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Research Article

On the Almost Periodic Solutions of Differential Equations on Hilbert Spaces

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For the differential equation $u'(t) = Au(t) + f(t)$, $t \geq 0$ on a Hilbert space H, we find the necessary and sufficient conditions that the above-mentioned equation has a unique almost periodic solution. Some applications are also given.

1. Introduction

In this paper we are concerned with the almost periodicity of solutions of the differential equation

$$u'(t) = Au(t) + f(t) \quad t \in \mathbb{R}, \tag{1.1}$$

where A is a linear, closed operator on a Hilbert space H and f is a function from $\mathbb{R}$ to H. The asymptotic behavior and, in particular, the almost periodicity of solutions of (1.1) has been a subject of intensive study for recent decades; see, for example, [1–5] and references therein. A particular condition for almost periodicity is the countability of the spectrum of the solution. In this paper we investigate the almost periodicity of mild solutions of (1.1), when A is a linear, unbounded operator on a Hilbert space H. We use the Hilbert space $\mathrm{AP}(\mathbb{R}, H)$ introduced in [4], defined by what follows. Let $(\cdot,\cdot)$ be the inner product of H, and let $AP_b(\mathbb{R}, E)$ be the space of all almost periodic functions from $\mathbb{R}$ to H. The completion of $AP_b(\mathbb{R}, E)$ is then a Hilbert space with the inner product defined by

$$\langle f, g\rangle := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (f(s), g(s))\,ds. \tag{1.2}$$

First, we establish the relationship between the Bohr transforms of the almost periodic solutions of (1.1) and those of the inhomogeneity f. We then give a necessary and sufficient condition so that (1.1) admits a unique almost periodic solution for each almost periodic inhomogeneity f. As applications, in Section 4 we show a short proof of the Gearhart's theorem. If A is generator of a strongly continuous semigroup $T(t)$, then $1 \in \varrho(T(1))$ if and only if $2k\pi i \in \varrho(A)$ and $\sup_{k\in\mathbb{Z}} \|(2k\pi i - A)^{-1}\| < \infty$.

2. The Hilbert Space of Almost Periodic Functions

Let us fix some notations. Recall that a bounded, uniformly continuous function f from $\mathbb{R}$ to a Banach space H is almost periodic, if the set $\{S(t)f : t \in \mathbb{R}\}$ is relatively compact in $\mathrm{BUC}(\mathbb{R}, H)$, the space of bounded uniformly continuous functions with sup-norm topology. Let H be now a Hilbert space with $(\cdot,\cdot)$, and let $\|\cdot\|$ be the inner product and the norm in H, respectively. Let $AP_b(\mathbb{R}, H)$ be the space of all almost periodic functions from $\mathbb{R}$ to H. In $AP_b(\mathbb{R}, H)$ the following expression

$$\langle f, g\rangle := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (f(s), g(s))ds \tag{2.1}$$

exists and defines an inner product. Hence, $AP_b(\mathbb{R}, H)$ is a pre-Hilbert space and its completion, denoted by $AP(\mathbb{R}, H)$, is a Hilbert space. The inner product and the norm in $AP(\mathbb{R}, H)$ are denoted by $\langle f, g\rangle$ and $\|\cdot\|_{AP}$, respectively.

For each function $f \in AP(\mathbb{R}, H)$, the Bohr transform is defined by

$$a(\lambda, f) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(s)e^{-i\lambda s}ds. \tag{2.2}$$

The set

$$\sigma(f) := \{\lambda \in \mathbb{R} : a(\lambda, f) \neq 0\} \tag{2.3}$$

is called the Bohr spectrum of f. It is well known that $\sigma(f)$ is countable for each function $f \in AP(\mathbb{R}, H)$, and the Fourier-Bohr series of f is

$$\sum_{\lambda\in\sigma(f)} a(\lambda, f)e^{i\lambda t}, \tag{2.4}$$

and it converges to f in the norm topology of $AP(\mathbb{R}, E)$. The following Parseval's equality also holds

$$\|f\|^2_{AP(\mathbb{R},H)} = \sum_{\lambda\in\sigma(f)} \|a(\lambda, f)\|. \tag{2.5}$$

For more information about the almost periodic functions and properties of the Hilbert space $AP(\mathbb{R}, H)$, we refer readers to [2, 4].

Let $W_2(AP)$ be the space consisting of all almost periodic functions f, such that $f' \in AP(\mathbb{R}, H)$. $W_2(AP)$ is then a Hilbert space with the norm

$$\|f\|^2_{W_2(AP)} := \|f\|^2_{AP(\mathbb{R},H)} + \|f'\|^2_{AP(\mathbb{R},H)}. \tag{2.6}$$

Note that the $W_2(AP)$-topology is stronger than the sup-norm topology (see [6]). We will use the following lemma.

Lemma 2.1. *If F is a function in $W_2(AP)$ and $f = F'$, then we have*

$$a(\lambda, f) = \lambda i \cdot a(\lambda, F). \tag{2.7}$$

Proof. If $\lambda \neq 0$, using the integration by part we have

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T} e^{-i\lambda s} f(s)ds &= \frac{1}{2T}F(t)e^{-i\lambda t}\Big|_{-T}^{T} + \frac{i\lambda}{2T}\int_{-T}^{T} F(s)e^{-i\lambda s}ds \\ &= \frac{F(T)e^{-i\lambda T} - F(-T)e^{i\lambda T}}{2T} + i\lambda\frac{1}{2T}\int_{-T}^{T} F(s)e^{-i\lambda s}ds.\end{aligned} \tag{2.8}$$

Let $T \to \infty$, and note that $F(t)$ is bounded, we have (2.7).

If $\lambda = 0$, then

$$a(0, f) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(s)ds = \lim_{T\to\infty}\frac{F(T) - F(-T)}{2T} = 0, \tag{2.9}$$

which also satisfies (2.7). □

Finally, for a linear, closed operator A in a Hilbert space H, we denote the domain, the range, the spectrum, and the resolvent set of A by $D(A)$, $\mathrm{Range}(A)$, $\sigma(A)$, and $\varrho(A)$, respectively.

3. Almost Periodic Mild Solutions of Differential Equations

We now turn to the differential equation

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R}. \tag{3.1}$$

First we define two types of solutions to (3.1).

Definition 3.1. (1) A continuous function u is called a mild solution of (3.1) if

$$u(t) = u(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds, \tag{3.2}$$

for all $t \in \mathbb{R}$.

(2) A function u is a classical solution of (3.1), if $u(t) \in D(A)$, u is continuously differentiable, and (3.1) holds for $t \in \mathbb{R}$.

Remark 3.2. The mild solution to (3.1) defined by (3.2) is really an extension of classical solution in the sense that every classical solution is a mild solution and conversely, if a mild solution is continuously differentiable, then it is a classical solution.

If A is the generator of a C_0 semigroup $T(t)$, then a continuous function $u : \mathbb{R} \to E$ is a mild solution of (1.1) if and only if it has the form (see [7])

$$u(t) = T(t-s)u(s) + \int_s^t T(t-r)f(r)dr, \quad \text{for } s < t. \tag{3.3}$$

We now consider the almost periodic mild solutions of (3.1). The following proposition describes the connection between the Bohr transforms of such solutions and those of $f(t)$.

Proposition 3.3. *Suppose $f \in AP(\mathbb{R}, H)$ and u is an almost periodic mild solution of* (3.1). *Then*

$$(\lambda i - A)a(\lambda, u) = a(\lambda, f), \tag{3.4}$$

for every $\lambda \in \mathbb{R}$.

Proof. Suppose λ is a nonzero real number. Multiplying each side of (3.2) with $e^{-i\lambda t}$ and taking definite integral from $-T$ to T on both sides, we have

$$\begin{aligned}\int_{-T}^{T} e^{-i\lambda t}u(t)dt = \int_{-T}^{T} e^{-i\lambda t}u(0)dt + A\int_{-T}^{T} e^{-i\lambda t}\int_0^t u(s)ds\,dt \\ + \int_{-T}^{T} e^{-i\lambda t}\int_0^t f(s)ds\,dt.\end{aligned} \tag{3.5}$$

Here we used the fact that $\int_a^b Au(t)dt = A\int_a^b u(t)dt$ for a closed operator A. It is easy to see that

$$\int_{-T}^{T} e^{-i\lambda t}u(0)dt = -\frac{e^{-i\lambda T}u(0) - e^{i\lambda T}u(0)}{i\lambda} \tag{3.6}$$

and, applying integration by part for any integrable function $g(t)$, we have

$$\begin{aligned}\int_{-T}^{T} e^{-i\lambda t}\int_0^t g(s)ds\,dt &= e^{-i\lambda t}\int_0^t g(s)ds\Big|_{-T}^{T} + \frac{1}{i\lambda}\int_{-T}^{T} e^{-i\lambda t}g(t)dt \\ &= e^{-i\lambda T}\int_0^T g(t)dt - e^{i\lambda T}\int_0^{-T} g(t)dt + \frac{1}{i\lambda}\int_{-T}^{T} e^{-i\lambda t}g(t)dt.\end{aligned} \tag{3.7}$$

Using (3.7) for $g(t) = u(t)$ and $g(t) = f(t)$ in (3.5), respectively, we have

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t}u(t)dt &= -\frac{e^{-i\lambda T}u(0) - e^{i\lambda T}u(0)}{i\lambda 2T} \\ &\quad + \frac{e^{-i\lambda T}}{2T}\left(A\int_0^T u(t)dt + \int_0^T f(t)dt\right) \\ &\quad - \frac{e^{i\lambda T}}{2T}\left(A\int_0^{-T} u(t)dt + \int_0^{-T} f(t)dt\right) \\ &\quad + \frac{1}{i\lambda 2T}\left(A\int_{-T}^{T} e^{-i\lambda t}u(t)dt + \int_{-T}^{T} e^{-i\lambda t}f(t)dt\right) \\ &= I_1 + I_2 + I_3,\end{aligned} \tag{3.8}$$

where

$$I_1 = -\frac{e^{-i\lambda T}u(0) - e^{i\lambda T}u(0)}{i\lambda 2T} \longrightarrow 0 \tag{3.9}$$

as $T \to \infty$;

$$\begin{aligned}I_2 &= \frac{e^{-i\lambda T}}{2T}\left(A\int_0^T u(t)dt + \int_0^T f(t)dt\right) - \frac{e^{i\lambda T}}{2T}\left(A\int_0^{-T} u(t)dt + \int_0^{-T} f(t)dt\right) \\ &= \frac{e^{-i\lambda T}}{2T}(u(T) - u(0)) - \frac{e^{i\lambda T}}{2T}(u(-T) - u(0)) \longrightarrow 0\end{aligned} \tag{3.10}$$

as $T \to \infty$, and

$$I_3 = \frac{1}{i\lambda}\left(\frac{1}{2T}A\int_{-T}^{T} e^{-i\lambda t}u(t)dt + \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t}f(t)dt\right). \tag{3.11}$$

Let $u_T := (1/2T)\int_{-T}^{T} e^{-i\lambda t}u(t)dt$. It is clear that

$$\lim_{T\to\infty} u_T = a(\lambda, u), \tag{3.12}$$

and from (3.11), we have

$$\begin{aligned}
Au_T &= \frac{1}{2T} A \int_{-T}^{T} e^{-i\lambda t} u(t)\,dt \\
&= i\lambda I_3 - \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t} f(t)\,dt \\
&= i\lambda (u_T - I_1 - I_2) - \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t} f(t)\,dt \\
&\longrightarrow i\lambda a(\lambda, u) - a(\lambda, f) \quad \text{as } T \longrightarrow \infty.
\end{aligned} \tag{3.13}$$

Since A is a closed operator, from (3.12) and (3.13) we obtain $a(\lambda, u) \in D(A)$ and $Au(\lambda, u) = i\lambda a(\lambda, u) - a(\lambda, f)$, from which (3.4) is followed.

Finally, if $\lambda = 0$, let $u_T = (1/2T)\int_{-T}^{T} u(s)ds$. Then, $\lim_{t\to\infty} u_T = a(0, u)$ and, using the definition of u in (3.2),

$$\begin{aligned}
Au_T &= \frac{1}{2T} A \int_{-T}^{T} u(s)\,ds = \frac{u(T) - u(-T)}{2T} - \frac{1}{2T}\int_{-T}^{T} f(s)\,ds \\
&\longrightarrow -a(0, f) \quad \text{as } T \longrightarrow \infty.
\end{aligned} \tag{3.14}$$

Again, since A is a closed operator, it implies $a(a, u) \in D(A)$ and $Au(0, u) = -a(0, f)$, from which (3.4) is followed, and this completes the proof. □

Note that Proposition 3.3 also holds in a Banach space. We are now going to look for conditions that (3.1) has an almost periodic mild solution.

Theorem 3.4. *Suppose f is an almost periodic function, which is in $W_2(AP)$. Then the following statements are equivalent.*

(i) *Equation* (3.1) *has an almost periodic mild solution, which is in $W_2(AP)$.*

(ii) *For every $\lambda \in \sigma(f)$, $a(\lambda, f) \in Range(A)$ and there exists a series $\{x_\lambda\}_{\lambda\in\sigma(f)}$ in H satisfying $(i\lambda - A)x_\lambda = a(\lambda, f)$, for which the following holds*

$$\sum_{\lambda\in\sigma(f)} \|x_\lambda\|^2 < \infty, \qquad \sum_{\lambda\in\sigma(f)} |\lambda|^2 \|x_\lambda\|^2 < \infty. \tag{3.15}$$

Proof. (i)⇒(ii) Let $u(t)$ be an almost periodic solution to (3.1), which is in $W_2(AP)$. By Proposition 3.3, $(i\lambda - A)a(\lambda, u) = a(\lambda, f)$. Hence $a(\lambda, f) \in \text{Range}(A)$ for all $\lambda \in \sigma(f)$.

Put now $x_\lambda := a(\lambda, u)$ for $\lambda \in \sigma(f)$. Then it satisfies $(i\lambda - A)x_\lambda = a(\lambda, f)$. Moreover, $i\lambda x_\lambda = a(\lambda, u')$; hence,

$$\begin{aligned}
\sum_{\lambda\in\sigma(f)} \|x_\lambda\|^2 &= \|u\|_{AP}^2, \\
\sum_{\lambda\in\sigma(f)} |\lambda|^2 \|x_\lambda\|^2 &= \|u'\|_{AP}^2,
\end{aligned} \tag{3.16}$$

which imply (3.15).

(ii)⇒(i) Let $\{x_\lambda\}_{\lambda\in\sigma(f)}$ be a series in H satisfying $(i\lambda - A)x_\lambda = a(\lambda, f)$, for which (3.15) holds. Put

$$\begin{aligned} f_N(t) &:= \sum_{\lambda\in\sigma(f),|\lambda|<N} e^{i\lambda t}a(\lambda, f), \\ u_N(t) &:= \sum_{\lambda\in\sigma(f),|\lambda|<N} e^{i\lambda t}x_\lambda. \end{aligned} \tag{3.17}$$

It is then easy to find their norms:

$$\|u_N\|^2 = \sum_{\lambda\in\sigma(f),|\lambda|<N} \|x_\lambda\|^2, \qquad \left\|u_N'\right\|^2 = \sum_{\lambda\in\sigma(f),|\lambda|<N} |\lambda|^2\|x_\lambda\|^2. \tag{3.18}$$

From (3.15) it implies that $u_N \to u$ and $u_N' \to v$ as $N \to \infty$ for some function u and v in the topology of $AP(\mathbb{R}, H)$. Since the differential operator is closed, we obtain $u \in W_2$, $u' = v$ and $\lim_{N\to\infty} u_N = u$ in the topology of $W_2(AP)$. Hence, u is almost periodic. It remains to show that u is a mild solution of (1.1). In order to do that, note u_N is a classical solution of (3.1), and hence, a mild one, that is,

$$u_N(t) = u_N(0) + A\int_0^t u_N(s)ds + \int_0^t f_N(s)ds. \tag{3.19}$$

For each $t \in \mathbb{R}$, we have

$$\lim_{N\to\infty}\int_0^t f_N(s)ds = \int_0^t f(s)ds, \qquad \lim_{N\to\infty}\int_0^t u_N(s)ds = \int_0^t u(s)ds, \tag{3.20}$$

and, using (3.19),

$$\begin{aligned} \lim_{T\to\infty} A\int_0^t u_N(s)ds &= \lim_{T\to\infty}\left(u_N(t) - u_N(0) - \int_0^t f_N(s)ds\right) \\ &= u(t) - u(0) - \int_0^t f(s)ds. \end{aligned} \tag{3.21}$$

Since A is a closed operator, we obtain $\int_0^t u(s)ds \in D(A)$ and

$$A\int_0^t u(s)ds = u(t) - u(0) - \int_0^t f(s)ds, \tag{3.22}$$

which shows that u is a mild solution of (1.1) and the proof is complete. □

Note that if condition (ii) in Theorem 3.4 holds, (3.1) may have two or more almost periodic mild solutions. We are going to find conditions such that for each almost periodic function f, (3.1) has a unique almost periodic mild solution. We are now in the position to state the main result.

Theorem 3.5. *Suppose A is a closed operator on a Hilbert space H and M is a closed subset of $\mathbb{R}$. The following are equivalent.*

(i) *For each function $f \in W_2(AP)$ with $\sigma(f) \subseteq M$,* (3.1) *has a unique almost periodic mild solution u in $W_2(AP)$ with $\sigma(u) \subseteq M$.*

(ii) *For each $\lambda \in M$, $i\lambda \in \varrho(A)$ and*

$$\sup_{\lambda \in M} \left\| (i\lambda - A)^{-1} \right\| < \infty. \tag{3.23}$$

Proof. (i)⇒(ii) Let $W_2(AP)_{|M}$ be the subspace of all functions f in $W_2(AP)$ with $\sigma(f) \in M$. Then $W_2(AP)_{|M}$ is a Hilbert space by nature. Let x be any vector in H, let λ be a number in M, and let $f(t) = e^{i\lambda t}x$. Then $f \in W_2(AP)_{|M}$ and hence, (3.1) has a unique almost periodic solution u. By Theorem 3.4, $x = a(\lambda, f) \in \mathrm{Range}(i\lambda - A)$, hence $(i\lambda - A)$ is surjective for all $\lambda \in M$. On the other hand, $(i\lambda - A)$ is injective; otherwise, $u_2(t) = u(t) + e^{i\lambda t}x$, where x is a nonzero vector in H satisfying $(i\lambda - A)x = 0$, would be another almost periodic mild solution to (3.1) with $\sigma(u_2) = \sigma(u) \subseteq M$. Hence $(i\lambda - A)$ is bijective and $i\lambda \in \varrho(A)$ for all $\lambda \in M$.

In $W_2(AP)_{|M}$ we define the operator L by what follows. For each $f \in W_2(AP)_{|M}$, $L(f)$ is the unique almost periodic mild solution to (1.1) corresponding to f. By the assumption, L is everywhere defined. We will prove that L is a bounded operator by showing L is closed in $W_2(AP)_{|M}$. Let $f_n \to f$ and $Lf_n \to u$ in $W_2(AP)_{|M}$, where

$$(Lf_n)(t) = (Lf_n)(0) + A\int_0^t (Lf_n)(s)ds + \int_0^t f_n(s)ds. \tag{3.24}$$

For each $t \in \mathbb{R}$, we have $\lim_{n\to\infty} Lf_n(t) = u(t)$, $\lim_{N\to\infty}\int_0^t f_n(s)ds = \int_0^t f(s)ds$, and $\lim_{n\to\infty}\int_0^t Lf_n(s)ds = \int_0^t u(s)ds$. Moreover, from (3.24) we have

$$\begin{aligned} A\int_0^t (Lf_n)(s)ds &= (Lf_n)(t) - (Lf_n)(0) - \int_0^t f_n(s)ds \\ &\xrightarrow{n\to\infty} u(t) - u(0) - \int_0^t f(s)ds, \end{aligned} \tag{3.25}$$

for each $t \in \mathbb{R}$. Since A is a closed operator, $\int_0^t u(s)ds \in D(A)$ and

$$A\int_0^t u(s)ds = u(t) - u(0) - \int_0^t f(s)ds, \tag{3.26}$$

which means u is a mild solution to (3.1) corresponding to f. Thus, $f \in D(L)$, $Lf = u$ and hence, L is closed.

Next, for any $x \in H$ and $\lambda \in M$, put $f(t) = e^{i\lambda t}x$, then $u(t) = e^{i\lambda t}(2k\pi i - A)^{-1}x$ is the unique almost periodic solution to (3.1), that is, $u = Lf$. Using the boundedness of operator L, we obtain

$$(|\lambda| + 1)\left\|(i\lambda - A)^{-1}x\right\| = \|u\|_{W_2(AP)} \preccurlyeq \|L\|\|u\|_{W_2(AP)} = \|L\|(|\lambda| + 1)\|x\|, \tag{3.27}$$

which implies

$$\left\|(i\lambda - A)^{-1}x\right\| \preccurlyeq \|L\| \cdot \|x\|. \tag{3.28}$$

for any $x \in E$ and any $\lambda \in M$. Thus, (3.33) holds.

(ii)⇒(i) Suppose f is a function in $W_2(AP)_{|M}$. Put $x_\lambda := (i\lambda - A)^{-1}a(\lambda, f)$. Then

$$\begin{aligned}
\sum_{\lambda\in\sigma(f)} \|x_\lambda\|^2 &\preccurlyeq \sup_{\lambda\in\sigma(f)} \left\|(i\lambda - A)^{-1}\right\|^2 \sum_{\lambda\in\sigma(f)} \|a(\lambda, f)\|^2 \\
&\preccurlyeq \sup_{\lambda\in M}\left\|(i\lambda - A)^{-1}\right\|^2 \|f\|^2 < \infty, \\
\sum_{\lambda\in\sigma(f)} \lambda^2\|x_\lambda\|^2 &\preccurlyeq \sup_{\lambda\in\sigma(f)} \left\|(i\lambda - A)^{-1}\right\|^2 \sum_{\lambda\in\sigma(f)} \lambda^2\|a(\lambda, f)\|^2 \\
&= \sup_{\lambda\in M}\left\|(i\lambda - A)^{-1}\right\|^2 \|f'\|^2 < \infty.
\end{aligned} \tag{3.29}$$

By Proposition 3.3, (3.1) has an almost periodic mild solution in $W_2(AP)_{|M}$. That solution is unique, since its Bohr transforms are uniquely determined by $a(\lambda, u) = (i\lambda - A)^{-1}a(\lambda, f)$ for all $\lambda \in M$. □

We can apply Theorem 3.5 to some particular sets for M. First, if $M = \mathbb{R}$, we have the following.

Corollary 3.6. *Suppose A is a closed operator on a Hilbert space H. The following are equivalent.*

(i) *For each function $f \in W_2(AP)$, (3.1) has a unique 1-periodic mild solution in $W_2(AP)$.*

(ii) *$i\mathbb{R} \subseteq \varrho(A)$ and*

$$\sup_{\lambda\in\mathbb{R}}\left\|(i\lambda - A)^{-1}\right\| \le \infty. \tag{3.30}$$

Let now $L_2(0, 1)$ be the Hilbert space of integrable functions f from $(0, 1)$ to H with the norm

$$\|f\|^2_{L_2(0,1)} = \int_0^1 \|f(t)\|^2 dt < \infty. \tag{3.31}$$

If $M = \{2k\pi : k \in \mathbb{Z}\}$, then the space $W_2(AP)_{|M}$ becomes $W_2^1(1)$, the space of all periodic functions f of period 1 with $f' \in L_2(0, 1)$. $W_2^1(1)$ is then a Hilbert space with the norm

$$\|f\|^2_{W_2^1(1)} = \|f\|^2_{L_2(0,1)} + \|f'\|^2_{L_2(0,1)}. \tag{3.32}$$

Corollary 3.7. *Suppose A is a closed operator on a Hilbert space H. The following are equivalent.*

(i) *For each function $f \in W_2^1(1)$, (3.1) has a unique 1-periodic mild solution in $W_2^1(1)$.*

(ii) *For each* $k \in \mathbb{Z}$, $2ki\pi \in \varrho(A)$ *and*

$$\sup_{k\in\mathbb{Z}} \left\| (2ki\pi - A)^{-1} \right\| < \infty. \tag{3.33}$$

4. Application: A C_0-Semigroup Case

If A generates a C_0-semigroup $(T(t))_{t\geq 0}$, then (see [7, Theorem 2.5]), mild solutions of (3.1) can be expressed by

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau)d\tau, \tag{4.1}$$

for $t \geq s$. If f is a 1-periodic function, then it is easy to see that the above solution u is 1-periodic if and only if $u(1) = u(0)$. Hence, to consider 1-periodic solution, it suffices to consider u in $[0,1]$ and in this interval we have

$$u(t) = T(t)u(0) + \int_0^t T(t-s)f(s)ds. \tag{4.2}$$

We obtain the following results, in which we show the Gearhart's theorem (the equivalence (iv)⇔(v)) with a short proof.

Theorem 4.1. *Let A generate a C_0-semigroup $(T(t))$ on a Hilbert H, then the following are equivalent.*

(i) *For each function $f \in L_2(0,1)$,* (3.1) *has a unique 1-periodic mild solution.*

(ii) *For each function $f \in W_2^1(1)$,* (3.1) *has a unique 1-periodic classical solution.*

(iii) *For each function $f \in W_2^1(1)$,* (3.1) *has a unique 1-periodic solution contained in $W_2^1(1)$.*

(iv) *For each $k \in \mathbb{Z}$, $2k\pi i \in \varrho(A)$ and*

$$\sup_{k\in\mathbb{Z}} \left\| (2k\pi i - A)^{-1} \right\| < \infty. \tag{4.3}$$

(v) $1 \in \rho(T(1))$.

Proof. The equivalence (iii)⇔(iv) is shown in Corollary 3.7, (i)⇔(ii) can be easily proved by using standard arguments, (i)⇔(v) has been shown in [8], and (ii)⇒(iii) is obvious. So, it remains to show the inclusion (iii)⇒(ii).

Let f be any function in $W_2^1(1)$ and let $u(t)$ be the unique mild solution of (3.1), which is in $W_2^1(1)$. Since for each $f \in W_2^1(1)$, the function $g(t) := \int_0^t T(t-s)f(s)ds$ is continuously differentiable and $g(t) \in D(A)$ for all $t \in [0,1]$ (see [9]), to show u is a classical solution, it suffices to show $u(0) \in D(A)$.

From the above observation and from formula (4.2), the function $t \mapsto T(t)u(0) = u(t) - \int_0^t T(t-s)f(s)ds$ is differentiable almost everywhere on $[0,1]$. It follows that $T(t)u(0) \in D(A)$ for almost everywhere t (since $t \mapsto T(t)x$ is differentiable at t_0 if and only if $T(t_0)x \in D(A)$). Hence, $T(1)u(0) \in D(A)$. By formula (4.2), $u(1)$, and thus, $u(0) = u(1)$, belongs to $D(A)$. The uniqueness of this 1-periodic classical solution is obvious and the proof is complete. □

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