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Research Article

Analytic Solutions of an Iterative Functional Differential Equation near Resonance

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We investigate the existence of analytic solutions of a class of second-order differential equations involving iterates of the unknown function $x''(z) + cx'(z) = x(az + bx(z))$ in the complex field $\mathbb{C}$. By reducing the equation with the Schröder transformation to the another functional differential equation without iteration of the unknown function $\lambda^2 g''(\lambda z)g'(z) - \lambda g'(\lambda z)g''(z) + c(g'(z))^2(\lambda g'(\lambda z) - ag'(z)) = (g'(z))^3(g(\lambda^2 z) - ag(\lambda z))$, we get its local invertible analytic solutions.

1. Introduction

Functional differential equations with state dependent delay have attracted the attentions of many authors in the last years because of their extensive applications (e.g., [1–4]). However, there are only a few papers dealing with functional differential equation with state derivative dependent delay. In [5] Eder studied the functional differential equation $x'(t) = x(x(t))$. V. R. Petahov [6] proved the existence of solutions of equation

$$x''(t) = ax(x(t)). \tag{1.1}$$

In [7–9], the authors studied the existence of analytic solutions of the following second-order iterative functional differential equations:

$$\begin{gathered} x''(z) = x\left(az + bx'(z)\right), \\ x''\left(x^{[r]}(z)\right) = c_0 z + c_1 x(z) + \cdots + c_m x^{[m]}(z), \\ x''(z) = x(az + bx(z)), \end{gathered} \tag{1.2}$$

respectively. Since such equations are quite different from the usual differential equations, the standard existence and uniqueness theorems cannot be applied directly. It is therefore of interest to find some or all of their solutions. In this paper, we will discuss the existence of analytic solutions for another second-order functional differential equation with a state derivative dependent delay:

$$x''(z) + cx'(z) = x(az + bx(z)), \tag{1.3}$$

where a, b, and c are complex numbers. When $c = 0$, (1.3) change into functional differential equation $x''(z) = x(az + bx(z))$, it has been studied in [9]. For the general equation (1.3) the same idea, however, cannot be applied. Therefore, in Section 3 we first reduce (1.3) to an iterative functional differential equation. Then, as in [9], the author reduces again this iterative functional differential equation with the Shchröder transformation, that is, $x(z) = (1/b)(g(\lambda g^{-1}(z)) - az)$ to a functional differential equation with proportional delay (*which is called the auxiliary equation*). Lastly, according to the position of an indeterminate constant λ in complex plane, we discuss the existence of analytic solutions.

In the next section, we will seek explicit analytic solutions of (1.3) in the form of power functions, in the case $b = 0$.

2. Explicit Analytic Solutions

In case $b = 0$, (1.3) changes into functional differential equation

$$x''(z) + cx'(z) = x(az). \tag{2.1}$$

For the above equation we have the following proposition.

Proposition 2.1. *Suppose* $0 < |a| \leq 1$. *Then* (2.1) *has an analytic solution* $x(z)$ *in a neighborhood of the origin, satisfying* $x(0) = \mu$ *and* $x'(0) = \eta \in \mathbb{C}$.

Proof. Let

$$x(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{2.2}$$

be the expansion of formal solution $x(z)$ of (2.1). Substituting (2.2) into (2.1), we get

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}z^n = \sum_{n=0}^{\infty} (a^n a_n - c(n+1)a_{n+1})z^n. \tag{2.3}$$

By comparing the coefficients, we have

$$\begin{gathered} 2a_2 = a_o - ca_1 = \mu - c\eta, \\ a_{n+2} = \frac{a^n}{(n+2)(n+1)}a_n - \frac{c}{(n+2)}a_{n+1}. \end{gathered} \tag{2.4}$$

In view of $|a| \leq 1$ and c is complex number, thus there exists a positive number m, so that

$$\left|\frac{a^n}{(n+2)(n+1)}\right| \leq m, \qquad \left|\frac{c}{(n+2)}\right| \leq m. \tag{2.5}$$

Thus if we define recursively a sequence $\{B_n\}_{n=0}^{\infty}$ by $B_0 = |\mu|$, $B_1 = |\eta|$,

$$B_{n+2} = m(B_n + B_{n+1}), \quad n = 0, 1, 2, \ldots, \tag{2.6}$$

then one can show that by induction

$$|a_n| \leq B_n, \quad n = 0, 1, 2, \ldots. \tag{2.7}$$

Now if we define

$$M(z) = \sum_{n=0}^{\infty} B_n z^n, \tag{2.8}$$

then

$$\begin{aligned} M(z) &= B_0 + B_1 z + z^2 \sum_{n=1}^{\infty} B_{n+2} z^n \\ &= B_0 + B_1 z + m \sum_{n=1}^{\infty} (B_n + B_{n+1}) z^{n+2} \\ &= \left(mz^2 + mz\right) M(z) + (|\eta| - m|\mu|) z + |\mu|, \end{aligned} \tag{2.9}$$

that is,

$$\left(mz^2 + mz - 1\right) M(z) + (|\eta| - m|\mu|) z + |\mu| = 0. \tag{2.10}$$

Let

$$Q(z, w) = \left(mz^2 + mz - 1\right) w + (|\eta| - m|\mu|) z + |\mu| \tag{2.11}$$

for (z, w) from a neighborhood of $(0, |\mu|)$. Since $Q(0, |\mu|) = 0$, $Q'_w(0, |\mu|) = -1 \neq 0$, there exists a unique function $w(z)$, analytic on a neighborhood of zero, such that $w(0) = |\mu|$, $w'(0) = |\eta|$ and satisfying the equality $Q(z, w(z)) = 0$. According to (2.8) and (2.10), we have $M(z) = w(z)$. It follows that the power series (2.8) converges on a neighborhood of the origin, which implies that the power series (2.2) also converges in a neighborhood of the origin. The proof is complete.

Thus, the desired solution of (2.1) is

$$x(z) = \mu + \eta z + \frac{\mu - c\eta}{2!} z^2 + \frac{(a + c^2)\eta - c\mu}{3!} z^3 + \cdots . \tag{2.12}$$

□

3. Analytic Solutions of the Auxiliary Equation

A distinctive feature of the (1.3) when $b \neq 0$ is that the argument of the unknown function is dependent on the state $x(z)$, and this is the case we will emphasize in this paper. We now discuss the existence of analytic solution of (1.3) by locally reducing the equation to another functional differential equation with proportional delays. Let

$$y(z) = az + bx(z). \tag{3.1}$$

Then for any number z, we have

$$x(z) = \frac{1}{b}\left(y(z) - az\right), \tag{3.2}$$

and so $x(y(z)) = (1/b)(y(y(z)) - ay(z))$. Therefore, in view of (1.3) and $x''(z) = (1/b)y''(z)$, we have

$$y''(z) + cy'(z) - a = y\left(y(z)\right) - ay(z). \tag{3.3}$$

To find analytic solution of (3.3), we first seek an analytic solution $g(z)$ of the auxiliary equation

$$\lambda^2 g''(\lambda z)g'(z) - \lambda g'(\lambda z)g''(z) + c(g'(z))^2(\lambda g'(\lambda z) - ag'(z)) = (g'(z))^3\left(g\left(\lambda^2 z\right) - ag(\lambda z)\right) \tag{3.4}$$

satisfying the initial value conditions

$$g(0) = \mu, \qquad g'(0) = \eta \neq 0, \tag{3.5}$$

where μ and η are complex numbers, and λ satisfies one of the following conditions:

(H1) $0 < |\lambda| < 1$;

(H2) $\lambda = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is a Brjuno number, that is, $B(\theta) = \sum_{k=0}^{\infty} \log q_{k+1}/q_k < \infty$, where $\{p_k/q_k\}$ denotes the sequence of partial fraction of the continued fraction expansion of θ;

(H3) $\lambda = e^{2\pi i q/p}$ for some integers $p \in \mathbb{N}$ with $p \geq 2$ and $q \in \mathbb{Z} \setminus \{0\}$, and $\alpha \neq e^{2\pi i l/k}$ for all $1 \leq k \leq p-1$ and $l \in \mathbb{Z} \setminus \{0\}$.

We observe that λ is inside the unit circle S^1 in case (H1) but on S^1 in the rest cases. More difficulties are encountered for λ on S^1 since the small divisor $\lambda^{n+2} - \lambda$ is involved in the latter (3.10). Under Diophantine condition: "*$\lambda = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and there exist constants $\zeta > 0$ and $\delta > 0$ such that $|\lambda^n - 1| \geq \zeta^{-1} n^{-\delta}$ for all $n \geq 1$*," the number $\lambda \in S^1$ is "far" from all roots of the unity and was considered in different settings [7–12]. Since then, we have been striving to give a result of analytic solutions for those λ "near" a root of the unity, that is, neither being roots of the unity nor satisfying the Diophantine condition. The Brjuno condition in (H2) provides such a chance for us. Moreover, we also discuss the so-called the resonance case, that is, the case of (H3).

Theorem 3.1. *If (H1) holds. Then, for the initial value conditions* (3.5), (3.4) *has an analytic solution of the form*

$$g(z) = \mu + \eta z + \sum_{n=2}^{\infty} b_n z^n \tag{3.6}$$

in a neighborhood of the origin.

Proof. Rewrite (3.4) in the form

$$\frac{\lambda^2 g''(\lambda z) g'(z) - \lambda g'(\lambda z) g''(z)}{(g'(z))^2} + c(\lambda g'(\lambda z) - a g'(z)) = g'(z)\Big(g\big(\lambda^2 z\big) - a g(\lambda z)\Big) \tag{3.7}$$

or

$$\lambda \left(\frac{g'(\lambda z)}{g'(z)}\right)' = g'(z)\Big(g\big(\lambda^2 z\big) - a g(\lambda z)\Big) - c(\lambda g'(\lambda z) - a g'(z)). \tag{3.8}$$

Therefore, in view of $g'(0) = \eta \neq 0$, we obtain

$$g'(\lambda z) = g'(z)\left(1 + \frac{1}{\lambda}\int_0^z g'(s)\Big(g\big(\lambda^2 s\big) - a g(\lambda s)\Big) - c\lambda g'(\lambda s) + acg'(s) ds\right). \tag{3.9}$$

We now seek a solution of (3.4) in the form of a power series (3.6). By defining $b_0 = \mu$ and $b_1 = \eta$ and then substituting (3.6) into (3.9), we see that the sequence $\{b_n\}_{n=2}^{\infty}$ is successively determined by the condition

$$\begin{aligned}\left(\lambda^{n+2} - \lambda\right)(n+2) b_{n+2} = \sum_{k=0}^{n} \Bigg(&\sum_{j=0}^{n-k} \frac{(j+1)(k+1)\left(\lambda^{2(n-j-k)} - a\lambda^{n-j-k}\right)}{n-k+1} b_{j+1} b_{k+1} b_{n-j-k} \\ &+ c(k+1)\left(a - \lambda^{n+1-k}\right) b_{k+1} b_{n+1-k} \Bigg), \quad n = 0, 1, 2, \ldots\end{aligned} \tag{3.10}$$

in a unique manner. Now we show that the resulting power series (3.6) converges in a neighborhood of the origin. First of all, we have $\lim_{n\to\infty}|1/(\lambda^{n+2}-\lambda)| = 1/|\lambda|$, thus there exists a positive number M, such that

$$\left|\frac{(j+1)(k+1)\left(\lambda^{2(n-j-k)}-a\lambda^{n-j-k}\right)}{(n-k+1)(n+2)(\lambda^{n+2}-\lambda)}\right| \leq M,$$
$$\left|\frac{c(k+1)\left(a-\lambda^{n+1-k}\right)}{(n+2)(\lambda^{n+2}-\lambda)}\right| \leq M. \tag{3.11}$$

If we define a sequence $\{D_n\}_{n=0}^{\infty}$ by $D_0 = |\mu|$, $D_1 = |\eta|$ and

$$D_{n+2} = M\sum_{k=0}^{n}\left(\sum_{j=0}^{n-k} D_{j+1}D_{k+1}D_{n-j-k} + D_{k+1}D_{n+1-k}\right), \quad n = 0,1,2,\ldots, \tag{3.12}$$

then in view of (3.10), we can show by induction that

$$|b_n| \leq D_n, \quad n = 0,1,2,\ldots. \tag{3.13}$$

Now if we define

$$G(z) = \sum_{n=0}^{\infty} D_n z^n, \tag{3.14}$$

then

$$\begin{aligned}
G^2(z) &= \left(|\mu| + \sum_{n=0}^{\infty} D_{n+1}z^{n+1}\right)\left(\sum_{n=0}^{\infty} D_n z^n\right) \\
&= |\mu|\sum_{n=0}^{\infty} D_n z^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n} D_{k+1}D_{n-k}z^{n+1} \\
&= |\mu|^2 + 2|\mu|\sum_{n=0}^{\infty} D_{n+1}z^{n+1} + \sum_{n=0}^{\infty}\sum_{k=0}^{n} D_{k+1}D_{n+1-k}z^{n+2}, \\
G^3(z) &= \left(|\mu| + \sum_{n=0}^{\infty} D_{n+1}z^{n+1}\right)\left(|\mu|\sum_{n=0}^{\infty} D_n z^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n} D_{k+1}D_{n-k}z^{n+1}\right) \\
&= |\mu|^2\sum_{n=0}^{\infty} D_n z^n + 2|\mu|\sum_{n=0}^{\infty}\sum_{k=0}^{n} D_{k+1}D_{n-k}z^{n+1} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{n-k} D_{k+1}D_{j+1}D_{n-j-k}z^{n+2}.
\end{aligned} \tag{3.15}$$

We get immediately

$$\begin{aligned} G^3(z) + G^2(z) &= 2|\mu|G^2(z) + \left(2|\mu| - |\mu|^2\right)G(z) - |\mu|^2 + \frac{1}{M}\sum_{n=0}^{\infty} D_{n+2} z^{n+2} \\ &= 2|\mu|G^2(z) + \left(2|\mu| - |\mu|^2\right)G(z) - |\mu|^2 + \frac{1}{M}(G(z) - |\mu| - |\eta| z). \end{aligned} \tag{3.16}$$

That is,

$$G^3(z) - (2|\mu| - 1)G^2(z) - \left(\frac{1}{M} + 2|\mu| - |\mu|^2\right)G(z) + \frac{1}{M}(|\eta| z + |\mu|) + |\mu|^2 = 0. \tag{3.17}$$

Let

$$R(z, w) = w^3 - (2|\mu| - 1)w^2 - \left(\frac{1}{M} + 2|\mu| - |\mu|^2\right)w + \frac{1}{M}(|\eta| z + |\mu|) + |\mu|^2 \tag{3.18}$$

for (z, w) from a neighborhood of $(0, |\mu|)$. Since $R(0, |\mu|) = 0$ and $R'_w(0, |\mu|) = -1/M \neq 0$, and the implicit function theorem, there exists a unique function $w(z)$, analytic in a neighborhood of zero, such that $w(0) = |\mu|$, $w'(0) = |\eta|$ and $R(z, w) = 0$. According to (3.14) and (3.17), we have $G(z) = w(z)$. It follows that the power series (3.14) converges in a neighborhood of the origin. So does (3.6). The proof is complete. □

Now, we discuss local invertible analytic solutions of auxiliary equation (3.4) in cases (H2). In order to study the existence of analytic solutions of (3.6) under the Brjuno condition, we first recall briefly the definition of Brjuno numbers and some basic facts. As stated in [13], for a real number θ, we let $[\theta]$ denote its integer part and $\{\theta\} = \theta - [\theta]$ its fractional part. Then every irrational number θ has a unique expression of the Gauss' continued fraction

$$\theta = d_0 + \theta_0 = d_0 + \frac{1}{d_1 + \theta_1} = \cdots, \tag{3.19}$$

denoted simply by $\theta = [d_0, d_1, \ldots, d_n, \ldots]$, where d_j's and θ_j's are calculated by the algorithm: (a) $d_0 = [\theta]$, $\theta_0 = \{\theta\}$ and (b) $d_n = [1/\theta_{n-1}]$, $\theta_n = \{1/\theta_{n-1}\}$ for all $n \geq 1$. Define the sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned} q_{-2} &= 1, \qquad q_{-1} = 0, \qquad q_n = d_n q_{n-1} + q_{n-2}, \\ p_{-2} &= 0, \qquad p_{-1} = 1, \qquad p_n = d_n p_{n-1} + p_{n-2}. \end{aligned} \tag{3.20}$$

It is easy to show that $p_n/q_n = [d_0, d_1, \ldots, d_n]$. Thus, for every $\theta \in \mathbb{R} \setminus \mathbb{Q}$ we associate, using its convergence, an arithmetical function $B(\theta) = \sum_{n\geq 0} \log q_{n+1}/q_n$. We say that θ is a Brjuno number or that it satisfies Brjuno condition if $B(\theta) < +\infty$. The Brjuno condition is weaker than the Diophantine condition. For example, if $d_{n+1} \leq c e^{d_n}$ for all $n \geq 0$, where $c > 0$ is a constant, then $\theta = [d_0, d_1, \ldots, d_n, \ldots]$ is a Brjuno number but is not a Diophantine number. So, the case (H2) contains both Diophantine condition and a part of α "near" resonance. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$

and $\{q_n\}_{n\in\mathbb{N}}$ be the sequence of partial denominators of the Gauss's continued fraction for θ. As in [13], let

$$A_k = \left\{ n \geq 0 \mid \|n\theta\| \leq \frac{1}{8q_k} \right\}, \qquad E_k = \max\left\{ q_k, \frac{q_{k+1}}{4} \right\}, \qquad \eta_k = \frac{q_k}{E_k}. \tag{3.21}$$

Let A_k^* be the set of integers $j \geq 0$ such that either $j \in A_k$ or for some j_1 and j_2 in A_k, with $j_2 - j_1 < E_k$, one has $j_1 < j < j_2$ and q_k divides $j - j_1$. For any integer $n \geq 0$, define

$$l_k(n) = \max\left\{ (1+\eta_k)\frac{n}{q_k} - 2,\ (m_n\eta_k + n)\frac{1}{q_k} - 1 \right\}, \tag{3.22}$$

where $m_n = \max\{j \mid 0 \leq j \leq n,\ j \in A_k^*\}$. We then define function $h_k : \mathbb{N} \to \mathbb{R}_+$ as follows:

$$h_k(n) = \begin{cases} \dfrac{m_n + \eta_k n}{q_k} - 1, & \text{if } m_n + q_k \in A_k^*, \\ l_k(n), & \text{if } m_n + q_k \notin A_k^*. \end{cases} \tag{3.23}$$

Let $g_k(n) := \max\{h_k(n), [n/q_k]\}$, and define $k(n)$ by the condition $q_{k(n)} \leq n \leq q_{k(n)+1}$. Clearly, $k(n)$ is nondecreasing. Now we are able to state the following result.

Lemma 3.2 (Davie's lemma [14]). *Let* $K(n) = n\log 2 + \sum_{k=0}^{k(n)} g_k(n)\log(2q_{k+1})$. *Then*

(a) *there is a universal constant* $\gamma > 0$ *(independent of* n *and* θ*) such that*

$$K(n) \leq n\big(B(\theta) + \gamma\big); \tag{3.24}$$

(b) $K(n_1) + K(n_2) \leq K(n_1 + n_2)$ *for all* n_1 *and* n_2*;*

(c) $-\log|\alpha^n - 1| \leq K(n) - K(n-1)$.

Theorem 3.3. *Suppose (H2) holds. Then* (3.4) *has an analytic solution of the form* (3.6) *in a neighborhood of the origin such that* $g(0) = \mu$, $g'(0) = \eta$.

Proof. As in the proof of Theorem 3.1, we seek a power series solution of the form (3.6). Set $b_0 = \mu$ and $b_1 = \eta$, (3.10) holds again. From (3.10) we get

$$|b_{n+2}| \leq \frac{M}{|\lambda^{n+1} - 1|}\sum_{k=0}^{n}\left(\sum_{j=0}^{n-k} |b_{j+1}||b_{k+1}||b_{n-j-k}| + |b_{k+1}||b_{n+1-k}| \right), \quad n = 0, 1, 2, \ldots. \tag{3.25}$$

To construct a governing series of (3.6), we consider the implicit functional equation

$$R(z, \varphi) =: R(z, \varphi, \mu, \eta, M) = 0, \tag{3.26}$$

where R is defined in (3.18). Similarly to the proof of Theorem 3.1, using the implicit function theorem we can prove that (3.26) has a unique analytic solution $\varphi(z, \mu, \eta, M)$ in a neighborhood of the origin such that $\varphi(0,\mu,\eta,M) = |\mu|$ and $\varphi'_z(0,\mu,\eta,M) = |\eta|$. Thus $\varphi(z,\mu,\eta,M)$ in (3.26) can be expanded into a convergent series

$$\varphi(z,\mu,\eta,M) = \sum_{n=0}^{\infty} C_n z^n \tag{3.27}$$

in a neighborhood of the origin. Replacing (3.27) into (3.26) and comparing coefficients, we obtain that $C_0 = |\mu|, \quad C_1 = |\eta|$ and

$$C_{n+2} = M \sum_{k=0}^{n} \left(\sum_{j=0}^{n-k} C_{j+1} C_{k+1} C_{n-j-k} + C_{k+1} C_{n+1-k} \right), \quad n = 0,1,2,\ldots. \tag{3.28}$$

Note the power series (3.27) converges in a neighborhood of zero. Hence there is a positive constant T such that

$$C_n < T^n, \quad n = 1,2,\ldots. \tag{3.29}$$

Now by induction, we prove that

$$|b_n| \le C_n e^{K(n-1)}, \quad n = 1,2,\ldots, \tag{3.30}$$

where $K : \mathbb{N} \to \mathbb{R}$ is defined in Lemma 3.2. In fact, $|b_1| = C_1$. For inductive proof, we assume that $|b_m| \le C_m e^{K(m-1)}$. From (3.10) and Lemma 3.2 ,we know

$$\begin{aligned} |b_{m+1}| &\le \frac{M}{|\lambda^m - 1|} \sum_{k=0}^{m-1} \left(\sum_{j=0}^{m-1-k} \left|b_{j+1}\right| |b_{k+1}| \left|b_{m-1-j-k}\right| + |b_{k+1}||b_{m-k}| \right) \\ &\le \frac{M}{|\lambda^m - 1|} \sum_{k=0}^{m-1} \left(\sum_{j=0}^{m-1-k} C_{j+1} C_{k+1} C_{m-1-j-k} e^{K(j)+K(k)+K(m-2-j-k)} + C_{k+1} C_{m-k} e^{K(k)+K(m-1-k)} \right). \end{aligned} \tag{3.31}$$

Note that

$$\begin{aligned} K(j) + K(k) + K(m-2-j-k) \le K(m-2) \le K(m-1) \le \log|\lambda^m - 1| + K(m), \\ K(k) + K(m-1-k) \le K(m-1) \le \log|\lambda^m - 1| + K(m). \end{aligned} \tag{3.32}$$

Hence

$$\begin{aligned} |b_{m+1}| &\le e^{K(m)} \cdot M \sum_{k=0}^{m-1} \left(\sum_{j=0}^{m-1-k} C_{j+1} C_{k+1} C_{m-1-j-k} + C_{k+1} C_{m-k} \right) \\ &= C_{m+1} e^{K(m)} \end{aligned} \tag{3.33}$$

as desired. In view of (3.29) and Lemma 3.2, we know that $K(n) \le n(B(\theta)+\gamma)$ for some universal constant $\gamma > 0$. Then

$$|b_n| \le T^n e^{n(B(\theta)+\gamma)}. \tag{3.34}$$

That is,

$$\limsup_{n\to\infty}(|b_n|)^{1/n} \le \limsup_{n\to\infty}\left(T^n e^{n(B(\theta)+\gamma)}\right)^{1/n} \le T e^{B(\theta)+\gamma}. \tag{3.35}$$

This implies that the convergence radius of (3.6) is at least $\left(Te^{B(\theta)+\gamma}\right)^{-1}$. This concludes the proof. □

In case (H3), the constant λ is not only the unit circle in $\mathbb{C}$, but also a root of unity. In such a case, the resonant case, both the Diophantine and the Brjuno conditions are not satisfied. Let $\{D_n\}_{n=0}^{\infty}$ be a sequence define by $D_0 = |\mu|$, $D_1 = |\eta|$ and

$$D_{n+2} = \Gamma M \sum_{k=0}^{n}\left(\sum_{j=0}^{n-k} D_{j+1}D_{k+1}D_{n-j-k} + D_{k+1}D_{n+1-k}\right), \quad n = 0,1,2,\ldots, \tag{3.36}$$

where $\Gamma = \max\{1, |\lambda^i - 1|^{-1} : i = 1,2,\ldots,p-1\}$ and M is defined in Theorem 3.1.

Theorem 3.4. *Suppose that (H3) holds, let $\{b_n\}_{n=0}^{\infty}$ be determined by $b_0 = \mu$, $b_1 = \eta$ and*

$$\left(\lambda^{n+2} - \lambda\right)(n+2)b_{n+2} = \Theta(n,\lambda), \quad n = 1,2,\ldots, \tag{3.37}$$

where

$$\begin{aligned}\Theta(n,\lambda) = \sum_{k=0}^{n}\Bigg(&\sum_{j=0}^{n-k}\frac{(j+1)(k+1)\left(\lambda^{2(n-j-k)} - a\lambda^{n-j-k}\right)}{n-k+1} b_{j+1}b_{k+1}b_{n-j-k} \\ &+ c(k+1)\left(a - \lambda^{n+1-k}\right)b_{k+1}b_{n+1-j}\Bigg).\end{aligned} \tag{3.38}$$

If $\Theta(vp-1,\lambda) = 0$, $v = 1,2,\ldots$, then (3.4) *has an analytic solution $g(z)$ in a neighborhood of the origin such that $g(0) = \mu$, $g'(0) = \eta$, and $g^{(vp-1)}(0) = (vp-1)!\eta_{vp-1}$, where all η_{vp-1}'s are arbitrary constants satisfying the inequality $|\eta_{vp-1}| \le D_{vp-1}$, and sequence $\{D_n\}_{n=0}^{\infty}$ is defined in* (3.36). *Otherwise, if $\Theta(vp-1,\lambda) \ne 0$, $v = 1,2,\ldots$, then* (3.4) *has no any analytic solution $g(z)$ in a neighborhood of the origin.*

Proof. We seek a power series solution of (3.4) of the form (3.6), as in the proof of Theorem 3.1, where the equality in (3.10) or (3.29) is indispensable. If $\Theta(vp-1,\lambda) \ne 0$, $v = 1,2,\ldots$, then the equality in (3.37) does not hold for $n = vp-1$, since $\lambda^{vp} - 1 = 0$. In such a circumstance (3.4) has no formal solutions.

If $\Theta(vp-1,\lambda) = 0$ for all natural numbers v, the corresponding b_{vp-1} in (3.29) has infinitely many choices in $\mathbb{C}$; this is, the formal series solution (3.6) defines a family of solutions with infinitely many parameters. Choose $b_{vp-1} = \eta_{vp-1}$ arbitrarily such that

$$\left|\eta_{vp-1}\right| \leq D_{vp-1}, \quad v = 1,2,\ldots, \tag{3.39}$$

where D_{vp-1} is defined by (3.36). In what follows, we prove that the formal series solution (3.6) converges in a neighborhood of the origin. Observe that $\left|\lambda^{n+1}-1\right|^{-1} \leq \Gamma$ for $n \neq vp-1$. It follows from (3.29) that

$$|b_{n+2}| \leq \Gamma M \sum_{k=0}^{n}\left(\sum_{j=0}^{n-k}\left|b_{j+1}\right||b_{k+1}|\left|b_{n-j-k}\right| + |b_{k+1}||b_{n+1-k}|\right), \quad n = 0,1,2,\cdots \tag{3.40}$$

for $n \neq vp-1$, $v = 1,2,\ldots$. Further, we can prove that

$$|b_n| \leq D_n, \quad n = 1,2,\ldots. \tag{3.41}$$

Set

$$\psi(z,\mu,\eta,\Gamma M) = \sum_{n=0}^{\infty} D_n Z^n. \tag{3.42}$$

It is easy to check that (3.42) satisfies the implicit functional equation

$$R(z,\psi,\mu,\eta,\Gamma M) = 0, \tag{3.43}$$

where R is defined in (3.18). Moreover, similarly to the proof of Theorem 3.1, we can prove that (3.43) has a unique analytic solution $\psi(z,\mu,\eta,\Gamma M)$ in a neighborhood of the origin such that $\psi(0,\mu,\eta,\Gamma M) = |\mu|$, and $\psi'_z(0,\mu,\eta,\Gamma M) = |\eta|$. Thus (3.42) converges in a neighborhood of the origin. Therefore, the series (3.6) converges in a neighborhood of the origin. The proof is complete. □

Theorem 3.5. *Suppose the conditions of Theorem 3.1 or Theorem 3.3 or Theorem 3.4 are satisfied. Then* (1.3) *has an analytic solution* $x(z) = (1/b)(g(\lambda g^{-1}(z)) - az)$ *in a neighborhood of the number* μ*, where* $g(z)$ *is an analytic solution of* (3.4).

Proof. In view of Theorem 3.1 or Theorem 3.3 or Theorem 3.4, we may find a sequence $\{b_n\}_{n=0}^{\infty}$ such that the function $g(z)$ of the form (3.6) is an analytic solution of (3.4) in a neighborhood of the origin. Since $g'(0) = \eta \neq 0$, the function $g^{-1}(z)$ is analytic in a neighborhood of $g(0) = \mu$. If we now define $x(z) = (1/b)(g(\lambda g^{-1}(z)) - az)$ then

$$
\begin{aligned}
x'(z) &= \frac{1}{b}\left(\frac{\lambda g'(\lambda g^{-1}(z))}{g'(g^{-1}(z))} - a\right) \\
x''(z) &= \frac{\lambda^2 g''(\lambda g^{-1}(z))g'(g^{-1}(z)) - \lambda g'(\lambda g^{-1}(z))g''(g^{-1}(z))}{b[g'(g^{-1}(z))]^3},
\end{aligned}
\tag{3.44}
$$

then

$$
\begin{aligned}
x''(z) + cx'(z) &= \frac{\lambda^2 g''(\lambda g^{-1}(z))g'(g^{-1}(z)) - \lambda g'(\lambda g^{-1}(z))g''(g^{-1}(z))}{b[g'(g^{-1}(z))]^3} \\
&\quad - \frac{cg'(g^{-1}(z))^2[\lambda g'(\lambda g^{-1}(z)) - ag'(g^{-1}(z))]}{b[g'(g^{-1}(z))]^3} \\
&= \frac{[g'(g^{-1}(z))]^3[g(\lambda^2 g^{-1}(z)) - ag(\lambda g^{-1}(z))]}{b[g'(g^{-1}(z))]^3} \\
&= x(az + bx(z))
\end{aligned}
\tag{3.45}
$$

as required. □

4. Analytic Solution of (1.3)

In the previous section, we have shown that under the conditions of Theorem 3.1 or Theorem 3.3 or Theorem 3.4, (3.3) has an analytic solution $y(z) = g(\lambda g^{-1}(z))$ in a neighborhood of the number μ, where $g(z)$ is an solution of (3.4). Since the function $y(z) = g(\lambda g^{-1}(z))$ can be determined by (3.10), it is possible to calculate, at last in theory, the explicit form of $y(z)$, an analytic solution of (1.3), in a neighborhood of the fixed point μ of $y(z)$ by means of (3.2). However, knowing that an analytic solution of (1.3) exists, we can take an alternative route as follows. Assume that $x(z)$ is of the form

$$
x(z) = x(\mu) + x'(\mu)(z-\mu) + \frac{x''(\mu)}{2!}(z-\mu)^2 + \cdots, \tag{4.1}
$$

we need to determine the derivatives $x^{(n)}(\mu)$, $n = 0, 1, 2, \ldots$. First of all, in view of (1.3) and (3.2), we have

$$x(\mu) = \frac{y(\mu) - a\mu}{b} = \frac{g(\lambda g^{-1}(\mu)) - a\mu}{b} = \frac{(1-a)\mu}{b},$$
$$x'(z) = \frac{\lambda}{b}\frac{g'(\lambda g^{-1}(z))}{g'(g^{-1}(z))} - \frac{a}{b}, \tag{4.2}$$
$$x'(\mu) = \frac{\lambda - a}{b},$$

respectively. Furthermore,

$$x''(z) = x(az + bx(z)) - cx'(z),$$
$$x''(\mu) = \frac{(1-a)\mu}{b} - \frac{(\lambda - a)c}{b} = \frac{(1-a)\mu - (\lambda - a)c}{b}. \tag{4.3}$$

Next by calculating the derivatives of both sides of (1.3), we obtain successively

$$x'''(z) = x'(az + bx(z))\left(a + bx'(z)\right) - cx''(z), \tag{4.4}$$

so that

$$x'''(\mu) = \frac{\lambda(\lambda - a) - c(1-a)\mu + (\lambda - a)c^2}{b}. \tag{4.5}$$

Recall the formula for the higher derivatives of composition. Namely, for $n \geq 1$,

$$\begin{aligned} x^{(k+2)}(z) &= x^{(k)}(h(z)) - cx^{(k+1)}(z) \\ &= \sum_{i=1}^{k} P_{i,k}\left(h'(z), h''(z), \ldots, h^{(i)}(z)\right)x^{(i)}(h(z)) - cx^{(k+1)}(z), \end{aligned} \tag{4.6}$$

where $h(z) = az + bx(z)$, and $P_{i,k}$ is a polynomial with nonnegative coefficients.

We have

$$x^{(k+2)}(\mu) = \sum_{i=1}^{k} P_{i,k}\left(h'(\mu), h''(\mu), \ldots, h^{(i)}(\mu)\right)x^{(i)}(h(\mu)) - cx^{(k+1)}(\mu) =: \Delta_{k+2}. \tag{4.7}$$

Thus the desired solutions is

$$x(z) = \frac{(1-a)\mu}{b} + \frac{\lambda - a}{b}(z - \mu) + \frac{(1-a)\mu - (\lambda - a)c}{2!b}(z - \mu)^2 + \sum_{n=3}^{\infty}\frac{\Delta_n}{n!}(z - \mu)^n. \tag{4.8}$$

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