : \mathbb{Y} \longrightarrow \mathbb{Y}$ is completely continuous.*

Proof. Since $K_p(I-Q)N$ is continuous and $K_p(I-Q)Nf(t+\nu-1, y(t+\nu-1), \Delta y(t+\nu-2))$ is a finite sum for $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$, $K_p(I-Q)N$ is completely continuous. □

3. Existence Results

Observe by Lemma 6 that for $t \in [0, b+1]_{\mathbb{N}_0}$,

$$\Delta^\nu y(t) = \Delta \frac{1}{\Gamma(1-\nu)} \sum_{s=\nu-2}^{t-(1-\nu)} (t-s-1)^{\underline{-\nu}} y(s). \tag{33}$$

We let $y \in \operatorname{dom}(\Phi)$ and

$$z(t+\nu-1) = \frac{1}{\Gamma(1-\nu)} \sum_{s=\nu-1}^{t-(1-\nu)} (t-s-1)^{\underline{-\nu}} y(s), \tag{34}$$

then

$$z(\nu-1) = \frac{1}{\Gamma(1-\nu)} \sum_{s=\nu-1}^{0-(1-\nu)} (-s-1)^{\underline{-\nu}} y(s) = y(\nu-1), \tag{35}$$

$$\begin{aligned} z(\nu) &= \frac{1}{\Gamma(1-\nu)} \sum_{s=\nu-1}^{1-(1-\nu)} (1-s-1)^{\underline{-\nu}} y(s) \\ &= (1-\nu) y(\nu-1) + y(\nu), \end{aligned} \tag{36}$$

$$\begin{aligned} z(\nu+1) &= \frac{1}{\Gamma(1-\nu)} \sum_{s=\nu-1}^{2-(1-\nu)} (2-s-1)^{\underline{-\nu}} y(s) \\ &= \frac{(2-\nu)(1-\nu)}{2!} y(\nu-1) + (1-\nu) y(\nu) + y(\nu+1), \\ &\vdots \end{aligned} \tag{37}$$

$$\begin{aligned} z(\nu+b) &= \frac{1}{\Gamma(1-\nu)} \sum_{s=\nu-1}^{b+1-(1-\nu)} (b+1-s-1)^{\underline{-\nu}} y(s) \\ &= \frac{(b-\nu+1)(b-\nu)\cdots(1-\nu)}{(b+1)!} y(\nu-1) \\ &\quad + \frac{(b-\nu)(b-\nu-1)\cdots(1-\nu)}{b!} y(\nu) \\ &\quad + \cdots + (1-\nu) y(b+\nu-1) + y(b+\nu). \end{aligned} \tag{38}$$

That is, $z = By$, where $z = (z(\nu-1), z(\nu), \ldots, z(\nu+b))^{\dagger}$, $y = (y(\nu-1), y(\nu), \ldots, y(\nu+b))^{\dagger}$.

One has

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1-\nu & 1 & 0 & \cdots & 0 \\ \dfrac{(2-\nu)(1-\nu)}{2!} & 1-\nu & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \dfrac{(b-\nu+1)(b-\nu)\cdots(1-\nu)}{(b+1)!} & \dfrac{(b-\nu+1)(b-\nu)(b-\nu-1)\cdots(1-\nu)}{b!} & \cdots & \cdots & 1 \end{pmatrix}_{(b+2)\times(b+2)}. \quad (39)$$

By Lemma 7, $B^{\dagger}B$ is a positive definite matrix. Let $\lambda_{\min}$ and $\lambda_{\max}$ denote, respectively, the minimum and the maximum eigenvalues of $B^{\dagger}B$.

Since $z = By$, we may easyly see that

$$\begin{aligned} &\lambda_{\min}\left(|y(\nu-1)|^2 + |y(\nu)|^2 + \cdots + |y(\nu+b)|^2\right) \\ &\quad\le \left(|z(\nu-1)|^2 + |z(\nu)|^2 + \cdots + |z(\nu+b)|^2\right) \\ &\quad\le \lambda_{\max}\left(|y(\nu-1)|^2 + |y(\nu)|^2 + \cdots + |y(\nu+b)|^2\right), \end{aligned} \quad (40)$$

furthermore,

$$\begin{aligned} &\left(\frac{\lambda_{\min}}{b+2}\right)^{1/2} \max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}} |y(t)| \\ &\quad\le \max_{t\in[0,b+1]_{\mathbb{N}_0}} |z(t+\nu-1)| \\ &\quad\le \left(\lambda_{\max}(b+2)\right)^{1/2} \max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}} |y(t)|. \end{aligned} \quad (41)$$

Theorem 10. *Let $f : [0, b+1]_{\mathbb{N}_0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function in the second and the third variables. Assume that $\lambda \in [0,1]$ and*

(A_1) *there exist nonnegative constants $a, d, l, m,$ and n and constants $k, r \in [0,1)$ which satisfy $a + 2d < (1/(b+1))(\lambda_{\min}/(b+2))^{1/2}$ such that for all $(x, y) \in \mathbb{R}^2, t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$,*

$$f(t,x,y) \le a|x| + d|y| + l|x|^k + m|y|^r + n, \quad (42)$$

(A_2) *there exists $M_1 > 0$ such that for $y \in \mathrm{dom}(\Phi)$, if $|\Delta^{\nu-1}y(t)| > M_1$ for any $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$, then*

$$\sum_{s=0}^{b+1} G(s) f(s+\nu-1, y(s+\nu-1), \Delta y(s+\nu-2)) \ne 0, \quad (43)$$

(A_3) *there exists $M^* > 0$ such that for any $c \in \mathbb{R}$, if $|c| > M^*$, then either*

$$\begin{aligned} &c\sum_{s=0}^{b+1} G(s) f\Big(s+\nu-1, c(s+\nu-1)^{\underline{\nu-1}}, \\ &\qquad c(\nu-1)(s+\nu-2)^{\underline{\nu-2}}\Big) < 0, \end{aligned} \quad (44)$$

or else

$$\begin{aligned} &c\sum_{s=0}^{b+1} G(s) f\Big(s+\nu-1, c(s+\nu-1)^{\underline{\nu-1}}, \\ &\qquad c(\nu-1)(s+\nu-2)^{\underline{\nu-2}}\Big) > 0. \end{aligned} \quad (45)$$

Then the boundary value problem (1), (2) has at least one solution in $\mathbb{Y}$.

Proof. Set $\Omega_1 = \{y \in \mathrm{dom}(\Phi) \setminus \mathrm{Ker}(\Phi) \mid \Phi y = \lambda N y$ for some $\lambda \in (0,1]\}$. Then for $y \in \Omega_1$, $\Phi y = \lambda N y$, thus $Ny \in \mathrm{Im}(\Phi) = \mathrm{Ker}(Q)$, and hence

$$\sum_{s=0}^{b+1} G(s) f(s+\nu-1, y(s+\nu-1), \Delta y(s+\nu-2)) = 0. \quad (46)$$

From (A_2), there exists $t_0 \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$ such that $|\Delta^{\nu-1}y(t_0)| \le M_1$.

Next, for $y \in \Omega_1$, we have $\Delta\Delta^{\nu-1}y(t) = \lambda f(t+\nu-1, y(t+\nu-1), \Delta y(t+\nu-2))$, and hence

$$\begin{aligned} &\sum_{s=t_0}^{t-1} \Delta\Delta^{\nu-1} y(t) \\ &\quad= \sum_{s=t_0}^{t-1} \lambda f(t+\nu-1, y(t+\nu-1), \Delta y(t+\nu-2)), \end{aligned} \quad (47)$$

then we get

$$\begin{aligned}
&|z(t+\nu-1)| \\
&\quad\leq |z(t_0+\nu-1)| \\
&\qquad+\sum_{s=t_0}^{t-1}|f(s+\nu-1,y(s+\nu-1),\Delta y(s+\nu-2))| \\
&\quad\leq M_1+\sum_{s=t_0}^{t-1}\left[a|y|+d|\Delta y|+l|y|^k+m|\Delta y|^r+n\right] \\
&\quad\leq M_1+\sum_{s=0}^{b}\left[(a+2d)|y|+l|y|^k+2^r m|y|^r+n\right] \\
&\quad\leq M_1+(b+1)\left[(a+2d)\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}|y(t)|\right. \\
&\qquad+l\left(\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}|y(t)|\right)^k \\
&\qquad\left.+2^r m\left(\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}|y(t)|\right)^r+n\right],
\end{aligned} \tag{48}$$

then

$$\begin{aligned}
&\max_{t\in[0,b+1]_{\mathbb{N}_0}}|z(t+\nu-1)| \\
&\quad\leq M_1+\lambda(b+1) \\
&\qquad\times\left[(a+2d)\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}|y(t)|\right. \\
&\qquad+l\left(\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}|y(t)|\right)^k \\
&\qquad\left.+2^r m\left(\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}|y(t)|\right)^r+n\right],
\end{aligned} \tag{49}$$

from (41), then we can get

$$\begin{aligned}
&\max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}}|y(t)| \\
&\quad\leq M_1+(b+1)\left[l\left(\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}|y(t)|\right)^k\right. \\
&\qquad\left.+2^r m\left(\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}|y(t)|\right)^r+n\right] \\
&\qquad\times\left((\lambda_{\min}/(b+2))^{1/2}-(b+1)(a+2d)\right)^{-1}.
\end{aligned} \tag{50}$$

Furthermore, since $y(\nu-2)=0$, $\Delta^\alpha y(\nu+b-\alpha+1)=\gamma\Delta^\alpha y(\nu+\xi-\alpha)$ and for any $t\in[\nu-\alpha-2,\nu+b-\alpha+1]_{\mathbb{N}_{\nu-\alpha-2}}$,

$$\begin{aligned}
\Delta^\alpha y(t)&=\Delta\Delta^{\alpha-1}y(t)=\Delta\left[\frac{1}{\Gamma(1-\alpha)}\sum_{s=\nu-2}^{t+\alpha-1}(t-s-1)^{\underline{-\alpha}}y(s)\right] \\
&=\frac{1}{\Gamma(1-\alpha)}\sum_{s=\nu-1}^{t+\alpha}(t-s)^{\underline{-\alpha}}y(s)-\frac{1}{\Gamma(1-\alpha)} \\
&\quad\times\sum_{s=\nu-1}^{t+\alpha-1}(t-s-1)^{\underline{-\alpha}}y(s) \\
&=\frac{-\alpha}{\Gamma(1-\alpha)}\sum_{s=\nu-2}^{t+\alpha}(t-s-1)^{\underline{-\alpha-1}}y(s)
\end{aligned} \tag{51}$$

holds, and we may see that $y(\nu+b+1)$ can be expressed by a linear combination with $y(\nu-1), y(\nu),\ldots,y(\nu+b)$. Therefore, $\|y\|=\max_{t\in[\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}}}|y(t)|$ is bounded. Thus, Ω_1 is bounded.

Let $\Omega_2=\{y\in\mathrm{Ker}(\Phi)\mid Ny\in\mathrm{Im}(\Phi)\}$. For $y\in\Omega_2, y\in\mathrm{Ker}(\Phi)$, since $\mathrm{Im}(\Phi)=\mathrm{Ker}(Q)$, then $QNy=0$, thus

$$\begin{aligned}
&\sum_{s=0}^{b+1}G(s)f\left(s+\nu-1,c(s+\nu-1)^{\underline{\nu-1}},\right. \\
&\qquad\left.c(\nu-1)(s+\nu-2)^{\underline{\nu-2}}\right)=0.
\end{aligned} \tag{52}$$

From Lemma 5 and in view of $y\in\mathrm{Ker}(\Phi)$, we see that $|\Delta^{\nu-1}y(t)|=|c||\Delta^{\nu-1}t^{\underline{\nu-1}}|=|c||\Gamma(\nu)|$, then from (A_2) we get $|c||\Gamma(\nu)|\leq M_1$; that is, $|c|\leq(M_1/|\Gamma(\nu)|)$. Hence, Ω_2 is bounded. Next, according to (A_3), for any $c\in\mathbb{R}$, if $|c|>M^*$, then either

$$\begin{aligned}
&c\delta_0\sum_{s=0}^{b+1}G(s)f\left(s+\nu-1,c(s+\nu-1)^{\underline{\nu-1}},\right. \\
&\qquad\left.c(\nu-1)(s+\nu-2)^{\underline{\nu-2}}\right)<0,
\end{aligned} \tag{53}$$

or

$$\begin{aligned}
&c\delta_0\sum_{s=0}^{b+1}G(s)f\left(s+\nu-1,c(s+\nu-1)^{\underline{\nu-1}},\right. \\
&\qquad\left.c(\nu-1)(s+\nu-2)^{\underline{\nu-2}}\right)>0.
\end{aligned} \tag{54}$$

If (53) holds, set

$$\Omega_3=\left\{y\in\mathrm{Ker}(\Phi)\mid-\lambda J^{-1}y+(1-\lambda)QNy=0,\lambda\in[0,1]\right\}, \tag{55}$$

where $J:\mathrm{Im}(Q)\to\mathrm{Ker}(\Phi)$ is the linear isomorphism given by $J(c)=ct^{\underline{\nu-1}}$, $\forall c\in\mathbb{R}$, and $t\in[0,b+1]_{\mathbb{N}_0}$. Since $y\in c_0t^{\underline{\nu-1}}\in\Omega_3$, $c_0\in\mathbb{R}$, then

$$\begin{aligned}
\lambda c_0&=(1-\lambda)\delta_0\sum_{s=0}^{b+1}G(s)f\left(s+\nu-1,c(s+\nu-1)^{\underline{\nu-1}},\right. \\
&\qquad\left.c(\nu-1)(s+\nu-2)^{\underline{\nu-2}}\right).
\end{aligned} \tag{56}$$

If $\lambda = 1$, then $c_0 = 0$. Otherwise, if $|c_0| > M^*$, in view of (53), one has

$$0 \le \lambda c_0^2 = c_0 (1-\lambda) \delta_0 \sum_{s=0}^{b+1} G(s) f\left(s+\nu-1, c(s+\nu-1)^{\underline{\nu-1}}, c(\nu-1)(s+\nu-2)^{\underline{\nu-2}}\right) < 0, \quad (57)$$

which is a contradiction. Thus $\Omega_3 \subset \{y \in \operatorname{Ker}(\Phi) \mid y = ct^{\underline{\nu-1}}, |c| \le M^*\}$ is bounded.

If (54) holds, then the set

$$\Omega_3 = \left\{y \in \operatorname{Ker}(\Phi) \mid \lambda J^{-1} y + (1-\lambda) QNy = 0, \lambda \in [0,1]\right\}, \quad (58)$$

where J is mentioned as above. By the analogous argument, we can show that Ω_3 is bounded too.

Now, we will prove that all the conditions of Theorem 1 are satisfied. Set Ω to be bounded open set of $\mathbb{Y}$ such that $\cup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. By Lemma 9, $K_p(I-Q)N : \overline{\Omega} \to \mathbb{Y}$ is compact; hence N is L-compact on $\overline{\Omega}$; then by the above argument, we have

(i) $\Phi x \ne \lambda N x$ for every $(x,\lambda) \in [(\operatorname{dom}(\Phi) \setminus \operatorname{Ker}(\Phi)) \cap \partial\Omega] \times (0,1)$,

(ii) $Nx \notin \operatorname{Im}(\Phi)$ for every $x \in \operatorname{Ker}(\Phi) \cap \partial\Omega$,

then the conditions (i), (ii) of Theorem 1 are hold.

At last we will prove that the condition (iii) of Theorem 1 is satisfied. In fact, let $H(y,\lambda) \ne 0$ for all $y \in \operatorname{Ker}(\Phi) \cap \partial\Omega$, thus, by the homotopy property of degree as follows:

$$\begin{aligned} &\deg\left(JQN|_{\operatorname{Ker}(\Phi)}, \Omega \cap \operatorname{Ker}(\Phi), 0\right) \\ &\quad = \deg\left(H(\cdot,0), \Omega \cap \operatorname{Ker}(\Phi), 0\right) \\ &\quad = \deg\left(\pm I, \Omega \cap \operatorname{Ker}(\Phi), 0\right) \ne 0. \end{aligned} \quad (59)$$

Then by Theorem 1, $\Phi y = Ny$ has at least one solution in $\operatorname{dom}(\Phi) \cap \overline{\Omega}$; therefore the boundary value problem (1), (2) has at least one solution in $\mathbb{Y}$. □

Remark 11. We may obtain similarly result when the third variable of f is instead by fractional difference of $y(t)$ with the order β which satisfies $0 < \beta \le \nu - 1$.

Example 12. Consider the boundary value problem

$$\begin{gathered} \Delta^{3/2} y(t) = f\left(t+\frac{1}{2}, y\left(t+\frac{1}{2}\right), \Delta y\left(t-\frac{1}{2}\right)\right), \quad t \in [0,2]_{\mathbb{N}_0}, \\ y\left(-\frac{1}{2}\right) = 0, \qquad \Delta^{1/4} y\left(\frac{13}{4}\right) = \gamma \Delta^{1/4} y\left(\frac{9}{4}\right), \end{gathered} \quad (60)$$

where $f(t+\nu-1, y(t+\nu-1), \Delta y(t+\nu-2)) = (1/8)|y| + (1/64)\Delta y \cos y + \cos(y\Delta y)$.

It is clearly that f is continuous. $\nu = 3/2$, $\alpha = 1/4$, $b = \xi = 1$, and $\gamma = 13/12$, then $\nu - \alpha - 1 = (1/4) > 0$ and $(\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}} = \gamma(\nu+\xi-\alpha)^{\underline{\nu-\alpha-1}}$; therefore, the boundary value problem (1), (2) is at resonance case. By $\nu = 3/2$, we have

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{8} & -\frac{1}{2} & 1 \end{pmatrix}_{3\times 3}. \quad (61)$$

By calculating, $\lambda_{\min} = 0.4344$. Choosing $a = 1/8$, $d = (1/64)$, $l = m = r = k = 0$, and $n = 1$, we can get $a + 2d = 1/8 + 2 \times (1/64) \le (1/(b+1))(\lambda_{\min}/(b+2))^{1/2} = (1/2)(0.4344/3)^{1/2}$, furthermore, for all $(y, \Delta y) \in \mathbb{R}^2, t \in [0, b+1]_{\mathbb{N}_0}$,

$$\begin{aligned} &\left|f(t+\nu-1, y(t+\nu-1), \Delta y(t+\nu-2))\right| \\ &\quad \le \frac{1}{8}|y| + \frac{1}{64}|\Delta y| + 1. \end{aligned} \quad (62)$$

Therefore, the condition (A_1) of Theorem 10 holds. Moreover, we can choose $M_1 = 1620, M^* = 10$; then the condition (A_2), (A_3) of Theorem 10 hold too. In fact, by the expression of f, it is clear that if $|z| = |\Delta^{\nu-1} y| > 1620$, then $|y(\nu-1)| = |z(\nu-1)| > 1620$. Therefore, we can see

$$\begin{aligned} &\sum_{s=0}^{2} G(s) f(s+\nu-1, y(s+\nu-1), \Delta y(s+\nu-2)) \\ &= G(0)\left[\frac{1}{8}|y(\nu-1)| + \frac{1}{64}\Delta y(\nu-2)\cos(y(\nu-1)) + \cos(y(\nu-1)\Delta y(\nu-2))\right] \\ &\quad + G(1)\left[\frac{1}{8}|y(\nu)| + \frac{1}{64}\Delta y(\nu-1)\cos(y(\nu)) + \cos(y(\nu)\Delta y(\nu-1))\right] \\ &\quad + G(2)\left[\frac{1}{8}|y(\nu+1)| + \frac{1}{64}\Delta y(\nu)\cos(y(\nu+1)) + \cos(y(\nu+1)\Delta y(\nu))\right] \\ &= G(0)\left[\frac{1}{8}|y(\nu-1)| + \frac{1}{64}y(\nu-1)\cos(y(\nu-1)) + \cos(y(\nu-1)\Delta y(\nu-2))\right] \\ &\quad + G(1)\left[\frac{1}{8}|y(\nu)| + \frac{1}{64}y(\nu)\cos(y(\nu)) - \frac{1}{64}y(\nu-1)\cos(y(\nu)) + \cos(y(\nu)\Delta y(\nu-1))\right] \end{aligned}$$

$$
\begin{aligned}
&+G(2)\left[\frac{1}{8}|y(\nu+1)|+\frac{1}{64}y(\nu+1)\cos(y(\nu+1))\right.\\
&\qquad\left.\frac{1}{64}y(\nu)\cos(y(\nu+1))+\cos(y(\nu+1)\Delta y(\nu))\right]\\
&\geq\left[G(0)\left(\frac{1}{8}-\frac{1}{64}\right)-\frac{1}{64}G(1)\right]|y(\nu-1)|\\
&\quad+\left[G(1)\left(\frac{1}{8}-\frac{1}{64}\right)-\frac{1}{64}G(2)\right]|y(\nu)|\\
&\quad+G(2)\left(\frac{1}{8}-\frac{1}{64}\right)|y(\nu+1)|-(G(0)+G(1)+G(2))\\
&>\frac{11}{64\times 6\times 24}\Gamma\left(\frac{5}{4}\right)|y(\nu-1)|+\frac{1}{64\times 6}\Gamma\left(\frac{5}{4}\right)|y(\nu)|\\
&\quad-(G(0)+G(1)+G(2))\\
&>\frac{11}{64\times6\times24}\Gamma\left(\frac{5}{4}\right)|y(\nu-1)|-\left(\frac{5}{24\times4}+\frac{1}{6}+1\right)\Gamma\left(\frac{5}{4}\right)\\
&>\frac{1}{64\times 18}\Gamma\left(\frac{5}{4}\right)|y(\nu-1)|-\frac{5}{4}\Gamma\left(\frac{5}{4}\right)>0.
\end{aligned}
\tag{63}
$$

Thus, for every $y\in\mathbb{R}$, $s\in[0,2]_{\mathbb{N}_0}$, $\sum_{s=0}^{2}G(s)f(s+\nu-1, y(s+\nu-1)$, and $\Delta y(s+\nu-2))\neq 0$, so (A_2) holds, and if $|c|>(19/\sqrt{\pi})$, then for $t\in[0,b+1]_{\mathbb{N}_0}$,

$$
\left[\frac{1}{8}(t+\nu-1)^{\underline{\nu-1}}-\frac{1}{64}(\nu-1)(t+\nu-2)^{\underline{\nu-1}}\right]c-1>0 \quad \text{for } c>\frac{19}{\sqrt{\pi}}
\tag{64}
$$

or

$$
\left[\frac{1}{8}(t+\nu-1)^{\underline{\nu-1}}-\frac{1}{64}(\nu-1)(t+\nu-2)^{\underline{\nu-1}}\right]c+1<0 \quad \text{for } c<-\frac{19}{\sqrt{\pi}}.
\tag{65}
$$

Furthermore,

$$
\begin{aligned}
&cf\left(t+\nu-1,c(t+\nu-1)^{\underline{\nu-1}},c(\nu-1)(t+\nu-2)^{\underline{\nu-2}}\right)\\
&\quad=\frac{1}{8}c(t+\nu-1)^{\underline{\nu-1}}+\frac{1}{64}c(\nu-1)\\
&\quad\times(t+\nu-2)^{\underline{\nu-2}}\cos\left(c(t+\nu-1)^{\underline{\nu-1}}\right)\\
&\quad+\cos\left[\left(c(t+\nu-1)^{\underline{\nu-1}}\right)\left(c(\nu-1)(t+\nu-2)^{\underline{\nu-2}}\right)\right]\\
&\quad>0,\quad \text{for } c>\frac{19}{\sqrt{\pi}},
\end{aligned}
\tag{66}
$$

or

$$
\begin{aligned}
&cf\left(t+\nu-1,c(t+\nu-1)^{\underline{\nu-1}},\right.\\
&\qquad\left.c(\nu-1)(t+\nu-2)^{\underline{\nu-2}}\right)<0,\quad \text{for } c<-\frac{19}{\sqrt{\pi}},
\end{aligned}
\tag{67}
$$

so (53) or (54) holds; that is, (A_3) of Theorem 10 holds. Then, all the assumptions of Theorem 10 hold. Thus, with Theorem 10, the boundary value problem (60) has at least one solution in $\mathbb{Y}$.

Acknowledgment

This project was supported by the National Natural Science Foundation of China (11161049).

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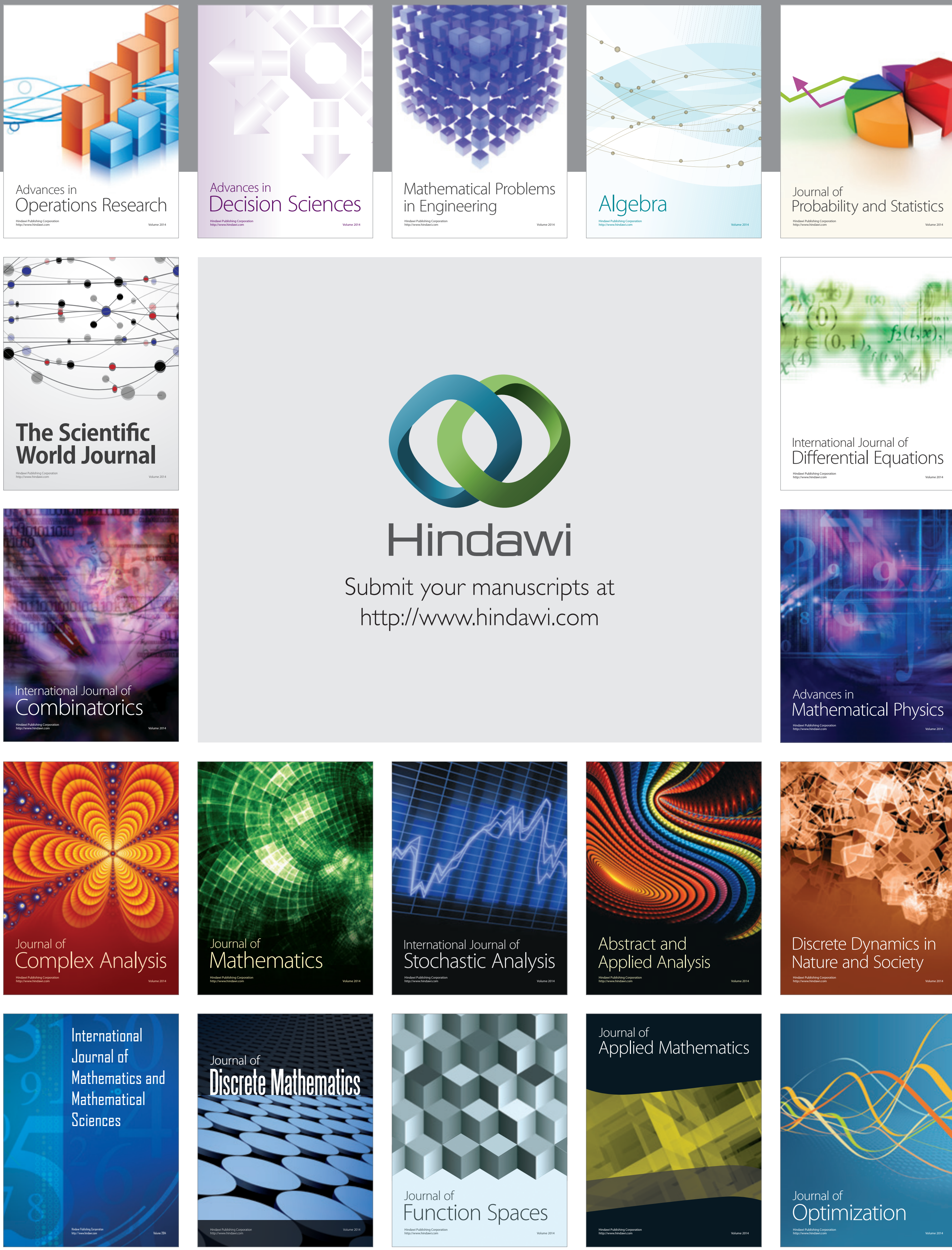

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