EXISTENCE, UNIQUENESS, AND QUASILINEARIZATION RESULTS FOR NONLINEAR DIFFERENTIAL EQUATIONS ARISING IN VISCOELASTIC FLUID FLOW

F. TALAY AKYILDIZ AND K. VAJRAVELU

Received 23 December 2005; Revised 13 April 2006; Accepted 17 April 2006

Solutions for a class of nonlinear second-order differential equations arising in steady Poiseuille flow of an Oldroyd six-constant model are obtained using the quasilinearization technique. Existence, uniqueness, and analyticity results are established using Schauder theory. Numerical results are presented graphically and salient features of the solutions are discussed.

1. Introduction

Because of their various applications during the past several years, generalizations of the Navier-Stokes model to highly nonlinear constitutive laws have been proposed and studied (see [4, 5, 7]). Several different models have been introduced to explain such nonstandard features, as normal stress effect, rod climbing, shear thinning, and shear thickening. Among the differential-type models, Oldroyd models received special attention [2]. These models are rather complex from the point of view of partial differential equations theory. Nevertheless, several authors in fluid mechanics are now engaged with the equations of motion of non-Newtonian fluids of Oldroyd two-, three-, six-, and eight-constant models. Several authors [2, 6] considered an Oldroyd three-constant model which is a special case of the Oldroyd six-constant model. This has been used recently by Baris [1] for dealing with the steady and slow flow in the wedge between intersecting planes, one fixed and the other one moving.

The Cauchy stress $\mathbf{T}$ in an incompressible Oldroyd six-constant-type fluid is related to the fluid motion by

$$\mathbf{T} = -\mathbf{pI} + \mathbf{S}, \tag{1.1}$$

Hindawi Publishing Corporation
Differential Equations and Nonlinear Mechanics
Volume 2006, Article ID 71717, Pages 1–9
DOI 10.1155/DENM/2006/71717

(for details see [2]), where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility. The extra stress tensor $\mathbf{S}$ is defined by

$$\mathbf{S}+\lambda_1\frac{D\mathbf{S}}{Dt}+\frac{\lambda_3}{2}(\mathbf{S}\mathbf{A}_1+\mathbf{A}_1\mathbf{S})+\frac{\lambda_5}{2}(\operatorname{tr}\mathbf{S})\mathbf{A}_1=\mu\left(\mathbf{A}_1+\lambda_2\frac{D\mathbf{A}_1}{Dt}+\lambda_4\mathbf{A}_1^2\right), \tag{1.2}$$

where μ, λ_1, λ_2, λ_3, λ_4, λ_5 are six material constants. $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor defined by

$$\mathbf{A}_1=\operatorname{grad}\mathbf{v}+(\operatorname{grad}\mathbf{v})^T, \tag{1.3}$$

where $D\mathbf{S}/Dt$ is the upper-convected derivative of $\mathbf{S}$ and is defined as

$$\frac{D\mathbf{S}}{Dt}=\frac{\partial\mathbf{S}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{S}-\mathbf{L}\mathbf{S}-\mathbf{S}\mathbf{L}^T \quad (\mathbf{L}=\operatorname{grad}\mathbf{v}). \tag{1.4}$$

Recently, Wang et al. [8] studied magnetohydrodynamic steady Poiseuille channel flow of an Oldroyd six-constant fluid and obtained the numerical solution using the predictor corrector method. However, they did not show existence and uniqueness results.

In this paper, we study the existence, uniqueness, and behavior of exact solutions of second-order nonlinear differential equations arising in Oldroyd six-constant fluid flows in a channel. Furthermore, we obtain numerical solutions by using the quasilinearization technique.

2. Formulation of the problem

In this paper, steady plane shearing flows are considered for which the equation for the fluid flow (for details see Wang et al. [8]) is

$$\frac{d}{dy}\left\{\frac{\mu(du/dy)+\mu\alpha_1(du/dy)^3}{1+\alpha_2(du/dy)^2}\right\}-\frac{dp}{dx}=0, \tag{2.1}$$

where

$$\alpha_1=\lambda_1\lambda_4-(\lambda_3+\lambda_5)(\lambda_4-\lambda_2), \qquad \alpha_2=\lambda_1\lambda_3-(\lambda_3+\lambda_5)(\lambda_3-\lambda_1). \tag{2.2}$$

We leave the issue of boundary conditions for later.

Defining nondimensional variables

$$\begin{gathered} y^*=\frac{y}{H}, \qquad x^*=\frac{x}{H}, \qquad u^*=\frac{u}{H}, \\ \alpha_1^*=\frac{\alpha_1}{(U/H)^2}, \qquad \alpha_2^*=\frac{\alpha_2}{(U/H)^2}, \qquad p^*=\frac{p}{(\mu U/H)}, \end{gathered} \tag{2.3}$$

and substituting (2.3) in (2.1), we obtain (after dropping the stars)

$$\frac{d}{dy}\left\{\frac{(du/dy)+\alpha_1(du/dy)^3}{1+\alpha_2(du/dy)^2}\right\}-\frac{dp}{dx}=0. \tag{2.4}$$

The appropriate no-slip boundary conditions are

$$u(0)=u(1)=0. \tag{2.5}$$

First, we define $L=((du/dy)+\alpha_1(du/dy)^3)/(1+\alpha_2(du/dy)^2)$ so that

$$\frac{dL}{dy}=\frac{dp}{dx}. \tag{2.6}$$

Now, (2.6) can be solved for du/dy in terms of L. In order to do this we assume the transformation

$$\overline{\frac{du}{dy}}=\alpha_1\frac{du}{dy}-\frac{\alpha_2}{3}L. \tag{2.7}$$

This transformation effectively gets rid of the quadratic first derivative term yielding

$$\overline{\frac{du}{dy}}^3-3R\overline{\frac{du}{dy}}-B=0, \tag{2.8}$$

where

$$R=\frac{1}{9}\alpha_2^2L^2-\frac{\alpha_1}{3},\qquad B=\alpha_1^2L-\frac{\alpha_1\alpha_2}{3}L+\frac{2}{27}\alpha_2^3L^3. \tag{2.9}$$

The solution of this is

$$\overline{\frac{du}{dy}}=\chi+\varphi, \tag{2.10}$$

where

$$\chi=\sqrt[3]{\frac{B}{2}+\frac{\sqrt{B^2-4R^3}}{2}},\qquad \varphi=\sqrt[3]{\frac{B}{2}-\frac{\sqrt{B^2-4R^3}}{2}}. \tag{2.11}$$

We note that (2.8) always has one real solution irrespective of the value of B^2-4R^3. Also, if $(B^2-4R^3)\le 0$, then it is easy to see that (2.8) has three real solutions, hence there is no unique solution, so, throughout this paper, we assume that $(B^2-4R^3)>0$.

Using (2.7) in (2.10), we get

$$\frac{du}{dy}=\frac{\chi}{\alpha_1}+\frac{\varphi}{\alpha_1}+\frac{\alpha_2}{3\alpha_1}L. \tag{2.12}$$

From (2.6), we have

$$L=\frac{dp}{dx}y+c. \tag{2.13}$$

Integrating (2.12) and substituting into(2.13), we obtain

$$u(y)=\frac{1}{\alpha_1}\int_0^y \chi(\xi)d\xi+\frac{1}{\alpha_1}\int_0^y \varphi(\xi)d\xi+\frac{\alpha_2}{3\alpha_1}\left(\frac{1}{2}\frac{dp}{dx}y^2+cy\right). \tag{2.14}$$

3. Existence and uniqueness results

THEOREM 3.1. *There exists a classical solution of (2.4) which can be written as*

$$\frac{d^2u}{dy^2}-\frac{(dp/dx)\left(1+\alpha_2(du/dy)^2\right)^2}{1+3\alpha_1(du/dy)^2-\alpha_2(du/dy)^2+\alpha_1\alpha_2(du/dy)^4}=0 \tag{3.1}$$

with

$$u(0)=u(1)=0. \tag{3.2}$$

Proof. We employ the Schauder fixed point theorem. First, from (2.6), we see that the solution can be written as (2.14). Let **B** be the Banach space of continuous functions $u(y)$ on the interval $0\le y\le 1$ which vanish at 0 and 1 with the norm

$$\|u\|=\sup_{0\le y\le 1}|u(y)|. \tag{3.3}$$

Define $\mathbb{F}:\mathbf{B}\to\mathbf{B}$, where $(Fu)(y)$ is equal to the right-hand side of (2.14). □

3.1. A priori bounds. The Schauder fixed point theorem requires us to show that $\mathbb{F}$ is a continuous mapping of a convex compact subset of **B** into itself. To do this we need to derive estimates on $(\mathbb{F}u)(y)$ and $(\mathbb{F}u)'(y)$. Since $dp/dx=k$ (constant), k is known, and $y\in[0,1]$, we have from (2.13) that $L=ky+c$. This gives us an estimate of $(\mathbb{F}u)(y)$ and $(\mathbb{F}u)'(y)$. From the triangle inequality, we get

$$\left|\frac{\chi}{\alpha_1}+\frac{\varphi}{\alpha_1}+\frac{\alpha_2}{3\alpha_1}L\right|\le\left|\frac{\chi}{\alpha_1}\right|+\left|\frac{\varphi}{\alpha_1}\right|+\left|\frac{\alpha_2}{3\alpha_1}L\right|, \tag{3.4}$$

where

$$\left|\frac{\chi}{\alpha_1}\right|=\frac{1}{|\alpha_1|}\left|\sqrt[3]{\frac{1}{2}\alpha_1^2L-\frac{1}{6}\alpha_1\alpha_2L+\frac{1}{27}\alpha_2^3+\frac{1}{2}\sqrt{\alpha_1^2L-\frac{1}{3}\alpha_1\alpha_2L+\frac{2}{27}\alpha_2^3-4\left(\frac{1}{9}\alpha_2^2L^2-\frac{\alpha_1}{3}\right)}}\right|,$$

$$\left|\frac{\varphi}{\alpha_1}\right|=\frac{1}{|\alpha_1|}\left|\sqrt[3]{\frac{1}{2}\alpha_1^2L-\frac{1}{6}\alpha_1\alpha_2L+\frac{1}{27}\alpha_2^3-\frac{1}{2}\sqrt{\alpha_1^2!!\frac{1}{3}\alpha_1\alpha_2L+\frac{2}{27}\alpha_2^3-4\left(\frac{1}{9}\alpha_2^2L^2-\frac{\alpha_1}{3}\right)}}\right|. \tag{3.5}$$

Since $0\le y\le 1$, and c and k are constants, we know that L is bounded. On the other hand α_1 and α_2 are dimensionless parameters (see (2.2)-(2.3)) and from (2.7), we see that χ

and φ are bounded. Therefore, $|\chi/\alpha_1|$ and $|\varphi/\alpha_1|$ are bounded by $C_1/|\alpha_1|$ and $C_2/|\alpha_1|$, respectively. Hence

$$\left|(\mathbb{F}w)(y)\right| \le \int_0^y \left(\frac{C_1}{|\alpha_1|} + \frac{C_2}{|\alpha_1|}\right) d\xi + \left|\frac{\alpha_2 k}{3\alpha_1}\right| \left(\frac{1}{2} + c\right) \le C_4. \tag{3.6}$$

Since C_4 is independent of the function w, we see $\mathbb{F} : Z \to V$, where

$$\mathbf{V} = \{u \in Z \mid \|w\| \le C_4\} \tag{3.7}$$

is a subset of $\mathbf{B}$ and hence

$$(\mathbb{F}w)'(y) = \frac{\chi}{\alpha_1} + \frac{\varphi}{\alpha_1} + \frac{\alpha_2}{3\alpha_1} L. \tag{3.8}$$

Similarly, it is easy to show that

$$\left|(\mathbb{F}w)'(y)\right| \le C_5. \tag{3.9}$$

Since C_5 is independent of w, we have $\mathbb{F} : Z \to V_c$, where

$$V_c = \{w \in Z \mid \|w\| \le C_4,\ \|w'\| \le C_5\}, \tag{3.10}$$

which is convex and compact via the Ascoli-Arzela theorem. Consequently, we have $\mathbb{F} : Z \to Z$. The continuity of $\mathbb{F}$ is an elementary calculation based on the estimates, and it is easy to see from (3.5) that

$$\left\|\mathbb{F}u_1 - \mathbb{F}u_2\right\| \le C_{10}, \tag{3.11}$$

where $C_{10} = C_{10}(C_4, C_5, \ldots, \alpha_1, \alpha_2)$.

Theorem 3.2. *The solution $u(y)$ of (3.1) and (3.2) is unique.*

Proof. The proof is by contradiction. We assume that (3.1) has two solutions u and v satisfying the conditions (3.2). Set $z = u - v$. We get

$$\begin{aligned} z''(y) + kz'(y) &\frac{(u'+v')\left(\alpha_2^2 a_1 u'^2 v'^2 + \alpha_2^2 u'^2 + 2\alpha_2 u'^2 v'^2 - a_2 u'^2 - 2\alpha_2 + a_1 - (\alpha_2^2 + a_2) v'^2\right)}{\left(1 + a_1 u'^2 + a_2 u'^4\right)\left(1 + a_1 v'^2 + a_2 v'^4\right)} \\ &= 0, \end{aligned} \tag{3.12}$$

where $a_1 = 3\alpha_1 - \alpha_2$ and $a_2 = \alpha_1 \alpha_2$. We can write this equation in the form

$$z'' + F(u', v') kz' = 0 \tag{3.13}$$

with boundary conditions

$$z(0) = 0, \qquad z(1) = 0. \tag{3.14}$$

Equation (3.13) can be solved easily to get

$$z = e_1 + e_2 \int_0^y e^{-\int kF dt} dt. \tag{3.15}$$

Using the boundary condition (3.14), we find that $z = 0$. This proves the theorem. □

4. Results and discussion

We use the quasilinearization method which has been explained in detail in [3]. The quasilinear process equations for our differential equation are

$$\begin{aligned} \left(\frac{d^2u}{dy^2}\right)_{k+1} &= \frac{(dp/dx)(1+\alpha_2 N_k^2)^2}{M} + \frac{dp}{dx}(1+\alpha_2 N_k^2)^2 \\ &\quad \times \left\{ -\frac{(6\alpha_1 N_k + 4\alpha_1\alpha_2 N_k^3 - 2\alpha_2 N_k)}{M^2} + \frac{4\alpha_2 N_k}{(1+\alpha_2 N_k^2)^2 M} \right\} \\ &\quad \times \left(\left(\frac{du}{dy}\right)_{k+1} - \left(\frac{du}{dy}\right)_k\right), \quad k = 0, 1, \dots, \\ & u_k(0) = 0, \qquad u_k(1) = 0, \end{aligned} \tag{4.1}$$

where

$$N = \frac{du}{dx}, \qquad M = 1 + 3\alpha_1 N_k^2 + \alpha_1\alpha_2 N_k^4 - \alpha_2 N_k^2. \tag{4.2}$$

By means of the finite difference method a linear algebraic equation system is derived and solved for each iterative step. A sequence of functions $\mathbf{u}_0(y), \mathbf{u}_1(y), \dots$ is determined in the following manner: if an initial estimate $\mathbf{u}_0(y)$ is given, then $\mathbf{u}_1(y), \mathbf{u}_2(y), \dots$ are calculated successively as the solution of the boundary-value problem (4.1). The solution is assumed to converge when the difference between two successive iterations is less than the infinitesimal number $\varepsilon = 1 \times 10^{-10}$.

In Figures 4.1 and 4.2, we show the effects of the parameters (α_1, α_2), and the pressure gradient on the velocity field. In these figures, we also compared our results with the results of Wang et al. [8]. For small values of α_2, there is no appreciable difference between the two solutions. However, if α_2 is large enough, these two solutions are different; this mathematical problem is of interest and will be the subject of our future investigation. If α_1 is not large, these two solutions are identical as shown in Figures 4.3 and 4.4. Here, the parameters α_1, α_2 represent material constants; when they are zero, the model reduces to the linear Oldroyd-B model. Hence, we can regard the effects of the parameters α_1, α_2 on the velocity field as due to nonlinearity.

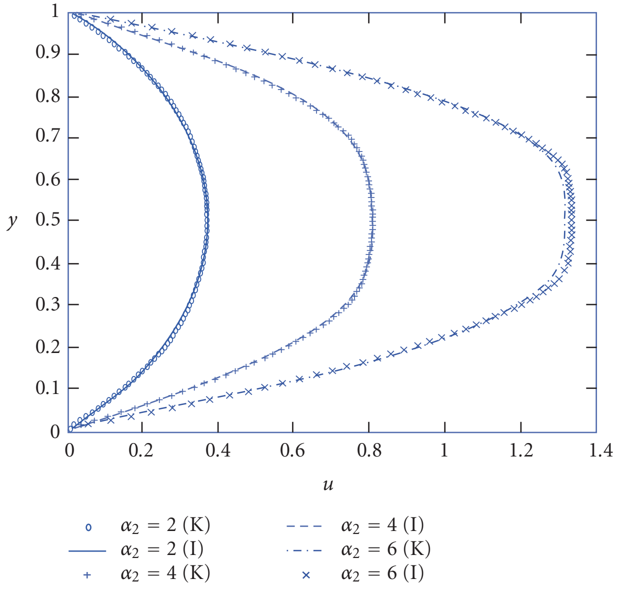

Figure 4.1. Velocity profiles for various values of α_2 and fixed $\alpha_1 = 1$ (I: Wang et al. solution; K: our solution) for $dp/dx = -2$.

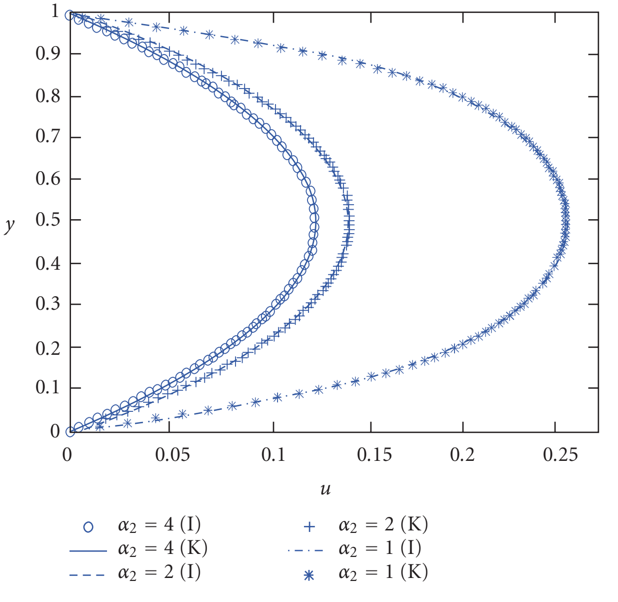

Figure 4.2. Velocity profiles for various values of α_2 and fixed $\alpha_1 = 1$ (I: Wang et al. solution; K: our solution) for $dp/dx = -1$.

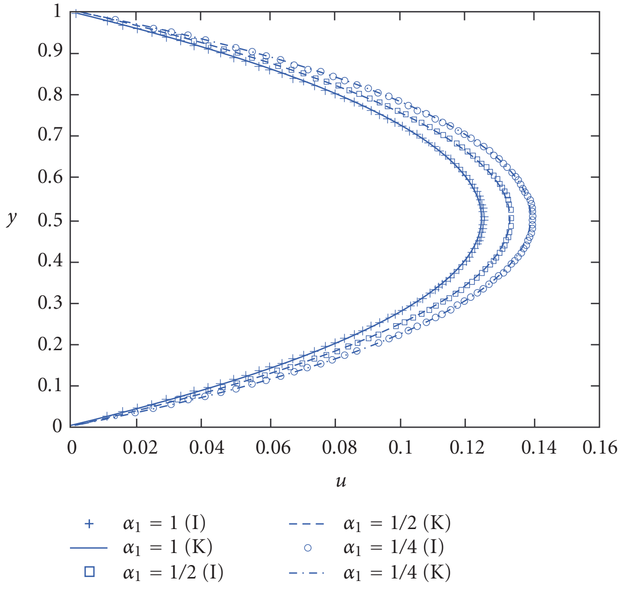

Figure 4.3. Velocity profiles for various values of α_1 and fixed $\alpha_2 = 1$ (I: Wang et al. solution; K: our solution) for $dp/dx = -1$.

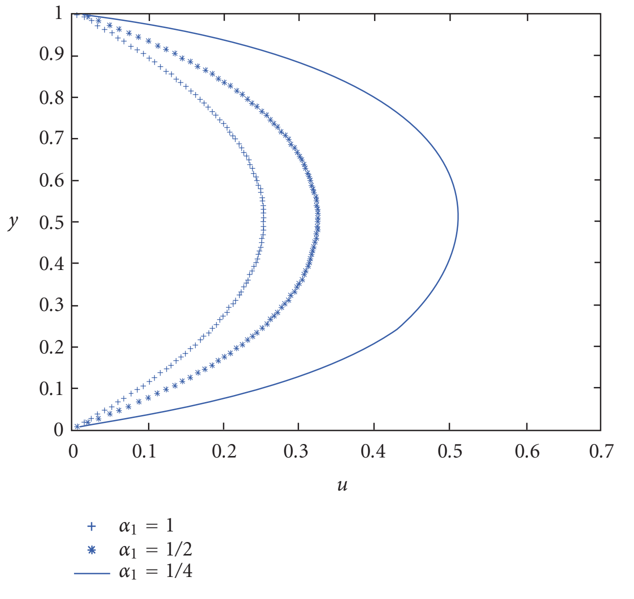

Figure 4.4. Velocity profiles for various values of α_1 and fixed $\alpha_2 = 1$ (I: Wang et al. solution; K: our solution) for $dp/dx = -2$.

Acknowledgments

The authors' thanks are due to Professor Ivar Stakgold and the referee for constructive comments which led to the definitive improvement in the paper.

References

[1] S. Baris, *Flow of an Oldroyd 6-constant fluid between intersecting planes, one of which is moving*, Acta Mechanica **147** (2001), no. 1–4, 125–135.

[2] H. A. Barnes, J. F. Hutton, and K. Walters, *An Introduction to Rheology*, Elsevier, Amsterdam, 1989.

[3] R. E. Bellman and R. E. Kalaba, *Quasilinearization and Nonlinear Boundary-Value Problems*, Modern Analytic and Computional Methods in Science and Mathematics, vol. 3, Elsevier, New York, 1965.

[4] J. G. Oldroyd, *On the formulation of rheological equations of state*, Proceedings of the Royal Society. London. Series A **200** (1950), 523–541.

[5] K. R. Rajagopal, *Mechanics of non-Newtonian fluids*, Recent Developments in Theoretical Fluid Mechanics (Paseky, 1992) (G. P. Galdi and J. Necas, eds.), Pitman Research Notes in Mathematical Series, vol. 291, Longman Scientific & Technical, Harlow, 1993, pp. 129–162.

[6] K. R. Rajagopal and R. K. Bhatnagar, *Exact solutions for some simple flows of an Oldroyd-B fluid*, Acta Mechanica **113** (1995), no. 1–4, 233–239.

[7] C. Truesdell and W. Noll, *The non-linear field theories of mechanics*, Handbuch der Physik, Band III/3 (W. Flugge, ed.), Springer, Berlin, 1965, pp. 1–602.

[8] Y. Wang, T. Hayat, and K. Hutter, *On non-linear magnetohydrodynamic problems of an Oldroyd 6-constant fluid*, International Journal of Non-Linear Mechanics **40** (2005), no. 1, 49–58.

F. Talay Akyildiz: Department of Mathematics, Arts and Science Faculty, Ondokuz Mayis University, 55139 Kurupelit Samsun, Turkey
E-mail address: akyildiz@omu.edu.tr

K. Vajravelu: Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA
E-mail address: vajravel@mail.ucf.edu