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Research Article

Double Integral Operators Concerning Starlike of Order β

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Double integral operators which were considered by S. S. Miller and P. T. Mocanu (Integral Transform. Spec. Funct. **19**(2008), 591–597) are discussed. In order to show the analytic function $f(z)$ is starlike of order β in the open unit disk $\mathbb{U}$, the theory of differential subordinations for analytic functions is applied. The object of the present paper is to discuss some interesting conditions for $f(z)$ to be starlike of order β in $\mathbb{U}$ concerned with second-order differential inequalities and double integral operators.

1. Introduction, Definition and Preliminaries

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of functions $f(z)$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}. \tag{1.1}$$

For a positive integer n and $a \in \mathbb{C}$, we define the following classes of analytic functions:

$$\mathcal{H}[a, n] = \left\{ f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,\ z \in \mathbb{U} \right\} \tag{1.2}$$

and

$$\mathcal{A}_n = \left\{ f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots,\ z \in \mathbb{U} \right\} \tag{1.3}$$

with $\mathcal{A}_1 = \mathcal{A}$.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order β in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in \mathbb{U}), \tag{1.4}$$

for some $\beta(0 \leqq \beta < 1)$. We denote by $\mathcal{S}^*(\beta)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are starlike of order β in $\mathbb{U}$.

By the familiar principle of differential subordinations between analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$.

We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}). \tag{1.5}$$

In particular, if $g(z)$ is univalent in $\mathbb{U}$, then it is known that

$$f(z) \prec g(z), \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.6}$$

To obtain some results of this paper, we need the following two lemmas concerning the differential subordinations.

Lemma 1.1 (see [1, Hallenbeck and Ruscheweyh]). *Let $h(z)$ be a convex function with $h(0) = a$ and let $\operatorname{Re}\gamma > 0$. If $p(z) \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \tag{1.7}$$

then

$$p(z) \prec q(z) \prec h(z), \tag{1.8}$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}\, dt. \tag{1.9}$$

This result is sharp.

Lemma 1.2 (see [2, Al-Amiri and Mocanu]). *Let n be a positive integer, and let α be real with $0 \leqq \alpha < n$. Let $q(z) \in \mathcal{H}$ with $q(0) = 0, q'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zq''(z)}{q'(z)} + 1 > \frac{\alpha}{n}. \tag{1.10}$$

If $p \in \mathcal{H}[0, n]$ *satisfies*

$$zp'(z) - \alpha p(z) \prec nzq'(z) - \alpha q(z), \tag{1.11}$$

then $p(z) \prec q(z)$ *and this result is sharp.*

By making use of these lemmas, Miller and Mocanu [3] have investigated some second-order differential inequality that implies starlikeness and deduced the following lemma.

Lemma 1.3. *Let* $f(z) \in \mathcal{A}_n$ *and let* $0 \leqq \alpha < n$*. If* $f(z)$ *satisfies*

$$\left|zf''(z) - \alpha\left(f'(z) - 1\right)\right| < n - \alpha, \tag{1.12}$$

then $f(z)$ *is starlike.*

Furthermore, by using Lemma 1.3, Miller and Mocanu [3] obtained some result concerning the double integral starlike operator as follows.

Lemma 1.4. *Let* $0 \leqq \alpha < n$ *and let* $g(z) \in \mathcal{H}$*. If* $g(z)$ *satisfies*

$$\left|g(z)\right| \leqq n - \alpha, \tag{1.13}$$

then the function $f(z)$ *given by*

$$f(z) = z + z^{n+1} \iint_0^1 g(rsz) r^{n-\alpha-1} s^n \, dr \, ds \tag{1.14}$$

is starlike.

2. Some Second-Order Differential Inequalities for Starlike of Order β

In this section, we deduced some conditions concerning the second-order differential inequality to show that $f(z)$ is starlike of order β in $\mathbb{U}$.

Theorem 2.1. *Let* $f(z) \in \mathcal{A}_n$ *and let* $0 \leqq \alpha < n$ *and* $0 \leqq \beta < 1$*. If* $f(z)$ *satisfies*

$$\left|zf''(z) - \alpha\left(f'(z) - 1\right)\right| < \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta}, \tag{2.1}$$

then $f(z)$ *is starlike of order* β *in* $\mathbb{U}$*.*

Proof. We can rewrite the inequality (2.1) in terms of subordination as

$$zf''(z) - \alpha\left(f'(z) - 1\right) \prec \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta} z. \tag{2.2}$$

If we set

$$\begin{aligned} P(z) &= f'(z) - (1+\alpha)\frac{f(z)}{z} \\ &= -\alpha + (n-\alpha)a_{n+1}z^n + (n+1-\alpha)a_{n+2}z^{n+1} + \cdots \in \mathcal{H}[-\alpha, n], \end{aligned} \tag{2.3}$$

then the subordination (2.2) becomes

$$P(z) + zP'(z) \prec -\alpha + \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta}z. \tag{2.4}$$

Applying Lemma 1.1 to this first-order differential subordination, we obtain

$$P(z) \prec \frac{1}{nz^{1/n}}\int_0^z \left\{-\alpha + \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta}t\right\} t^{1/n-1}\, dt = -\alpha + \frac{(1-\beta)(n-\alpha)}{n+1-\beta}z, \tag{2.5}$$

or equivalently

$$f'(z) - (1+\alpha)\frac{f(z)}{z} \prec -\alpha + \frac{(1-\beta)(n-\alpha)}{n+1-\beta}z. \tag{2.6}$$

If we consider

$$q(z) = \frac{1-\beta}{n+1-\beta}z \quad \left(q(0) = 0,\ q'(0) = \frac{1-\beta}{n+1-\beta} \neq 0\right) \tag{2.7}$$

and

$$p(z) = \frac{f(z)}{z} - 1 = a_{n+1}z^n + a_{n+2}z^{n+1} + \cdots \in \mathcal{H}[0, n], \tag{2.8}$$

then the subordination (2.6) can be written as

$$zp'(z) - \alpha p(z) \prec \frac{(1-\beta)(n-\alpha)}{n+1-\beta}z = nzq'(z) - \alpha q(z). \tag{2.9}$$

Since $0 \leqq \alpha < n$ and the function $q(z)$ satisfies

$$\operatorname{Re}\frac{zq''(z)}{q'(z)} + 1 = 1 > \frac{\alpha}{n}, \tag{2.10}$$

using Lemma 1.2, we obtain the subordination $p(z) \prec q(z)$, or

$$\frac{f(z)}{z} - 1 \prec \frac{1-\beta}{n+1-\beta}z. \tag{2.11}$$

It follows from the subordination (2.6) that

$$\left| f'(z) - (1+\alpha)\frac{f(z)}{z} \right| < \alpha + \frac{(1-\beta)(n-\alpha)}{n+1-\beta} = \frac{n(1+\alpha-\beta)}{n+1-\beta}, \tag{2.12}$$

while from the subordination (2.11) that

$$\left| \frac{f(z)}{z} \right| > 1 - \frac{1-\beta}{n+1-\beta} = \frac{n}{n+1-\beta}. \tag{2.13}$$

Combining these last two inequalities, we see that

$$\begin{aligned} \frac{n}{n+1-\beta}\left| \frac{zf'(z)}{f(z)} - (1+\alpha) \right| &< \left| \frac{f(z)}{z} \right| \left| \frac{zf'(z)}{f(z)} - (1+\alpha) \right| \\ &= \left| f'(z) - (1+\alpha)\frac{f(z)}{z} \right| < \frac{n(1+\alpha-\beta)}{n+1-\beta}, \end{aligned} \tag{2.14}$$

which simplifies to

$$\left| \frac{zf'(z)}{f(z)} - (1+\alpha) \right| < \frac{n(1+\alpha-\beta)}{n+1-\beta} \cdot \frac{n+1-\beta}{n} = 1+\alpha-\beta. \tag{2.15}$$

This gives us that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > (1+\alpha) - (1+\alpha-\beta) = \beta, \tag{2.16}$$

which proves that $f(z)$ is starlike of order β in $\mathbb{U}$. □

We introduce the following example for Theorem 2.1.

Example 2.2. For the function $f(z) = z + ((1-\beta)/(n+1-\beta))z^{n+1}$ $(0 \leqq \beta < 1)$, we have

$$\begin{aligned} \left| zf''(z) - \alpha(f'(z) - 1) \right| &= \left| \frac{n(n+1)(1-\beta)}{n+1-\beta}z^n - \alpha\frac{(n+1)(1-\beta)}{n+1-\beta}z^n \right| \\ &= \left| \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta} \right| |z|^n < \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta}. \end{aligned} \tag{2.17}$$

Furthermore, we see that

$$\begin{aligned}\operatorname{Re}\frac{zf'(z)}{f(z)} &= \operatorname{Re}\left(\frac{1+((n+1)(1-\beta)/(n+1-\beta))z^n}{1+((1-\beta)/(n+1-\beta))z^n}\right) \\ &> \frac{1-(n+1)(1-\beta)/(n+1-\beta)}{1-((1-\beta)/(n+1-\beta))} = \beta \quad (z \in \mathbb{U}).\end{aligned} \tag{2.18}$$

Remark 2.3. Letting $\beta = 0$ in Theorem 2.1, we obtain Lemma 1.3 given by Miller and Mocanu [3].

Also, setting $\alpha = 0$ in Theorem 2.1, we have the following.

Corollary 2.4. *Let* $f(z) \in \mathcal{A}_n$ *and let* $0 \leqq \beta < 1$. *If* $f(z)$ *satisfies*

$$\left|zf''(z)\right| < \frac{n(n+1)(1-\beta)}{n+1-\beta}, \tag{2.19}$$

then $f(z)$ *is starlike of order* β *in* $\mathbb{U}$.

For the case $n = 1$ in the above corollary, we find the following.

Remark 2.5. For $f(z) \in \mathcal{A}$, we have that

$$\left|zf''(z)\right| < \frac{2(1-\beta)}{2-\beta} \Longrightarrow f(z) \in \mathcal{S}^*(\beta). \tag{2.20}$$

The case $\beta = 0$ was first discussed by Obradović [4].

Next, by making use of Theorem 2.1, we obtain the following result concerning the double, integral operator for starlike of order β.

Theorem 2.6. *Let a function* $g(z) \in \mathcal{H}$ *satisfy*

$$\left|g(z)\right| \leqq \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta}, \tag{2.21}$$

for some $0 \leqq \alpha < n$ *and* $0 \leqq \beta < 1$. *Then the function* $f(z)$ *given by*

$$f(z) = z + z^{n+1}\iint_0^1 g(rsz)r^{n-\alpha-1}s^n \, dr\, ds \tag{2.22}$$

is starlike of order β *in* $\mathbb{U}$.

Proof. We first consider the function $f(z) \in \mathcal{A}_n$ satisfying the differential equation

$$zf''(z) - \alpha\left(f'(z) - 1\right) = z^n g(z). \tag{2.23}$$

Then, it is clear that

$$\left|zf''(z)-\alpha\left(f'(z)-1\right)\right| = |z|^n\left|g(z)\right| < \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta} \quad (z\in\mathbb{U}). \tag{2.24}$$

Thus, from Theorem 2.1, we see that the solution of the differential equation (2.23) must be starlike of order β. The solution of (2.23) can be obtained in two integrations. If we set $\varphi(z) = f'(z)-1$, then the equation (2.23) can be simplified to

$$z\varphi'(z)-\alpha\varphi(z) = z^n g(z), \tag{2.25}$$

which has the solution $\varphi(z)$ given by

$$\varphi(z) = z^{\alpha}\int_0^z g(\zeta)\zeta^{n-\alpha-1}\,d\zeta = z^n\int_0^1 g(rz)r^{n-\alpha-1}\,dr. \tag{2.26}$$

Since $\varphi(z) = f'(z)-1$, we have

$$f'(z)-1 = z^n\int_0^1 g(rz)r^{n-\alpha-1}\,dr, \tag{2.27}$$

that is,

$$f(z) = z+\int_0^z \zeta^n\left(\int_0^1 g(r\zeta)r^{n-\alpha-1}\,dr\right)d\zeta = z+z^{n+1}\iint_0^1 g(rsz)r^{n-\alpha-1}s^n\,dr\,ds. \tag{2.28}$$

□

Remark 2.7. Taking $\beta=0$ in Theorem 2.6, we find Lemma 1.4 given by Miller and Mocanu [3]. However, (1.14) and (2.22) are double integral operators of the same form.

Moreover, making $n=1$ and $\alpha=0$ in Theorem 2.6, we have the following.

Corollary 2.8. *If $g(z)\in\mathcal{H}$ and*

$$\left|g(z)\right| \leqq \frac{2(1-\beta)}{2-\beta} \quad (z\in\mathbb{U}) \tag{2.29}$$

for some $0\leqq\beta<1$, then

$$f(z) = z+z^2\iint_0^1 g(rsz)s\,dr\,ds \in \mathcal{S}^*(\beta). \tag{2.30}$$

As examples of Corollary 2.8, we get the following.

Example 2.9. For the function $g(z) = 2(1-\beta)/(2-\beta)$ $(0 \leqq \beta < 1)$, we find

$$f(z) = z + \frac{1-\beta}{2-\beta} z^2 \in \mathcal{S}^*(\beta), \tag{2.31}$$

because

$$\operatorname{Re} \frac{zf'(z)}{f(z)} = \operatorname{Re}\left(\frac{1 + (2(1-\beta)/(2-\beta))z}{1 + ((1-\beta)/(2-\beta))z} \right) > \frac{1 - (2(1-\beta)/(2-\beta))}{1 - ((1-\beta)/(2-\beta))} = \beta \quad (z \in \mathbb{U}). \tag{2.32}$$

Example 2.10. For the function $g(z) = (2(1-\beta)/(2-\beta))z$ $(0 \leqq \beta < 1)$, we have

$$f(z) = z + \frac{1-\beta}{3(2-\beta)} z^3. \tag{2.33}$$

Then, we see that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} = \operatorname{Re}\left(\frac{1 + ((1-\beta)/(2-\beta))z^2}{1 + ((1-\beta)/3(2-\beta))z^2} \right) > \frac{3}{5-2\beta} > \beta \quad (z \in \mathbb{U}), \tag{2.34}$$

that is,

$$f(z) = z + \frac{1-\beta}{3(2-\beta)} z^3 \in \mathcal{S}^*\left(\frac{3}{5-2\beta} \right) \subset \mathcal{S}^*(\beta). \tag{2.35}$$

3. Other Result for Starlikeness of Order β

To obtain that $f(z)$ is starlike of order β in Theorem 2.1, we showed that

$$\left| \frac{zf'(z)}{f(z)} - (1+\alpha) \right| < 1 + \alpha - \beta \quad (z \in \mathbb{U}). \tag{3.1}$$

In this section, to obtain that $f(z)$ is starlike of order β, we consider some second-order differential inequality concerning the order β and show the following inequality:

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta} \quad (z \in \mathbb{U}) \tag{3.2}$$

for some $0 < \beta < 1$.

Remark 3.1. Foe some $\beta(0<\beta<1)$, we see that

$$\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\beta}\right|<\frac{1}{2\beta}\Longleftrightarrow \operatorname{Re}\frac{zf'(z)}{f(z)}>\beta. \tag{3.3}$$

Now, we consider the following theorem.

Theorem 3.2. *Let $f(z)\in\mathcal{A}_n$ and let $0<\beta<1$. If $f(z)$ satisfies*

$$\left|zf''(z)+(1-2\beta)(f'(z)-1)\right|<\begin{cases}\dfrac{\beta(n+1)(n+1-2\beta)}{n+1-\beta} & \left(0<\beta<\dfrac{1}{2}\right)\\[2mm] \dfrac{(1-\beta)(n+1)(n+1-2\beta)}{n+1-\beta}, & \left(\dfrac{1}{2}\leqq\beta<1\right),\end{cases} \tag{3.4}$$

then

$$\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\beta}\right|<\frac{1}{2\beta}\quad (z\in\mathbb{U}), \tag{3.5}$$

or $f(z)$ is starlike of order β in $\mathbb{U}$.

Proof. (i) For the case $0<\beta<1/2$, the inequality (3.4) can be written as follows:

$$zf''(z)+(1-2\beta)(f'(z)-1)\prec\frac{\beta(n+1)(n+1-2\beta)}{n+1-\beta}z. \tag{3.6}$$

If we set

$$\begin{aligned}P(z)&=f'(z)-2\beta\frac{f(z)}{z}\\&=(1-2\beta)+(n+1-2\beta)a_{n+1}z^n+(n+2-2\beta)a_{n+2}z^{n+1}+\cdots\in\mathcal{H}[1-2\beta,n],\end{aligned} \tag{3.7}$$

then the subordination (3.6) becomes

$$P(z)+zP'(z)\prec(1-2\beta)+\frac{\beta(n+1)(n+1-2\beta)}{n+1-\beta}z. \tag{3.8}$$

Applying Lemma 1.1 as well as the proof of Theorem 2.1, we obtain that

$$P(z)\prec\frac{1}{nz^{1/n}}\int_0^z\left\{(1-2\beta)+\frac{\beta(n+1)(n+1-2\beta)}{n+1-\beta}t\right\}t^{1/\mathrm{n}-1}\,dt=(1-2\beta)+\frac{\beta(n+1-2\beta)}{n+1-\beta}z, \tag{3.9}$$

or equivalently, that

$$f'(z) - 2\beta\frac{f(z)}{z} \prec (1-2\beta) + \frac{\beta(n+1-2\beta)}{n+1-\beta}z. \tag{3.10}$$

Also, if the function $p(z)$ is defined by

$$p(z) = f'(z) - 1 = (n+1)a_{n+1}z^n + (n+2)a_{n+2}z^{n+1} + \cdots \in \mathcal{H}[0,n], \tag{3.11}$$

then the subordination (3.6) becomes

$$zp'(z) + (1-2\beta)p(z) \prec \frac{\beta(n+1)(n+1-2\beta)}{n+1-\beta}z, \tag{3.12}$$

namely,

$$p(z) + \frac{zp'(z)}{1-2\beta} \prec \frac{\beta(n+1)(n+1-2\beta)}{(1-2\beta)(n+1-\beta)}z. \tag{3.13}$$

Since $0 < 1-2\beta < 1$, we can use Lemma 1.1 as $\gamma = 1-2\beta$ and obtain

$$p(z) \prec \frac{1-2\beta}{nz^{(1-2\beta)/n}}\int_0^z \frac{\beta(n+1)(n+1-2\beta)}{(1-2\beta)(n+1-\beta)}t^{(1-2\beta)/n}\, dt = \frac{\beta(n+1)}{n+1-\beta}z, \tag{3.14}$$

that is,

$$f'(z) - 1 \prec \frac{\beta(n+1)}{n+1-\beta}z. \tag{3.15}$$

From the subordination (3.10), we find

$$\left|f'(z) - 2\beta\frac{f(z)}{z}\right| < (1-2\beta) + \frac{\beta(n+1-2\beta)}{n+1-\beta} = \frac{n(1-\beta)+(1-2\beta)}{n+1-\beta}, \tag{3.16}$$

while, from the subordination (3.15), we get

$$\left|f'(z)\right| > 1 - \frac{\beta(n+1)}{n+1-\beta} = \frac{n(1-\beta)+(1-2\beta)}{n+1-\beta}. \tag{3.17}$$

By combining these last two inequalities, we obtain that

$$\begin{aligned}\frac{n(1-\beta)+(1-2\beta)}{n+1-\beta}\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\beta}\right| &< \left|f'(z)\right|\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\beta}\right| \\ &= \frac{1}{2\beta}\left|f'(z)-2\beta\frac{f(z)}{z}\right| < \frac{n(1-\beta)+(1-2\beta)}{2\beta(n+1-\beta)} \quad (z\in\mathbb{U}),\end{aligned} \tag{3.18}$$

which simplifies to

$$\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\beta}\right| < \frac{1}{2\beta} \quad (z\in\mathbb{U}). \tag{3.19}$$

(ii) For the case $1/2 \leqq \beta < 1$, we can rewrite (3.4) in terms of the subordination as

$$zf''(z)+(1-2\beta)(f'(z)-1) \prec \frac{(1-\beta)(n+1)(n+1-2\beta)}{n+1-\beta}z. \tag{3.20}$$

Applying Lemma 1.1 in similar to case (i), we can obtain that

$$f(z)-2\beta\frac{f(z)}{z} \prec (1-2\beta)+\frac{(1-\beta)(n+1-2\beta)}{n+1-\beta}z. \tag{3.21}$$

Furthermore, if we set

$$q(z)=\frac{(1-\beta)(n+1)}{n+1-\beta}z \quad \text{and} \quad p(z)=f'(z)-1\in\mathcal{H}[0,n], \tag{3.22}$$

then the subordination (3.20) can be written as

$$zp'(z)-(2\beta-1)p(z) \prec nzq'(z)-(2\beta-1)q(z). \tag{3.23}$$

Since $0 \leqq 2\beta-1 < 1 \leqq n$ and the function $q(z)$ satisfies

$$q(0)=0, \quad q'(0)=\frac{(1-\beta)(n+1)}{n+1-\beta}\neq \quad \text{and} \quad \operatorname{Re}\frac{zf''(z)}{f'(z)}+1=1>\frac{2\beta-1}{n}, \tag{3.24}$$

we can use Lemma 1.2 and obtain the following fact:

$$zp'(z)-(2\beta-1)p(z) \prec nzq'(z)-(2\beta-1)q(z) \Longrightarrow p(z)\prec q(z). \tag{3.25}$$

This implies that

$$f'(z)-1 \prec \frac{(1-\beta)(n+1)}{n+1-\beta}z. \tag{3.26}$$

From subordinations (3.21) and (3.26), we find

$$\left|f'(z)-2\beta\frac{f(z)}{z}\right|<\left|(1-2\beta)-\frac{(1-\beta)(n+1-2\beta)}{n+1-\beta}\right|=\frac{\beta n}{n+1-\beta}\quad(z\in\mathbb{U}) \tag{3.27}$$

and

$$|f'(z)|>1-\frac{(1-\beta)(n+1)}{n+1-\beta}=\frac{\beta n}{n+1-\beta}\quad(z\in\mathbb{U}). \tag{3.28}$$

Therefore, we obtain

$$\begin{aligned}\frac{\beta n}{n+1-\beta}\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\beta}\right|&<|f'(z)|\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\beta}\right|\\&=\frac{1}{2\beta}\left|f'(z)-2\beta\frac{f(z)}{z}\right|<\frac{\beta n}{2\beta(n+1-\beta)}\quad(z\in\mathbb{U}),\end{aligned} \tag{3.29}$$

which simplifies to

$$\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\beta}\right|<\frac{1}{2\beta}\quad(z\in\mathbb{U}). \tag{3.30}$$

This completes the proof of this theorem. □

Making use of Theorem 3.2, we obtain the following result concerning the double integral operator for starlike of order β.

Theorem 3.3. *If* $g(z)\in\mathscr{H}$ *satisfies*

$$|g(z)|\leqq\begin{cases}\dfrac{\beta(n+1)(n+1-2\beta)}{n+1-\beta} & \left(0<\beta<\dfrac{1}{2}\right)\\[2ex]\dfrac{(1-\beta)(n+1)(n+1-2\beta)}{n+1-\beta} & \left(\dfrac{1}{2}\leqq\beta<1\right),\end{cases} \tag{3.31}$$

for some $0<\beta<1$*, then*

$$f(z)=z+z^{n+1}\iint_0^1 g(rsz)r^{n-2\beta}s^n\,dr\,ds, \tag{3.32}$$

is a starlike function of order β.

This theorem can be shown as well as the proof of Theorem 2.6.

Making $n = 1$ in Theorem 3.3, we have

Corollary 3.4. *If* $g(z) \in \mathcal{H}$ *satisfies*

$$|g(z)| \leqq \begin{cases} \dfrac{4\beta(1-\beta)}{2-\beta} & \left(0 < \beta < \dfrac{1}{2}\right) \\ \dfrac{4(1-\beta)^2}{2-\beta} & \left(\dfrac{1}{2} \leqq \beta < 1\right), \end{cases} \tag{3.33}$$

for some $0 < \beta < 1$, *then*

$$f(z) = z + z^2 \iint_0^1 g(rsz) r^{1-2\beta} s \, dr \, ds \in \mathcal{S}^*(\beta). \tag{3.34}$$

Let us consider an example for Corollary 3.4.

Example 3.5. If we consider $g(z) = 4\beta(1-\beta)/(2-\beta)$ $(0 < \beta < 1/2)$, then we find

$$f(z) = z + \frac{\beta}{2-\beta} z^2 \in \mathcal{S}^*\left(\frac{2-3\beta}{2(1-\beta)}\right) \subset \mathcal{S}^*(\beta), \tag{3.35}$$

because

$$\operatorname{Re} \frac{zf'(z)}{f(z)} = \operatorname{Re}\left(\frac{1 + (2\beta/(2-\beta))z}{1 + (\beta/(2-\beta))z}\right) > (2-3\beta)/2(1-\beta) > \beta \quad (z \in \mathbb{U}). \tag{3.36}$$

Further, if we take $g(z) = 4(1-\beta)^2/(2-\beta)$ $(1/2 \leqq \beta < 1)$, then we see that

$$f(z) = z + \frac{1-\beta}{2-\beta} z^2 \in \mathcal{S}^*(\beta). \tag{3.37}$$

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