Hindawi Publishing Corporation
Differential Equations and Nonlinear Mechanics
Volume 2007, Article ID 19529, 16 pages
doi:10.1155/2007/19529

Research Article

Variational Iteration Decomposition Method for Solving Eighth-Order Boundary Value Problems

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Received 18 September 2007; Revised 10 November 2007; Accepted 11 December 2007

Recommended by Yong Zhou

We implement a relatively new analytical technique, the variational iteration decomposition method (VIDM), for solving the eighth-order boundary value problems. The proposed method is an elegant combination of variational iteration method and decomposition method. The analytical results of the equations have been obtained in terms of convergent series with easily computable components. Numerical work is given to check the efficiency of the method. Comparisons are made to confirm the reliability and accuracy of the technique. The technique can be used as an alternative for solving nonlinear boundary value problems.

1. Introduction

In this paper, we consider the general eighth-order boundary value problem of the type

$$y^{(viii)}(x)+f(x)y(x)=g(x), \quad x\in[a,b], \tag{1.1}$$

with boundary conditions

$$\begin{aligned} &y(a)=\alpha_0, \qquad y(b)=\alpha_{1,} \qquad y^{(2)}(a)=\varepsilon_0, \qquad y^{(2)}(b)=\varepsilon_1,\\ &y^{(4)}(a)=\xi_0, \qquad y^{(3)}(b)=\xi_1, \qquad y^{(6)}(a)=\sigma_0, \qquad y^{(6)}(b)=\sigma_1. \end{aligned} \tag{1.2}$$

A class of characteristic-value problems of higher order (as higher as 24) is known to arise in hydrodynamic and hydromagnetic stability [1, 2]. In addition, it is well known that when a layer of fluid is heated from below and is subject to the action of rotation, instability may set in as overstability [1, 3, 4]. This instability may be modeled by an eighth-order ordinary differential equation with appropriate boundary conditions [1, 4, 5]. For more discussion about the eighth-order boundary value problems, see [1–3, 5–7] and the references therein. The literature of numerical analysis contains little on the solution of the eighth-order boundary value problems [6]. Research in this direction may be considered in its early stages. Theorems which list the conditions for the existence and uniqueness of solutions of such problems are contained in a comprehensive survey by Agarwal [8].

The boundary value problems of higher order have been investigated because of both of their mathematical importance and the potential for applications in hydrodynamic and hydromagnetic stability. Finite-difference method was employed in [2, 6] to find the solution of eighth-order boundary value problems. The obtained results were divergent at points adjacent to the boundary. In a later study, Siddiqi and Twizell [4] used octic polynomial spline for solving these problems. Twizell et al. [2, 4, 6] also solved some other higher-order problems and encountered the same deficiencies. The divergent results are due to the use of lower-order test function in the spline methods. The spline function values at the mid knots of the interpolation interval and the corresponding values of the even-order derivatives are related through consistency relations. However, the performance of the techniques used so far is well known that it provides the solution at grid points only. Modified Adomian decomposition method was used in [5] to find the analytical solution of linear and nonlinear boundary value problems of eighth order. Recently, Noor and Mohyud-Din applied homotopy perturbation method [9–15] and variational iteration method [9, 16–22] for solving higher-orders boundary value problems, see [3, 21–25] and the references therein. The obtained results were compared with the exact solutions. Inspired and motivated by the ongoing research in this area, we apply the variational iteration decomposition method (VIDM) to find solutions of eighth-order boundary value problems. It is worth mentioning that our proposed technique can handle any boundary value problem with a set of boundary conditions defined at any order derivatives and is an elegant combination of variational iteration method and decomposition method.

He [9, 16–19] developed the variational iteration method for solving linear, nonlinear, initial, and boundary value problems. It is worth mentioning that method was first considered by Inokuti et al. [20]. Since the beginning of 1980s, the Adomian decomposition method has been applied to a wide class of functional equations [5, 26–28]. In these methods, the solution is given in an infinite series usually converging to an accurate solution, see [5, 26–28] and the references therein. In this paper, we apply the variational iteration decomposition method (VIDM) which is an elegant combination of variational iteration method and the Adomian's decomposition method to solve eighth-order boundary value problems. This idea has been used by Abbasbandy [29, 30] for solving quadratic Riccati differential equation and Klein-Gordon equation. The basic motivation of this paper is to apply the variational iteration decomposition method (VIDM) for solving eighth-order boundary value problems. It is shown that the variational iteration decomposition method provides the solution in a rapid convergent series. To make the implementation of the proposed method simpler, we first rewrite eighth-order boundary value problem in an equivalent system of integral equations using a suitable transformation. This alternate transformation plays a pivotal and fundamental role in solving the boundary value problems. We use the VIDM to solve equivalent system of integral equations efficiently. The VIDM solves effectively, easily, and accurately a large class of linear, nonlinear, partial, deterministic, or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions. Several examples are given to illustrate the reliability and performance of the proposed method. We would like to emphasize that the VIDM may be considered as an important and significant improvement of the already developed methods.

2. Variational iteration method

To illustrate the basic concept of the technique, we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{2.1}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the forcing term. According to variational iteration method [9, 16–20], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\big(Lu_n(s) + N\tilde{u}_n(s) - g(s)\big)\,ds, \tag{2.2}$$

where λ is a Lagrange multiplier [16–20], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta\tilde{u}_n = 0$; (2.2) is called as a correct functional.

The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier.

The principles of variational iteration method and its applicability for various kinds of differential equations are given [16–19]. For the sake of simplicity and to convey the idea of the technique, we consider the following system of differential equations:

$$x_i'(t) = f_i(t, x_i), \quad i = 1, 2, 3, \dots, n, \tag{2.3}$$

subject to the boundary conditions

$$x_i(0) = c_i, \quad i = 1, 2, 3, \dots, n. \tag{2.4}$$

To solve the system by means of the variational iteration method, we rewrite the system (2.3) in the following form:

$$x_i'(t) = f_i(x_i) + g_i(t), \quad i = 1, 2, 3, \dots, n, \tag{2.5}$$

subject to the boundary conditions $x_i(0) = c_i$, $i = 1, 2, 3, \dots, n$ and g_i is defined in (2.1)

The correct functional for the nonlinear system (2.3) can be approximated as

$$
\begin{aligned}
x_1^{(k+1)}(t) &= x_1^{(k)}(t) + \int_0^t \lambda_1\left(x_1'^{(k)}(T), f_1\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)\right) - g_1(T)\right) dT, \\
x_2^{(k+1)}(t) &= x_2^{(k)}(t) + \int_0^t \lambda_2\left(x_2'^{(k)}(T), f_2\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)\right) - g_2(T)\right) dT, \\
&\vdots \\
x_n^{(k+1)}(t) &= x_n^{(k)}(t) + \int_0^t \lambda_n\left(x_n'^{(k)}(T), f_n\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)\right) - g_n(T)\right) dT,
\end{aligned}
\tag{2.6}
$$

where $\lambda_i = \pm 1$, $i = 1, 2, 3, \ldots, n$, are Lagrange multipliers, $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ denote the restricted variations.

For $\lambda_i = -1$, $i = 1, 2, 3, \ldots, n$, we have the following iterative schemes:

$$
\begin{aligned}
x_1^{(k+1)}(t) &= x_1^{(k)}(t) - \int_0^t \left(x_1'^{(k)}(T), f_1\left(x_1^{(k)}(T), x_2^{(k)}(T), \ldots, x_n^{(k)}(T)\right) - g_1(T)\right) dT, \\
x_2^{(k+1)}(t) &= x_2^{(k)}(t) - \int_0^t \left(x_2'^{(k)}(T), f_2\left(x_1^{(k)}(T), x_2^{(k)}(T), \ldots, x_n^{(k)}(T)\right) - g_2(T)\right) dT, \\
&\vdots \\
x_n^{(k+1)}(t) &= x_n^{(k)}(t) - \int_0^t \left(x_n'^{(k)}(T), f_n\left(x_1^{(k)}(T), x_2^{(k)}(T), \ldots, x_n^{(k)}(T)\right) - g_n(T)\right) dT.
\end{aligned}
\tag{2.7}
$$

If we start with the initial approximations $x_i(0) = c_i$, $i = 1, 2, 3, \ldots, n$, then the approximations can be completely determined; finally we approximate the solution

$$
x_i(t) = \lim_{k\to\infty} x_i^{(n)} \text{ by the } n\text{th term } x_i^{(n)}(t) \quad \text{for } i = 1, 2, 3, \ldots, n. \tag{2.8}
$$

3. Adomian's decomposition method

Consider the differential equation [5, 26–28]

$$
Lu + Ru + Nu = g, \tag{3.1}
$$

where L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of lesser order than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of (3.1) and using the given conditions, we obtain

$$u = f - L^{-1}(Ru) - L^{-1}(Nu), \tag{3.2}$$

where the function f represents the terms arising from integrating the source term g and by using the given conditions. Adomian's decomposition method [5, 26–28] defines the solution $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{3.3}$$

where the components $u_n(x)$ are usually determined recurrently by using the relation

$$\begin{aligned} u_0 &= f, \\ u_{k+1} &= L^{-1}(Ru_k) - L^{-1}(Nu_k), \quad k \geq 0. \end{aligned} \tag{3.4}$$

The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n, \tag{3.5}$$

where A_n are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in [5, 26–28] which yields

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right) F\left(\sum_{i=0}^{n} (\lambda^i u_i)\right)_{\lambda=0}, \quad n = 0, 1, 2, \ldots. \tag{3.6}$$

For further details about the Adomian's decomposition method, see [5, 26–28] and the references therein.

4. Variational iteration decomposition method (VIDM)

To illustrate the basic concept of the variational iteration decomposition method, we consider the following general differential (2.1):

$$Lu + Nu = g(x), \tag{4.1}$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ is the forcing term. According to variational iteration method [9, 16–22, 25], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left(Lu_n(s) + N\tilde{u}_n(s) - g(s)\right) ds, \tag{4.2}$$

where λ is a Lagrange multiplier [16–20], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, that is, $\delta\tilde{u}_n = 0$; (2.2) is called as a correct functional. We define the solution $u(x)$ by the series

$$u(x) = \sum_{i=0}^{\infty} u^{(i)}(x), \tag{4.3}$$

and the nonlinear term

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_i), \tag{4.4}$$

where A_n are the so-called Adomian's polynomials and can be generated for all types of nonlinearities according to the algorithm developed in [5, 26–28] which yields

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right)F(u(\lambda))\Big|_{\lambda=0}, \tag{4.5}$$

or equivalently,

$$\begin{aligned}
A_0 &= F(u_0),\\
A_1 &= u_1 F'(u_0),\\
A_2 &= u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0),\\
A_3 &= u_3 F'(u_0) + u_2 u_1 F''(u_0) - \frac{1}{3!}u_1^3 F'''(u_0),\\
A_4 &= u_4 F'(u_0) + \left(\frac{1}{2!}u_2^2 + u_3 u_1\right)F''(u_0) - \frac{1}{2!}u_1^2 u_2 F'''(u_0) + \frac{1}{4!}u_1^4 F^{(iv)}(u_0),\\
A_5 &= u_5 F'(u_0) + (u_2 u_3 + u_4 u_1)F''(u_0) + \left(\frac{1}{2!}u_1 u_2^2 + \frac{1}{2!}u_3 u_1^2\right)F'''(u_0)\\
&\quad - \frac{1}{3!}u_1^3 u_2 F^{(iv)}(u_0) + \frac{1}{5!}u_1^5 F^{(v)}(u_0)\\
&\quad\vdots
\end{aligned} \tag{4.6}$$

Hence, we obtain the following iterative scheme for finding the approximate solution:

$$u^{(n+1)}(x) = u^{(n)}(x) + \int_0^t \lambda\left(Lu^{(n)}(x) + \sum_{n=0}^{\infty} A_n - g(x)\right)dx. \tag{4.7}$$

This method is called as the variational iteration decomposition method (VIDM) and may be viewed as an important and significant improvement as compared with other similar methods.

5. Numerical applications

In this section, we first rewrite that the eighth-order boundary value problem is an equivalent system of integral equations by using a suitable transformation. The variational iteration decomposition method (VIDM) is applied to solve the resultant system of integral equations. The proposed method is an elegant combination of the variational iteration method and Adomian's decomposition method.

Example 5.1 [3, 5]. Consider the nonlinear boundary value problem of eighth-order as

$$y^{(viii)}(x) = e^{-x}y^2(x), \quad 0 < x < 1, \tag{5.1}$$

with boundary conditions

$$y(0) = y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = 1, \qquad y(1) = y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = e. \tag{5.2}$$

The exact solution is given by

$$y(x) = e^x. \tag{5.3}$$

Using the transformation

$$\begin{gathered}\frac{dy}{dx} = a(x), \qquad \frac{da}{dx} = b(x), \qquad \frac{db}{dx} = e(x), \\ \frac{de}{dx} = f(x), \qquad \frac{df}{dx} = g(x), \qquad \frac{dg}{dx} = h(x), \qquad \frac{dh}{dx} = z(x),\end{gathered} \tag{5.4}$$

we obtain the following system of differential equations:

$$\begin{gathered}\frac{dy}{dx} = a(x), \qquad \frac{da}{dx} = b(x), \qquad \frac{db}{dx} = e(x), \qquad \frac{de}{dx} = f(x), \\ \frac{df}{dx} = g(x), \qquad \frac{dg}{dx} = h(x), \qquad \frac{dh}{dx} = z(x), \qquad \frac{dz}{dx} = e^{-x}y^2(x).\end{gathered} \tag{5.5}$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1,\ i = 1,2,\dots,8$:

$$\begin{aligned}
y^{(k+1)}(x) &= 1 + \int_0^x a^{(k)}(x)dx, & a^{(k+1)}(x) &= A + \int_0^x b^{(k)}(x)\,dx, \\
b^{(k+1)}(x) &= 1 + \int_0^x e^{(k)}(x)dx, & e^{(k+1)}(x) &= B + \int_0^x f^{(k)}(x)\,dx, \\
f^{(k+1)}(x) &= 1 + \int_0^x g^{(k)}(x)dx, & g^{(k+1)}(x) &= C + \int_0^x h^{(k)}(x)\,dx, \\
h^{(k+1)}(x) &= 1 + \int_0^x z^{(k)}(x)dx, & z^{(k+1)}(x) &= D + \int_0^x e^{-x}\sum_{n=0}^{\infty} A_n\,dx,
\end{aligned} \tag{5.6}$$

where A_n are Adomian polynomials for nonlinear operator $F(y) = y^2(x)$ and can be generated for all types of nonlinearities according to the algorithm developed in [5, 26–28] which yields

$$
\begin{aligned}
A_0 &= F(y_0),\\
A_0 &= y_0^2(x),\\
A_1 &= y_1(x)F'(y_0),\\
A_1 &= 2y_0(x)y_1(x),\\
A_2 &= y_2F'(y_0) + \frac{y_1^2}{2!}F''(y_0),\\
A_2 &= 2y_0(x)y_2(x) + y_1^2(x)\\
&\vdots
\end{aligned}
\tag{5.7}
$$

Consequently, we obtain the following approximants:

$$
\begin{gathered}
y^{(0)}(x) = 1, \qquad a^{(0)}(x) = A, \qquad b^{(0)}(x) = 1, \qquad e^{(0)}(x) = B,\\
f^{(0)}(x) = 1, \qquad g^{(0)}(x) = C, \qquad h^{(0)}(x) = 1, \qquad z^{(0)}(x) = D,\\
y^{(1)}(x) = 1 + Ax, \qquad a^{(1)}(x) = A + x,\\
b^{(1)}(x) = 1 + Bx, \qquad e^{(1)}(x) = B + x,\\
f^{(1)}(x) = 1 + Cx, \qquad g^{(1)}(x) = C + x,\\
h^{(1)}(x) = 1 + Dx, \qquad z^{(1)}(x) = 1 - e^{-x},\\
y^{(2)}(x) = 1 + Ax + \frac{1}{2}x^2, \qquad a^{(2)}(x) = A + x + \frac{1}{2}Bx^2,\\
b^{(2)}(x) = 1 + Bx + \frac{1}{2}x^2, \qquad e^{(2)}(x) = B + x + \frac{1}{2}Cx^2,\\
f^{(2)}(x) = 1 + Cx + \frac{1}{2}x^2, \qquad g^{(2)}(x) = C + x\frac{1}{2}Dx^2,\\
h^{(2)}(x) = 1 + Dx - 1 + x + e^{-x}, \qquad z^{(2)}(x) = 1 - e^{-x} + A^2\left(2 + 2x + x^2 - 3e^{-x} - x^2e^{-x}\right),\\
y^{(3)}(x) = 1 + Ax + \frac{1}{2}x^2 + \frac{1}{3!}Bx^3, \qquad a^{(3)}(x) = A + x + \frac{1}{2}Bx^2 + \frac{1}{3!}x^3,\\
b^{(3)}(x) = 1 + Bx + \frac{1}{2}x^2 + \frac{1}{3!}Cx^3, \qquad e^{(3)}(x) = B + x + \frac{1}{2}Cx^2 + \frac{1}{3!}x^3,\\
f^{(3)}(x) = 1 + Cx + \frac{1}{2}x^2 + \frac{1}{3!}Dx^3, \qquad g^{(3)}(x) = C + x\frac{1}{2}Dx^2 + 1 - x + \frac{1}{2}x^2 - e^{-x}\\
\vdots
\end{gathered}
\tag{5.8}
$$

The series solution is given by

$$\begin{aligned}
y(x) = {} & 1+Ax-\frac{1}{2!}x^2+\frac{1}{6}Bx^3-\frac{1}{24}x^4+\frac{1}{120}Cx^5-\frac{1}{720}x^6+\frac{1}{5040}Dx^7 \\
& +\frac{1}{40320}x^8+\left(\frac{1}{18144}A-\frac{1}{362880}\right)x^9+\left(-\frac{1}{907200}A+\frac{1}{1209600}\right)x^{10} \\
& +\left(\frac{1}{1995840}B+\frac{1}{6652800}A+\frac{1}{5702400}\right)x^{11} \\
& +\left(-\frac{1}{59875200}A-\frac{1}{59875200}B+\frac{1}{31933440}\right)x^{12}+O(x^{13}).
\end{aligned} \tag{5.9}$$

Imposing the boundary conditions at $x=1$ leads to the following system of equations:

$$\begin{bmatrix}
\frac{3742317}{3742200} & \frac{4989601}{29937600} & \frac{1}{120} & \frac{1}{5040} \\
\frac{283}{907200} & \frac{302401}{302400} & \frac{1}{6} & \frac{1}{120} \\
\frac{61}{5040} & \frac{1}{5040} & 1 & \frac{1}{6} \\
\frac{37}{180} & \frac{1}{180} & 0 & \frac{37}{180}
\end{bmatrix}
\begin{bmatrix} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \end{bmatrix}
=
\begin{bmatrix}
e-\frac{246378989}{159667200} \\
e-\frac{1119787}{725760} \\
e-\frac{61951}{40320} \\
e-\frac{341}{240}
\end{bmatrix}. \tag{5.10}$$

The solution of the above algebraic system gives

$$A=0.999870193, \qquad B=1.001257423, \qquad C=0.988438914, \qquad D=1.086357080. \tag{5.11}$$

Consequently, the series solution is given as

$$\begin{aligned}
y(x) = {} & 1+0.999870193x+\frac{1}{2}x^2+0.1668762372x^3+\frac{1}{24}\mathrm{x}^4 \\
& +0.00823699095x^5+\frac{1}{720}x^6+0.000215547x^7 \\
& +\frac{1}{40320}x^8+2.755\times10^{-6}x^9-2.75\times10^{-7}x^{10}+2.51 \\
& \times10^{-8}x^{11}-2.1\times10^{-9}x^{12}+O(x^{13}),
\end{aligned} \tag{5.12}$$

which is exactly the same as obtained in [3] by using homotopy perturbation method and in [5] by modified Adomian's decomposition method.

Table 5.1 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration decomposition method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $y(x)$.

Table 5.1. Error estimates.

x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.000000
0.1	1.105170918	1.105158145	1.27E-5
0.2	1.221402758	1.221378444	2.43E-5
0.3	1.349858808	1.349825294	3.35E-5
0.4	1.491824698	1.491 785 229	3.94E-5
0.5	1.648721271	1.648679687	4.16E-5
0.6	1.822118800	1.822079168	3.96E-5
0.7	2.013752707	2.013718927	3.38E-5
0.8	2.225540928	2.225516346	2.45E-5
0.9	2.459603111	2.459590174	1.29E-5
1.0	2.718281828	2.718281829	1.00E-9

*Error = Exact solution-Series solution.

Example 5.2 [3, 5]. References Consider the following linear boundary value problem of eighth order:

$$y^{(viii)}(x) = -8xe^x + y(x), \qquad 0 < x < 1, \tag{5.13}$$

with boundary conditions

$$\begin{aligned} &y(0) = 1, \quad y''(0) = -1, \quad y^{(iv)}(0) = -3, \quad y^{(vi)}(0) = -5, \\ &y(1) = 0, \quad y''(1) = -2e, \quad y^{(iv)}(1) = -4e, \quad y^{(vi)}(1) = -6e. \end{aligned} \tag{5.14}$$

The exact solution of the problem is

$$y(x) = (1-x)e^x. \tag{5.15}$$

Using the transformation

$$\begin{aligned} &\frac{dy}{dx} = a(x), \quad \frac{da}{dx} = b(x), \quad \frac{db}{dx} = e(x), \quad \frac{de}{dx} = f(x), \\ &\frac{df}{dx} = g(x), \quad \frac{dg}{dx} = h(x), \quad \frac{dh}{dx} = z(x), \end{aligned} \tag{5.16}$$

we obtain the following system of differential equations:

$$\begin{aligned} &\frac{dy}{dx} = a(x), \quad \frac{da}{dx} = b(x), \quad \frac{db}{dx} = e(x), \quad \frac{de}{dx} = f(x), \\ &\frac{df}{dx} = g(x), \quad \frac{dg}{dx} = h(x), \quad \frac{dh}{dx} = z(x), \quad \frac{dz}{dx} = -8xe^x + y(x), \end{aligned} \tag{5.17}$$

with boundary conditions

$$\begin{aligned} &y(0) = 1, \quad a(0) = A, \quad b(0) = -1, \quad e(0) = B, \\ &f(0) = -3, \quad g(0) = C, \quad h(0) = -5, \quad z(0) = D. \end{aligned} \tag{5.18}$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1,\ i = 1,2,\ldots,8$:

$$\begin{aligned}
&y^{(k+1)}(x) = 1 + \int_0^x a^{(k)}(x)dx, && a^{(k+1)}(x) = A + \int_0^x b^{(k)}(x)dx,\\
&b^{(k+1)}(x) = -1 + \int_0^x e^{(k)}(x)dx, && e^{(k+1)}(x) = B + \int_0^x f^{(k)}(x)dx,\\
&f^{(k+1)}(x) = -3 + \int_0^x g^{(k)}(x)dx, && g^{(k+1)}(x) = C + \int_0^x h^{(k)}(x)dx,\\
&h^{(k+1)}(x) = -5 + \int_0^x z^{(k)}(x)dx, && z^{(k+1)}(x) = D + \int_0^x (-8xe^x + y^{(k)}(x)dx.
\end{aligned} \tag{5.19}$$

Consequently, we obtain the following approximants:

$$\begin{aligned}
y^{(0)}(x) &= 1,\\
a^{(0)}(x) &= A,\\
b^{(0)}(x) &= -1,\\
e^{(0)}(x) &= B,\\
f^{(0)}(x) &= -3,\\
g^{(0)}(x) &= C,\\
h^{(0)}(x) &= -5,\\
z^{(0)}(x) &= D,\\
y^{(1)}(x) &= 1 + Ax,\\
a^{(1)}(x) &= A - x,\\
b^{(1)}(x) &= -1 + Bx,\\
e^{(1)}(x) &= B - 3x,\\
f^{(1)}(x) &= -3 + Cx,\\
g^{(1)}(x) &= C - 5x,\\
h^{(1)}(x) &= -5 + Dx,\\
z^{(1)}(x) &= D - 8 + x + 8e^x - 8xe^x,\\
y^{(2)}(x) &= 1 + Ax - \frac{1}{2}x^2,\\
a^{(2)}(x) &= A - x + \frac{1}{2}Bx^2,\\
b^{(2)}(x) &= -1 + Bx - \frac{3}{2}x^2,\\
e^{(2)}(x) &= B - 3x + \frac{1}{2}Cx^2,
\end{aligned}$$

$$
\begin{aligned}
f^{(2)}(x) &= -3 + Cx - \frac{5}{2}x^2, \\
g^{(2)}(x) &= C - 5x + \frac{1}{2}Dx^2, \\
h^{(2)}(x) &= -5 + Dx - 16 + 8x + \frac{1}{2}x^2 + 16e^x - 8xe^x, \\
z^{(2)}(x) &= D - 8 + x + 8e^x - 8xe^x - 8 + \frac{1}{2}Ax^2 + 8e^x - 8xe^x, \\
y^{(3)}(x) &= 1 + Ax - \frac{1}{2}x^2 + \frac{1}{3!}Bx^3, \\
a^{(3)}(x) &= A - x + \frac{1}{2}Bx^2 - \frac{3}{3!}x^3, \\
b^{(3)}(x) &= -1 + Bx - \frac{3}{2}x^2 + \frac{1}{3!}Cx^3, \\
e^{(3)}(x) &= B - 3x + \frac{1}{2}Cx^2 - \frac{5}{3!}x^3, \\
f^{(3)}(x) &= -3 + Cx - \frac{5}{2}x^2 + \frac{1}{3!}Dx^3, \\
g^{(3)}(x) &= C - 5x + \frac{1}{2}Dx^2 - 24 - 16x + \frac{8}{2!}x^2 + \frac{1}{3!}x^3 + 24e^x - 8xe^x, \\
h^{(3)}(x) &= -5 + Dx - 16 + 8x + \frac{1}{2}x^2 + 16e^x - 8xe^x - 16 - 8x + \frac{1}{3!}Ax^3 + 16e^x - 8xe^x, \\
z^{(3)}(x) &= D - 8 + x + 8e^x - 8xe^x - 8 + \frac{1}{2}Ax^2 + 8e^x - 8xe^x - 8 - \frac{1}{3!}x^3 + 8e^x - 8xe^x, \\
y^{(4)}(x) &= 1 + Ax - \frac{1}{2}x^2 + \frac{1}{3!}Bx^3 - \frac{3}{4!}x^4, \\
a^{(4)}(x) &= A - x + \frac{1}{2}Bx^2 - \frac{3}{3!}x^3 + \frac{1}{4!}Cx^4, \\
b^{(4)}(x) &= -1 + Bx - \frac{3}{2}x^2 + \frac{1}{3!}Cx^3 + \frac{1}{4!}Dx^4, \\
e^{(4)}(x) &= B - 3x + \frac{1}{2}Cx^2 - \frac{5}{3!}x^3, \\
f^{(4)}(x) &= -3 + Cx - \frac{5}{2}x^2 + \frac{1}{3!}Dx^3 - 32 - 24x - \frac{16}{2!}x^2 + \frac{8}{3!}x^3 + \frac{1}{4!}x^4 + 32e^x - 8xe^x, \\
g^{(4)}(x) &= C - 5x + \frac{1}{2}Dx^2 - 24 - 16x + \frac{8}{2!}x^2 + \frac{1}{3!}x^3 + 24e^x - 8xe^x \\
&\quad - 24 - 16x - \frac{8}{2!}x^2 + \frac{1}{4!}Ax^4 + 24e^x - 8xe^x, \\
h^{(4)}(x) &= -5 + Dx - 16 + 8x + \frac{1}{2}x^2 + 16e^x - 8xe^x - 16 - 8x + \frac{1}{3!}Ax^3 \\
&\quad + 16e^x - 8xe^x - 16 - 8x - \frac{1}{4!}x^4 + 16e^x - 8xe^x, \\
z^{(4)}(x) &= D - 8 + x + 8e^x - 8xe^x - 8 + \frac{1}{2}Ax^2 + 8e^x - 8xe^x - 8 \\
&\quad - \frac{1}{3!}x^3 + 8e^x - 8xe^x - 8 + 8e^x - 8xe^x + \frac{1}{4!}Bx^4 \\
&\vdots
\end{aligned}
\tag{5.20}
$$

The series solution is given by

$$\begin{aligned} y(x) &= 1 + Ax - \frac{1}{2}x^2 + \frac{1}{6}Bx^3 - \frac{1}{8}x^4 + \frac{1}{120}Cx^5 + \frac{1}{144}x^6 \\ &\quad + \frac{1}{5040}Dx^7 - \frac{1}{5760}x^8 + \left(-\frac{1}{45360} + \frac{1}{362880}A\right)x^9 \\ &\quad - \frac{1}{403200}x^{10} + \left(-\frac{1}{4989600} + \frac{1}{39916800}B\right)x^{11} - \frac{1}{43545600}x^{12} + O(x^{13}). \end{aligned} \tag{5.21}$$

Imposing the boundary conditions at $x = 1$ leads to the following system of equations:

$$\begin{bmatrix} \frac{362881}{362880} & \frac{6652801}{39916800} & \frac{51891841}{6227020800} & \frac{1}{5040} \\ \frac{1}{5040} & \frac{362881}{362880} & \frac{6552801}{39916800} & \frac{1}{120} \\ \frac{1}{120} & \frac{1}{5040} & \frac{362881}{362880} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{120} & \frac{1}{5040} & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -\frac{2290654397}{6227020800} \\ -2e + \frac{108569359}{39916800} \\ -4e + \frac{2131091}{362880} \\ -6e + \frac{51871}{5040} \end{bmatrix}. \tag{5.22}$$

The solution of above system gives

$$\begin{aligned} A &= 6.771 \times 10^{-7}, & B &= -2.000006476, \\ C &= -3.99994303, & D &= -6.00036565. \end{aligned} \tag{5.23}$$

The series solution is given as

$$\begin{aligned} y(x) &= 1 - 6.771 \times 10^{-7}x - 0.50x^2 - .3333344127x^3 - \frac{1}{8}x^4 \\ &\quad - .033332858585x - \frac{1}{144}x^6 - .00119054874x^7 \\ &\quad - \frac{1}{5040}x^8 - 2.205 \times 10^{-5}x^9 - \frac{1}{403200}x^{10}2.505 \\ &\quad \times 10^{-7}x^{11} - \frac{1}{43545600}x^{12} + O(x^{13}), \end{aligned} \tag{5.24}$$

which is exactly the same as obtained in [3] by using homotopy perturbation method and in [5] by modified Adomian's decomposition method.

Table 5.2 exhibits the exact solution and the series solution along with the errors obtained by using the VIDM. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $y(x)$.

6. Conclusion

In this paper, we have used the variational iteration decomposition method (VIDM) which is mainly due to Abbasbandy for finding the solution of linear and nonlinear

Table 5.2. Error estimates.

x	Exact solution	Series solution	*Errors
0.0	1.00000000	1.0000000000	0.000000
0.1	0.99465383	0.9946538933	−6.71E-8
0.2	0.97712221	0.9771223332	−1.27E-7
0.3	0.94490117	0.9449013404	−1.75E-7
0.4	0.89509482	0.8950950252	−2.06E-7
0.5	0.82436064	0.8243608537	−2.18E-7
0.6	0.72884752	0.7288477280	−2.08E-7
0.7	0.60412581	0.6041259899	−1.78E-7
0.8	0.44510819	0.4451083155	−1.29E-7
0.9	0.24596031	0.2459603788	−6.77E-8
1.0	0.00000000	0.0000000000	0.000000

*Error = Exact Solution-Series Solution.

boundary value problems for eighth order. The method is used in a direct way without using linearization, perturbation, or restrictive assumptions. It may be concluded that VIDM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. Thus, we conclude that the variational iteration decomposition technique can be considered as an efficient method for solving linear and nonlinear problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result, the size reduction amounts to the improvement of performance of approach. This method is relatively new and may lead to some novel and innovative applications in solving linear and nonlinear problems.

Acknowledgments

The authors are highly grateful to both the referees and Professor Dr. Yong Zhou for their constructive comments. We would like to thank Dr. S. M. Junaid Zaidi, Rector CIIT for providing excellent research environment and facilities.

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