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Research Article

Approximate Traveling Wave Solutions of Coupled Whitham-Broer-Kaup Shallow Water Equations by Homotopy Analysis Method

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The homotopy analysis method (HAM) is applied to obtain the approximate traveling wave solutions of the coupled Whitham-Broer-Kaup (WBK) equations in shallow water. Comparisons are made between the results of the proposed method and exact solutions. The results show that the homotopy analysis method is an attractive method in solving the systems of nonlinear partial differential equations.

1. Introduction

In 1992, Liao [1] employed the basic ideas of the homotopy in topology to propose method for nonlinear problems, namely, homotopy analysis method (HAM), [2–6]. This method has many advantages over the classical methods; mainly, it is independent of any small or large quantities. So, the HAM can be applied no matter if governing equations and boundary/initial conditions contain small or large quantities or not. The HAM also avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculation, and avoidance of physically unrealistic assumptions. Furthermore, the HAM always provides us with a family of solution expressions in the auxiliary parameter h; the convergence region and rate of each solution might be determined conveniently by the auxiliary parameter h. This method has been successfully applied to solving many types of nonlinear problems [7–11].

A substantial amount of research work has been invested in the study of linear and nonlinear systems of partial differential equations (PDEs). Systems of nonlinear partial differential equations arise in many scientific models such as the propagation of shallow water waves and the Brusselator model of the chemical reaction-diffusion model.

Here, we consider the coupled Whitham-Broer-Kaup (WBK) equations which have been studied by Whitham [12], Broer [13], and Kaup [14]. The equations describe the propagation of shallow water waves, with different dispersion relations. The WBK equations are as follows:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\beta\frac{\partial^2 u}{\partial x^2}=0,$$
$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}(uv)+\alpha\frac{\partial^3 u}{\partial x^3}-\beta\frac{\partial^2 v}{\partial x^2}=0, \tag{1.1}$$

where $u=u(x,t)$ is the horizontal velocity, $v=v(x,t)$ is the height that deviates from equilibrium position of the liquid, and α, β are constants which are represented in different diffusion powers [15]. The exact solutions of $u(x,t)$ and $v(x,t)$ are given by [16]

$$u(x,t)=\lambda-2k\left(\alpha+\beta^2\right)^{0.5}\coth\left[k\left(x+x_0\right)-\lambda t\right],$$
$$v(x,t)=-2k^2\left(\alpha+\beta^2+\beta\left(\alpha+\beta^2\right)^{0.5}\right)\operatorname{csch}^2\left[k\left(x+x_0\right)-\lambda t\right], \tag{1.2}$$

where λ, k, and x_0 are arbitrary constants. Above system is a very good model to describe dispersive waves. If $\alpha=1$ and $\beta=0$, then the system represents the modified Boussinesq (MB) equations [16]. If $\alpha=0$ and $\beta\neq 0$, then the system represents the classical long-wave equations that describe shallow water wave with dispersion [15].

This Letter has been organized as follows. In Section 2, the basic concept of the HAM is introduced. In Section 3, we extend the application of the HAM to construct approximate solutions for the coupled WBK equations. Numerical experiments are presented in Section 4.

2. Basic concepts of HAM

Let us consider the following differential equation:

$$\mathcal{N}\left[w(\tau)\right]=0, \tag{2.1}$$

where $\mathcal{N}$ is a nonlinear operator; τ denotes independent variable; $w(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [5] constructs the so-called *zero-order deformation equation*:

$$(1-p)\mathcal{L}\left[\varphi(\tau;p)-w_0(\tau)\right]=ph\mathcal{N}\left[\varphi(\tau;p)\right], \tag{2.2}$$

where $p\in[0,1]$ is the embedding parameter; $h\neq 0$ is a nonzero auxiliary parameter; $\mathcal{L}$ is an auxiliary linear operator; $w_0(\tau)$ is an initial guess of $w(\tau)$; $\varphi(\tau;p)$ is an unknown function, respectively. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\varphi(\tau;0)=w_0(\tau),\qquad \varphi(\tau;1)=w(\tau), \tag{2.3}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(\tau;p)$ varies from the initial guess $w_0(\tau)$ to the solution $w(\tau)$. Expanding $\varphi(\tau;p)$ in Taylor series with respect to p, we have

$$\varphi(\tau;p) = w_0(\tau) + \sum_{m=1}^{+\infty} w_m(\tau)p^m, \tag{2.4}$$

where

$$w_m(\tau) = \frac{1}{m!}\frac{\partial^m \varphi(\tau;p)}{\partial p^m}\bigg|_{p=0}. \tag{2.5}$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameter h are so properly chosen, the series (2.4) converges at $p = 1$, then we have

$$w(\tau) = w_0(\tau) + \sum_{m=1}^{+\infty} w_m(\tau), \tag{2.6}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [5]. As $h = -1$, (2.2) becomes

$$(1-p)\mathcal{L}\left[\varphi(\tau;p) - w_0(\tau)\right] + p\mathcal{N}\left[\varphi(\tau;p)\right] = 0, \tag{2.7}$$

which is used mostly in the homotopy perturbation method, whereas the solution is obtained directly, without using Taylor series [17, 18].

According to definition (2.5), the governing equation can be deduced from the *zero-order deformation* (2.2). Define the vector

$$\vec{w}_n = \left\{w_0(\tau), w_1(\tau), \ldots, w_n(\tau)\right\}. \tag{2.8}$$

Differentiating (2.2) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-*order deformation equation*:

$$\mathcal{L}\left[w_m(\tau) - \chi_m w_{m-1}(\tau)\right] = hR_m(\vec{w}_{m-1}), \tag{2.9}$$

where

$$R_m(\vec{w}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}\left[\varphi(\tau;p)\right]}{\partial p^{m-1}}\bigg|_{p=0},$$
$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{2.10}$$

It should be emphasized that $w_m(\tau)$ for $m \ge 1$ is governed by the linear (2.9) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

3. Application

First, we consider the coupled Whitham-Broer-Kaup (WBK) equations (1.1), with the initial conditions

$$u(x,0) = \lambda - 2k(\alpha+\beta^2)^{0.5}\coth\left[k(x+x_0)\right],$$
$$v(x,0) = -2k^2\left(\alpha+\beta^2+\beta(\alpha+\beta^2)^{0.5}\right)\operatorname{csch}^2\left[k(x+x_0)\right]. \tag{3.1}$$

For application of the homotopy analysis method, we choose the initial approximations

$$u_0(x,t) = u(x,0) = \lambda - 2k(\alpha+\beta^2)^{0.5}\coth\left[k(x+x_0)\right],$$
$$v_0(x,t) = v(x,0) = -2k^2\left(\alpha+\beta^2+\beta(\alpha+\beta^2)^{0.5}\right)\operatorname{csch}^2\left[k(x+x_0)\right], \tag{3.2}$$

and the linear operator

$$\mathcal{L}\left[\varphi(x,t;p)\right] = \frac{\partial\varphi(x,t;p)}{\partial t} \tag{3.3}$$

with the property

$$\mathcal{L}(c) = 0, \tag{3.4}$$

where c is constant. From (1.1), we define a system of nonlinear operators as

$$\mathcal{N}_1\left[\varphi_1(x,t;p),\ \varphi_2(x,t;p)\right] = \frac{\partial\varphi_1(x,t;p)}{\partial t} + \varphi_1(x,t;p)\frac{\partial\varphi_1(x,t;p)}{\partial x} + \frac{\partial\varphi_2(x,t;p)}{\partial x} + \beta\frac{\partial^2\varphi_1(x,t;p)}{\partial x^2},$$
$$\mathcal{N}_2\left[\varphi_1(x,t;p),\ \varphi_2(x,t;p)\right] = \frac{\partial\varphi_2(x,t;p)}{\partial t} + \frac{\partial}{\partial x}\left(\varphi_1(x,t;p)\,\varphi_2(x,t;p)\right) + \alpha\frac{\partial^3\varphi_1(x,t;p)}{\partial x^3} - \beta\frac{\partial^2\varphi_2(x,t;p)}{\partial x^2}. \tag{3.5}$$

Using the above definition, we construct the *zero-order deformation equations*:

$$(1-p)\mathcal{L}\left[\varphi_1(x,t;p) - u_0(x,t)\right] = ph_1\mathcal{N}_1\left[\varphi_1(x,t;p),\varphi_2(x,t;p)\right],$$
$$(1-p)\mathcal{L}\left[\varphi_2(x,t;p) - v_0(x,t)\right] = ph_2\mathcal{N}_2\left[\varphi_1(x,t;p),\varphi_2(x,t;p)\right]. \tag{3.6}$$

Obviously, when $p = 0$ and $p = 1$,

$$\varphi_1(x,t;0) = u_0(x,t), \qquad \varphi_1(x,t;1) = u(x,t),$$
$$\varphi_2(x,t;0) = v_0(x,t), \qquad \varphi_2(x,t;1) = v(x,t). \tag{3.7}$$

Thus, as the embedding parameter p increases from 0 to 1, $\varphi_1(x,t;p)$ and $\varphi_2(x,t;p)$ vary from the initial approximations $u_0(x,t)$ and $v_0(x,t)$ to the solutions $u(x,t)$ and $v(x,t)$, respectively. Expanding $\varphi_1(x,t;p)$ and $\varphi_2(x,t;p)$ in Taylor series with respect to p, we have

$$\varphi_1(x,t;p) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)p^m,$$
$$\varphi_2(x,t;p) = v_0(x,t) + \sum_{m=1}^{+\infty} v_m(x,t)p^m, \tag{3.8}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi_1(x,t;p)}{\partial p^m}\bigg|_{p=0},$$
$$v_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi_2(x,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{3.9}$$

If the auxiliary linear operator, the initial approximations, and the auxiliary parameters h_1 and h_2 are so properly chosen, the above series converge at $p = 1$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t),$$
$$v(x,t) = v_0(x,t) + \sum_{m=1}^{+\infty} v_m(x,t), \tag{3.10}$$

which must be one of solutions of original system. Define the vectors

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \dots, u_n(x,t)\},$$
$$\vec{v}_n = \{v_0(x,t), v_1(x,t), \dots, v_n(x,t)\}. \tag{3.11}$$

We gain the mth-*order deformation equations*:

$$\mathcal{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = h_1 R_{1,m}(\vec{u}_{m-1}, \vec{v}_{m-1}),$$
$$\mathcal{L}\left[v_m(x,t) - \chi_m v_{m-1}(x,t)\right] = h_2 R_{2,m}(\vec{u}_{m-1}, \vec{v}_{m-1}), \tag{3.12}$$

subject to initial conditions

$$u_m(x,0) = 0, \qquad v_m(x,0) = 0, \tag{3.13}$$

where

$$R_{1,m}(\vec{u}_{m-1}, \vec{v}_{m-1}) = \frac{\partial u_{m-1}(x,t)}{\partial t} + \sum_{n=0}^{m-1} u_n(x,t)\frac{\partial u_{m-1-n}(x,t)}{\partial x} + \frac{\partial v_{m-1}(x,t)}{\partial x} + \beta\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2},$$
$$R_{2,m}(\vec{u}_{m-1}, \vec{v}_{m-1}) = \frac{\partial v_{m-1}(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(\sum_{n=0}^{m-1} u_n(x,t)v_{m-1-n}(x,t)\right) + \alpha\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3} - \beta\frac{\partial^2 v_{m-1}(x,t)}{\partial x^2},$$
$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{3.14}$$

Obviously, the solution of the mth-*order deformation equations* (3.12) for $m \ge 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h_1 \mathcal{L}^{-1}\left[R_{1,m}(\vec{u}_{m-1}, \vec{v}_{m-1})\right],$$
$$v_m(x,t) = \chi_m v_{m-1}(x,t) + h_2 \mathcal{L}^{-1}\left[R_{2,m}(\vec{u}_{m-1}, \vec{v}_{m-1})\right]. \tag{3.15}$$

For simplicity, we suppose $h_1 = h_2 = h$. From (3.2) and (3.15), we now successively obtain

$$
\begin{aligned}
u_1(x,t) &= 2\left(\alpha+\beta^2\right)^{0.5}\lambda h k^2 t \operatorname{csch}^2\left[k\left(x+x_0\right)\right],\\
v_1(x,t) &= 4\alpha + 4\beta\Big(\beta + \left(\alpha+\beta^2\right)^{0.5}\Big)\lambda h k^3 t \coth\left[k\left(x+x_0\right)\right]\operatorname{csch}^2\left[k\left(x+x_0\right)\right],\\
u_2(x,t) &= -2\left(\alpha+\beta^2\right)^{0.5}\lambda h k^2 t \operatorname{csch}^3\left[k\left(x+x_0\right)\right]\left(\lambda h k t \cosh\left[k\left(x+x_0\right)\right] - (1+h)\sinh\left[k\left(x+x_0\right)\right]\right),\\
v_2(x,t) &= -2\alpha - 2\beta\Big(\beta + \left(\alpha+\beta^2\right)^{0.5}\Big)\lambda h k^3 t \operatorname{csch}^4\left[k\left(x+x_0\right)\right]\\
&\qquad \times \left(\lambda h k t\left(2+\cosh\left[2k\left(x+x_0\right)\right]\right) - (1+h)\sinh\left[2k\left(x+x_0\right)\right]\right),\\
u_3(x,t) &= \frac{1}{3}\left(\alpha+\beta^2\right)^{0.5}\lambda h k^2 t \operatorname{csch}^4\left[k\left(x+x_0\right)\right]\\
&\qquad \times \left(-3(1+h)^2 + 4\lambda^2 h^2 k^2 t^2 + \left(3 + h\left(6 + h\left(3 + 2\lambda^2 k^2 t^2\right)\right)\right)\right.\\
&\qquad\quad \left.\times \cosh\left[2k\left(x+x_0\right)\right] - 6\lambda h(1+h)kt \sinh\left[2k\left(x+x_0\right)\right]\right),\\
v_3(x,t) &= \frac{1}{3}\alpha + \frac{1}{3}\beta\Big(\beta + \left(\alpha+\beta^2\right)^{0.5}\Big)\lambda h k^3 t \operatorname{csch}^5\left[k\left(x+x_0\right)\right]\\
&\qquad \times \left(\left(-3(1+h)^2 + 22\lambda^2 h^2 k^2 t^2\right)\cosh\left[k\left(x+x_0\right)\right] + \left(3 + h\left(6 + 3h + 2\lambda^2 h k^2 t^2\right)\right)\right.\\
&\qquad\quad \left.\times \cosh\left[3k\left(x+x_0\right)\right] - 6\lambda h(1+h)kt\left(3\sinh\left[k\left(x+x_0\right)\right] + \sinh\left[3k\left(x+x_0\right)\right]\right)\right)\\
&\ \ \vdots
\end{aligned}
\tag{3.16}
$$

We used 10 terms in evaluating the approximate solutions $u_{\text{app}} = \sum_{i=0}^{9} u_i$ and $v_{\text{app}} = \sum_{i=0}^{9} v_i$.

The series solutions contain the auxiliary parameter h. The validity of the method is based on such an assumption that the series (2.4) converges at $p = 1$. It is the auxiliary parameter h which ensures that this assumption can be satisfied. As pointed out by Liao [5], in general, by means of the so-called h-curve, it is straightforward to choose a proper value of h which ensures that the solution series is convergent. In this way, we choose $h = -0.8$ in following computational works.

4. Numerical experiments

We now obtain numerical solutions of the coupled Whitham-Broer-Kaup (WBK) equations. In order to verify the efficiency of the proposed method in comparison with exact solutions, we report the absolute errors for $k = 0.2$, $\lambda = 0.005$, $x_0 = 10$, and different values of α and β, in the following examples.

Example 4.1. Consider the WBK equations (1.1), with the initial conditions (3.1), and the exact solutions (1.2). In Table 1, we show the absolute error for $\alpha = 0.5$ and $\beta = 1$.

Example 4.2. When $\alpha = 1$ and $\beta = 0$, the WBK equations are reduced to the modified Boussinesq (MB) equations [16]. We show the absolute error for MB equations in Table 2.

Table 1: Absolute errors for $u(x,t)$ and $v(x,t)$ given by the HAM for $h = -0.8$, when $k = 0.2$, $\lambda = 0.005$, $x_0 = 10$, $\alpha = 0.5$, and $\beta = 1$.

x	$t = 1$		$t = 2$		$t = 3$	
	u	v	u	v	u	v
1	9.9250E-05	9.0532E-05	1.9973E-05	1.8221E-04	3.0144E-04	2.7505E-04
3	4.3988E-05	3.9582E-05	8.8512E-05	7.9651E-05	1.3358E-04	1.2021E-04
5	1.9644E-05	1.7569E-05	3.9526E-05	3.5351E-05	5.9649E-05	5.3350E-05
7	8.8025E-06	7.8509E-06	1.7711E-05	1.5797E-05	2.6727E-05	2.3839E-05

Table 2: Absolute errors for $u(x,t)$ and $v(x,t)$ given by the HAM for $h = -0.8$, when $k = 0.2$, $\lambda = 0.005$, $x_0 = 10$, $\alpha = 1$, and $\beta = 0$.

x	$t = 1$		$t = 2$		$t = 3$	
	u	v	u	v	u	v
1	8.1037E-05	3.3226E-05	1.6308E-04	6.6872E-05	2.4613E-04	1.0094E-04
3	3.5916E-05	1.4527E-05	7.2270E-05	2.9233E-05	1.0906E-04	4.4119E-05
5	1.6039E-05	6.4478E-06	3.2273E-05	1.2974E-05	4.8703E-05	1.9579E-05
7	7.1872E-06	2.8813E-06	1.4461E-05	5.7975E-06	2.1823E-05	8.7490E-06

Table 3: Absolute errors for $u(x,t)$ and $v(x,t)$ given by the HAM for $h = -0.8$, when $k = 0.2$, $\lambda = 0.005$, $x_0 = 10$, $\alpha = 0$, and $\beta = 0.5$.

x	$t = 1$		$t = 2$		$t = 3$	
	u	v	u	v	u	v
1	4.0519E-05	1.6613E-05	8.1538E-05	3.3436E-05	1.2306E-04	5.0472E-05
3	1.7958E-05	7.2634E-06	3.6135E-05	1.4616E-05	5.4533E-05	2.2059E-05
5	8.0197E-06	3.2239E-06	1.6137E-05	6.4871E-06	2.4352E-05	9.7899E-06
7	3.5936E-06	1.4407E-06	7.2306E-06	2.8988E-06	1.0911E-05	4.3745E-06

Example 4.3. When $\alpha = 0$ and $\beta = 0.5$, the WBK equations are reduced to the approximate long-wave (ALW) equations in shallow water [15]. Table 3 shows the absolute error in this case.

5. Conclusions

In this study, the homotopy analysis method (HAM) was used for finding the approximate traveling wave solutions of the Whitham-Broer-Kaup (WBK) equations in shallow water. A very good agreement between the results of the HAM and exact solutions was observed, which confirms the validity of the HAM. It should be emphasized that the HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between the HAM and other methods. Furthermore, as the HAM does not require discretization, it is not affected by computation round off errors, and large computer memory as well as consumed time which are issues in the calculation procedure. The results show that the HAM is powerful mathematical tool for solving systems of nonlinear partial differential equations having wide applications in engineering.

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