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Research Article

Oscillation Criteria for Even Order Neutral Equations with Distributed Deviating Argument

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We present new oscillation criteria for the even order neutral delay differential equations with distributed deviating argument $[r(t)\psi(x(t))Z^{(n-1)}(t)]' + \int_a^b p(t,\xi)f[x(g(t,\xi))]d\sigma(\xi) = 0$, $t \geq t_0$, where $Z(t) = x(t) + q(t)x(t-\tau)$. Assumptions in our theorems are less restrictive, whereas the proofs are significantly simpler compared to those by Wang et al. (2005).

1. Introduction

In this paper, we are concerned with the oscillation behavior of the even order neutral delay differential equations of the form

$$\left[r(t)\psi(x(t))Z^{(n-1)}(t)\right]' + \int_a^b p(t,\xi)f\left[x\left(g(t,\xi)\right)\right]d\sigma(\xi) = 0, \quad t \geq t_0, \tag{1.1}$$

where $Z(t) = x(t) + q(t)x(t-\tau)$, $\tau \geqslant 0$ and n *is an even positive integer*. We assume that

(A_1) $r, q \in C(I,R)$ and $0 \leq q(t) \leq 1$, $r(t) > 0$ for $t \in I$, $\int^\infty (1/r(s))ds = \infty$, $I = [t_0, \infty)$;

(A_2) $\psi \in C^1(R,R)$, $\psi(x) > 0$ for $x \neq 0$;

(A_3) $f \in C(R,R)$, $xf(x) > 0$ for $x \neq 0$;

(A_4) $p \in C(I \times [a,b], [0,\infty)$ and $p(t,\xi)$ is not eventually zero on any half linear $[t_u, \infty) \times [a,b]$, $t_u \geq t_0$;

(A_5) $g \in C(I \times [a,b], [0,\infty))$, $g(t,\xi) \leq t$ for $\xi \in [a,b]$, $g(t,\xi)$ has a continuous and positive partial derivative on $I \times [a,b]$ with respect to t and nondecreasing with respect to ξ, respectively, $\liminf_{t\to\infty} g(t,\xi) = \infty$ for $\xi \in [a,b]$;

(A_6) $\sigma \in C([a,b],R)$ is nondecreasing, and the integral of (1.1) is in the sense of Riemann-Stieltijes.

We restrict our attention to those solutions $x(t)$ of (1.1) which exist on some half linear $[t_x,\infty)$ and satisfy $\sup\{|x(t)| : t \geqslant t_x\} \neq 0$ for any $T \geqslant t_0$. As usual, such a solution of (1.1) is called oscillatory if the set of its zeros is unbounded from above; otherwise, it is said to be nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

The oscillatory behavior of solutions of higher-order neutral differential equations is of both theoretical and practical interest. There have been some results on the oscillatory and asymptotic behavior of even order neutral equations. We mention here [1–12]. The oscillation problem for nonlinear delay equation such as

$$\left[r(t)x'(t)\right]' + q(t)f(x(\sigma(t))) = 0, \quad t > t_0 \tag{1.2}$$

as well as for the the linear ordinary differential equation

$$\left[r(t)x'(t)\right]' + p(t)x'(t) + q(t)x(t) = 0, \quad t > t_0 \tag{1.3}$$

and the neutral delay differential equation

$$\left(x(t) + q(t)x(t-\sigma)\right)'' + p(t)x(t-\tau) = 0 \tag{1.4}$$

has been studied by many authors with different methods. In [13], Rogovchenko established some general oscillation criteria for second-order nonlinear differential equation:

$$\left(r(t)x'(t)\right)' + p(t)x'(t) + q(t)f(x(t)) = 0, \quad t \geq t_0. \tag{1.5}$$

In [14], the authors discussed the following neutral equations of the form

$$[x(t) + c(t)x(t-\tau)]^{(n)} + \int_a^b p(t,\xi)x\left[g(t,\xi)\right]d\sigma(\xi) = 0, \quad t \geq t_0 \tag{1.6}$$

and obtained the following results.

Theorem A (see [14, Theorem 2]). *Assume that there exist functions $H(t,s) \in C'(D;R), h(t,s) \in C(D_0;R)$, and $\rho(t) \in C'([t_0,\infty),(0,\infty))$, such that*

(I) $H(t,t) = 0, H(t,s) > 0$;

(II) $H'_s(t,s) \leq 0$, *and* $-H'_s(t,s) - H(t,s)(\rho'(s)/\rho(s)) = h(t,s)\sqrt{H(t,s)}$, *and*

$$0 < \inf_{s \geq t_0}\left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] \leq \infty; \tag{C_1}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} \frac{\rho(s)h^2(t,s)}{g^{n-2}(s,a)g'(s,a)}ds < \infty. \tag{C_2}$$

If there exists a function $\varphi(t) \in C([t_0,\infty),R)$ *satisfying*

$$\limsup_{t\to\infty} \frac{1}{H(t,u)} \int_u^t \left[H(t,s)\rho(s) \int_a^b p(s,\xi)(1-c[g(s,\xi)])d\sigma(\xi) - \frac{\rho(s)h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)} \right] ds \geq \varphi(u), \quad u \geq t_0, \tag{C_3}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \frac{g'(u,a)g^{n-2}(u,a)}{\rho(u)} \varphi_+^2(u)du = \infty, \quad \varphi_+(u) = \max_{u\geq t_0}\{\varphi(u),0\}, \tag{C_4}$$

then every solution of (1.6) *is oscillatory.*

We will use the function class W to study the oscillation criteria for (1.1). Let $D = \{(t,s) \mid t \geq s \geq t_0\}$, and $D_0 = \{(t,s)t > s \geq t_0\}$. We say that a continuous function $H(t,s) \subset C'(D,\Re)$ belongs to the class W if

(A_7) $H(t,t) = 0$ *and* $H(t,s) > 0$ *for* $-\infty < s < t < +\infty$;

(A_8) H has a continuous partial derivative $\partial H/\partial s$ satisfying, for some $h \in L_{\text{loc}}(D,R)$, the condition $\partial H/\partial s = -h(t,s)\sqrt{H(t,s)}$.

The purpose of this paper is to further improve Theorem A by Wang et al. [14], using a generalized Riccati transformation and developing ideas exploited by the Rogovchenko and Tuncay [13], we establish some new oscillation criteria for (1.1), which remove condition (C_2) in Theorem A by Wang et al. [14]; this complements and extends the results in [14].

In addition, we will make use of the following conditions.

(S_1) There exists a positive real number M such that $|f(\pm uv)| \geq Mf(u)f(v)$ for $uv > 0$.

Lemma 1.1. *If* $a > 0, b \geq 0$, *then*

$$-ax^2 + bx \leq -\frac{a}{2}x^2 + \frac{b^2}{2a}. \tag{1.7}$$

Lemma 1.2 (Kiguradze [15]). *Let u(t) be a positive and* n *times differentiable function on* R. *If* $u^{(n)}(t)$ *is of constant sign and identically zero on any ray* $[t,+\infty)$ *for* $(t_1 > 0)$, *then there exists a* $t_u \geq t_1$ *and an integer* l $(0 \leq l \leq n)$, *with* $n+l$ *even for* $u(t)u^{(n)}(t) \geq 0$ *or* $n+l$ *odd for* $u(t)u^{(n)}(t) \leq 0$, *and for* $t \geq t_u$,

$$u(t)u^{(k)}(t) > 0, \quad 0 \leq k \leq l; \quad (-1)^{k-1}u(t)u^{(k)}(t) > 0, \quad l \leq k \leq n. \tag{1.8}$$

Lemma 1.3 (Philos [16]). *Suppose that the conditions of Lemma 1.2 are satisfied, and*

$$u^{(n-1)}(t)u^{(n)}(t) \leq 0, \quad t \geq t_u, \tag{1.9}$$

then there exists a constant θ *in (0,1) such that for sufficiently large* t, *and there exists a constant* $M_\theta > 0$ *satisfying*

$$\left|u'\left(\frac{t}{2}\right)\right| \geq M_\theta t^{n-2}\left|u^{(n-1)}(t)\right|, \tag{1.10}$$

where $M_\theta = \theta/(n-2)!$.

2. When $f(x)$ Is Monotone

In this section, we will deal with the oscillation for (1.1) under the assumptions (A_1)–(A_8), (S_1) and the following assumption.

(A_9) $f'(x)$ exists, $f'(x) \geq K_1$ and $\psi(x) \leq L^{-1}$ for $x \neq 0$.

Theorem 2.1. *Let* (S_1), (A_1)–(A_9) *hold. Equation* (1.1) *is oscillatory provided that* $\rho(t) \in C^1([t_0,\infty), R)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)Q(s) - \frac{h^2(t,s)\rho(s)r(s)\beta}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)}\right]ds = \infty, \tag{2.1}$$

where

$$Q(t) = \rho(t)M\int_a^b p(t,\xi)f\left[1-q\left(g(t,\xi)\right)\right]d\sigma(\xi) - \frac{\left(\rho'(t)\right)^2 r(t)}{K_1 L M_\theta g^{n-2}(t,a)g'(t,a)\rho(t)}. \tag{2.2}$$

Proof. Suppose to the contrary that there exists a solution $x(t)$ of (1.1) such that

$$x(t) > 0, \quad x(t-\tau) > 0, \quad x\left[g(t,\xi)\right] > 0, \quad t \geq t_1, \quad \xi \in [a,b] \quad \text{for } t \geqslant t_1 \geqslant t_0. \tag{2.3}$$

From (1.1), we also have $Z(t) > 0$ and $[(r(t)\psi(x(t))Z^{(n-1)}(t)]' \leq 0$ for $t \geq t_1$.

It follows that the function $r(t)\psi(x(t))Z^{(n-1)}(t)$ is decreasing and we claim that

$$Z^{(n-1)}(t) \geq 0 \quad \text{for } t \geq t_1. \tag{2.4}$$

Otherwise, if there exist a $\tilde{t}_1 \geq t_1$ such that $Z^{(n-1)}(\tilde{t}_1) < 0$, then for all $t \geq \tilde{t}_1$,

$$r(t)\psi(x(t))Z^{(n-1)}(t) \leq r\left(\tilde{t}_1\right)\psi\left(x\left(\tilde{t}_1\right)\right)Z^{(n-1)}\left(\tilde{t}_1\right) = -C(C > 0), \tag{2.5}$$

which implies that $Z^{(n-1)}(t) \leq -C/r(t)\psi(x(t)),\ t \geq t_1$; integrating the above inequality from $\tilde{t}_1$ to t, we have

$$Z^{(n-2)}(t) \leq Z^{(n-2)}\left(\tilde{t}_1\right) - \mathrm{CL}\int_{\tilde{t}_1}^{t}\frac{1}{r(t)}ds. \tag{2.6}$$

Let $t \to \infty$; from (A_1), we get $\lim_{t\to\infty} Z^{(n-2)}(t) = -\infty$, which implies that $Z^{(n-1)}(t)$ and $Z^{(n-2)}(t)$ are negative for all large t; from Lemma 1.2, no two consecutive derivative can be eventually negative, for this would imply that $\lim_{t\to\infty} Z(t) = -\infty$, which is a contradiction. Hence $Z^{(n-1)}(t) \geq 0$ for $t \geq t_1$. Using this fact together with $x(t) \leq Z(t)$, we have that

$$x(t) \geq \left[1 - q(t)\right] Z(t), \quad t \geq t_1. \tag{2.7}$$

Now from (A_1), (S_1), and (2.7), we get

$$f\left[x\left(g(t,\xi)\right)\right] \geq Mf\left[1 - q\left(g(t,\xi)\right)\right] f\left[Z\left(g(t,\xi)\right)\right], \quad t \geq t_1, \tag{2.8}$$

and thus, from(1.1), we get

$$\begin{aligned} 0 = \left[r(t)\varphi(x(t))Z^{(n-1)}(t)\right]' + \int_a^b p(t,\xi) f\left[x\left(g(t,\xi)\right)\right] d\sigma(\xi) \geq \left[(r(t)\varphi(x(t))Z^{(n-1)}(t)\right]' \\ + M\int_a^b p(t,\xi) f\left[1 - q\left(g(t,\xi)\right)\right] f\left[Z\left(g(t,\xi)\right)\right] d\sigma(\xi). \end{aligned} \tag{2.9}$$

Further, observing that $g(t,\xi)$ is nondecreasing with respect to ξ and $Z^{(n-1)}(t) > 0$ for $t \geq t_1$, from Lemma 1.2, we have $Z'(t) \geq 0$, $t \geq t_1$, and so

$$Z\left(g(t,\xi)\right) \geq Z\left(g(t,a)\right), \quad t \geq t_1, \; \xi \in [a,b]. \tag{2.10}$$

So, $f[Z(g(t,\xi))] \geq f[Z(g(t,a))]$ for $t \geq t_1$ and $\xi \in [a,b]$. Thus

$$\left[r(t)\varphi(x(t))Z^{(n-1)}(t)\right]' + Mf\left[Z(g(t,a))\right] \int_a^b p(t,\xi) f\left[1 - q\left(g(t,\xi)\right)\right] d\sigma(\xi) \leq 0, \quad t \geqslant t_1. \tag{2.11}$$

Define

$$w(t) = \rho(t) \frac{r(t)\varphi(x(t))Z^{(n-1)}(t)}{f\left[Z\left(g(t,a)/2\right)\right]}, \quad t \geqslant t_1. \tag{2.12}$$

From (1.1), (2.11), and Lemma 1.3 we get

$$\begin{aligned} w'(t) = {} & \frac{\rho'(t)}{\rho(t)} w(t) + \rho(t) \frac{\left(r(t)\varphi(x(t))Z^{(n-1)}(t)\right)'}{f\left[Z\left(g(t,a)/2\right)\right]} \\ & - \rho(t) \frac{r(t)\varphi(x(t))Z^{(n-1)}(t)}{f^2\left[Z\left(g(t,a)/2\right)\right]} f'\left[Z\left(\frac{g(t,a)}{2}\right)\right] Z'\left(\frac{g(t,a)}{2}\right) \frac{1}{2} g'(t,a) \\ \leq {} & \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) M \int_a^b p(t,\xi) f\left[1 - q\left(g(t,\xi)\right)\right] d\sigma(\xi) - \frac{1}{2} \frac{K_1 L M_\theta g^{(n-2)}(t,a) g'(t,a)}{\rho(t) r(t)} w^2(t). \end{aligned} \tag{2.13}$$

Then, by Lemma 1.1 we get

$$
\begin{aligned}
w'(t) &\leq -\rho(t)M\int_a^b p(t,\xi)f\left[1-q(g(t,\xi))\right]d\sigma(\xi)+\frac{(\rho'(t))^2r(t)}{K_1LM_\theta g^{(n-2)}(t,a)g'(t,a)\rho(t)} \\
&\quad -\frac{1}{4}\frac{K_1LM_\theta g^{(n-2)}(t,a)g'(t,a)}{\rho(t)r(t)}w^2(t) \\
&= -Q(t)-\frac{1}{4}\frac{K_1LM_\theta g^{(n-2)}(t,a)g'(t,a)}{\rho(t)r(t)}w^2(t).
\end{aligned}
\tag{2.14}
$$

Let

$$
Q(t)=\rho(t)M\int_a^b p(t,\xi)f\left[1-q(g(t,\xi))\right]d\sigma(\xi)-\frac{(\rho'(t))^2r(t)}{K_1LM_\theta g^{n-2}(t,a)g'(t,a)\rho(t)}. \tag{2.15}
$$

That is,

$$
Q(t)\leq -w'(t)-\frac{K_1LM_\theta g^{n-2}(t,a)g'(t,a)}{4\rho(t)r(t)}w^2(t). \tag{2.16}
$$

Integrating by parts for any $t > T \geq t_1$, and using properties (A_7) and (A_8), we obtain

$$
\begin{aligned}
&\int_T^t H(t,s)Q(s)ds \\
&\quad\leq -\int_T^t H(t,s)w'(s)ds-\int_T^t H(t,s)\frac{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}{4\rho(s)r(s)}w^2(s) \\
&\quad= H(t,T)w(T)+\int_T^t w(s)\frac{\partial H(t,s)}{\partial s}ds-\int_T^t H(t,s)\frac{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}{4\rho(s)r(s)}w^2(s)ds \\
&\quad= H(t,T)w(T)-\int_T^t -w(s)\frac{\partial H(t,s)}{\partial s}ds-\int_T^t H(t,s)\frac{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}{4\rho(s)r(s)}w^2(s)ds \\
&\quad= H(t,T)w(T)-\int_T^t\left[h(t,s)\sqrt{H(t,s)}w(s)+H(t,s)\frac{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}{4\rho(s)r(s)}w^2(s)\right]ds \\
&\quad= H(t,T)w(T) \\
&\qquad -\int_T^t\left(\sqrt{\frac{H(t,s)K_1LM_\theta g^{n-2}(s,a)g'(s,a)}{4\beta\rho(s)r(s)}}w(s)+h(t,s)\sqrt{\frac{\beta\rho(s)r(s)}{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}}\right)^2ds \\
&\qquad +\frac{\beta}{M_\theta K_1L}\int_T^t\frac{h^2(t,s)\rho(s)r(s)}{g^{n-2}(s,a)g'(s,a)}ds \\
&\qquad -\frac{(\beta-1)K_1M_\theta L}{4\beta}\int_T^t\frac{H(t,s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)}w^2(s)ds.
\end{aligned}
\tag{2.17}
$$

We obtain

$$\begin{aligned}
&\int_T^t \left[H(t,s)Q(s) - \frac{\beta h^2(t,s)\rho(s)r(s)}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)} \right] ds \\
&\quad \le H(t,T)w(T) - \frac{(\beta-1)K_1 M_\theta L}{4\beta}\int_T^t \frac{H(t,s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)} w^2(s)ds \\
&\qquad - \int_T^t \left(\sqrt{\frac{H(t,s)K_1 L M_\theta g^{n-2}(s,a)g'(s,a)}{4\beta\rho(s)r(s)}}\, w(s) + h(t,s)\sqrt{\frac{\rho(s)r(s)}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)}} \right)^2 ds.
\end{aligned} \tag{2.18}$$

From (A_8), $H'(t,s) \le 0$, for $t_1 \ge t_0$, $H(t,t_1) \le H(t,t_0)$,

$$\int_{t_1}^t \left[H(t,s)Q(s) - \frac{\beta h^2(t,s)\rho(s)r(s)}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)} \right] ds \le H(t,t_1)w(t_1) \le H(t,t_0)w(t_1), \tag{2.19}$$

which implies that

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^t \left[H(t,s)Q(s) - \frac{\beta h^2(t,s)\rho(s)r(s)}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)} \right] ds \\
&\quad \le w(t_1) + \frac{1}{H(t,t_0)}\int_{t_0}^{t_1} \left[H(t,s)Q(s) - \frac{h^2(t,s)\rho(s)r(s)}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)} \right] ds \\
&\quad \le w(t_1) + \int_{t_0}^{t_1} Q(s)ds < \infty.
\end{aligned} \tag{2.20}$$

Let $t \to \infty$, and taking upper limits, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t \left[H(t,s)Q(s) - \frac{h^2(t,s)\rho(s)r(s)\beta}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)} \right] ds < \infty, \tag{2.21}$$

which contradicts the assumption (2.1). This complete the proof of Theorem 2.1. □

From Theorem 2.1, we have the following oscillation result.

Corollary 2.2. *If condition* (2.1) *of Theorem 2.1 is replaced by*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t [H(t,s)Q(s)]ds = \infty, \\
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t \frac{h^2(t,s)\rho(s)r(s)\beta}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)} ds < \infty,
\end{aligned} \tag{2.22}$$

where $Q(t)$ *is defined by* (2.2), *then* (1.1) *is oscillatory.*

Remark 2.3. By introducing various $H(t,s)$ from Theorem 2.1 or Corollary 2.2, we can obtain some oscillatory criteria of (1.1). For example, let $H(t,s) = (t-s)^{m-1}$, $t \geq s \geq t_0$, in which $m > 2$ is a integer. By choosing

$$h(t,s) = (t-s)^{(m-3)/2}(m-1), \tag{2.23}$$

it is clear that the conditions of (A_7) and (A_8) hold; then, from Theorem 2.1 and Corollary 2.2, we have the following.

Corollary 2.4. *Assume that there exists a function $\rho(t) \in C'([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \left[(t-s)^{m-1}Q(s)\right] - \frac{\rho(s)r(s)\beta}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)}(t-s)^{m-3}(m-1)^2 ds = \infty, \tag{2.24}$$

where $Q(t)$ is defined by (2.2), *then* (1.1) *is oscillatory.*

Corollary 2.5. *Assume that there exists a function $\rho(t) \in C'([t_0,\infty),(0,\infty))$ such that*

$$\begin{gathered}\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1}Q(s)ds = \infty, \\ \limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \frac{\rho(s)r(s)\beta}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)}(t-s)^{m-3}(m-1)^2 ds < \infty,\end{gathered} \tag{2.25}$$

where $Q(t)$ is defined by (2.2), *then* (1.1) *is oscillatory.*

Theorem 2.6. *Assume that the conditions of Theorem 2.1 hold, and*

$$0 < \inf_{s\geq t_0}\left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] \leq \infty. \tag{2.26}$$

If there exists a function $\varphi(t) \in C([t_0,\infty),\mathfrak{R})$ satisfying

$$\limsup_{t\to\infty} \frac{1}{H(t,u)} \int_{u}^{t} \left[H(t,s)Q(s) - \frac{h^2(t,s)\rho(s)r(s)\beta}{K_1 L M_\theta g^{n-2}(s,a)g'(s,a)}\right] ds \geq \varphi(u), \quad u \geq t_0, \tag{2.27}$$

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{g^{n-2}(u,a)g'(u,a)}{\rho(u)r(u)} \varphi_+^2(u)du = \infty, \quad \varphi_+(u) = \max_{u\geq t_0}\{\varphi(u),(0)\}, \tag{2.28}$$

where $Q(t)$ is defined by (2.2), *then* (1.1) *is oscillatory.*

Proof. Assume that there exists a nonoscillatory solution $x(t)$ of (1.1) on $[t_0,\infty)$, such that $x(t)\neq 0$ on $[t_0,\infty)$. Without loss of generality, assume that $x(t)>0$, $t\geq t_0$. Then, proceeding as in the proof of Theorem 2.1, for $t>u\geq t_1\geq t_0$, we have

$$\begin{aligned}&\frac{1}{H(t,u)}\int_u^t\left[H(t,s)Q(s)-\frac{\beta h^2(t,s)\rho(s)r(s)}{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}\right]ds\\&\quad\leq w(u)-\frac{1}{H(t,u)}\frac{(\beta-1)K_1M_\theta L}{4\beta}\int_u^t\frac{H(t,s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)}w^2(s)ds.\end{aligned}\tag{2.29}$$

Let $t\to\infty$, and taking upper limits, we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,u)}\int_u^t\left[H(t,s)Q(s)-\frac{\beta h^2(t,s)\rho(s)r(s)}{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}\right]ds\\&\quad\leq w(u)-\liminf_{t\to\infty}\frac{1}{H(t,u)}\frac{(\beta-1)K_1M_\theta L}{4\beta}\int_u^t\frac{H(t,s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)}w^2(s)ds,\end{aligned}\tag{2.30}$$

thus, from (2.27), we have

$$w(u)\geq\varphi(u)+\liminf_{t\to\infty}\frac{1}{H(t,u)}\frac{(\beta-1)K_1M_\theta L}{4\beta}\int_u^t\frac{H(t,s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)}w^2(s)ds,\tag{2.31}$$

then $w(u)\geq\varphi(u)$, and

$$\liminf_{t\to\infty}\frac{1}{H(t,u)}\int_u^t\frac{H(t,s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)}w^2(s)ds<\frac{4\beta}{(\beta-1)K_1M_\theta L}\big(w(u)-\varphi(u)\big)<\infty.\tag{2.32}$$

Now we can claim that

$$\int_{t_1}^\infty\frac{g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)}w^2(s)ds<\infty,\quad t<t_1.\tag{2.33}$$

In fact, assume the contrary, that

$$\int_{t_1}^\infty\frac{g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)}w^2(s)ds=\infty,\quad t<t_1.\tag{2.34}$$

From (2.26), there exists a constant $\rho>0$ such that

$$\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]>\rho>0,\tag{2.35}$$

this is

$$\lim_{t\to\infty} \inf \frac{H(t,s)}{H(t,t_0)} > \rho > 0, \tag{2.36}$$

and there exists a $T_2 \geq t_1$ such that $H(t,T)/H(t,t_0) \geq \rho$, for all $t \geq T_2$. On the other hand, by virtue of (2.34), for any positive number α, there exists a $T_1 \geq t_1$, such that, for all $t \geq T_1$

$$\int_{t_1}^{t} \frac{g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)} w^2(s)ds > \frac{\alpha}{\rho}. \tag{2.37}$$

Using integration by parts, we conclude that, for all $t \geq T > t_1$,

$$\begin{aligned}
&\frac{1}{H(t,t_1)}\int_{t_1}^{t} \frac{H(t,s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)} w^2(s)ds \\
&\quad= \frac{1}{H(t,t_1)}\int_{t_1}^{t} H(t,s)d\left(\int_{t_1}^{s} \frac{g^{n-2}(u,a)g'(u,a)}{\rho(u)r(u)} w^2(s)du\right) \\
&\quad= \frac{1}{H(t,t_1)}\int_{t_1}^{t} \left(\int_{t_1}^{s} \frac{g^{n-2}(u,a)g'(u,a)}{\rho(u)r(u)} w^2(s)du\right)\left(-\frac{\partial H}{\partial s}\right)ds \\
&\quad\geq \frac{\alpha}{\rho}\frac{H(t,T)}{H(t,t_1)} \geq \alpha.
\end{aligned} \tag{2.38}$$

Since α is an arbitrary positive constant,

$$\lim_{t\to\infty} \inf \frac{1}{H(t,t_1)}\int_{t_1}^{t} \frac{H(t,s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)} w^2(s)ds = \infty, \tag{2.39}$$

which contradicts (2.32), consequently, (2.33) holds, and, by virtue of $\omega(u) \geq \varphi(u)$ for $u \geq t_1 \geq t_0$,

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \frac{g'(u,a)g^{n-2}(u,a)}{\rho(u)r(u)} \varphi_+^2(u)du \leq \lim_{t\to\infty} \sup \int_{t_0}^{t} \frac{g'(u,a)g^{n-2}(u,a)}{\rho(u)r(u)} \omega_+^2(u)du < \infty, \tag{2.40}$$

which contradicts (2.28), and therefore, (1.1) is oscillatory. □

Remark 2.7. Choosing H as in Remark 2.3, it is not difficult to see that condition (2.26) is satisfied because, for any $s \geq t_0$,

$$\lim_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty} \frac{(t-s)^{n-1}}{(t-t_0)^{n-1}} = 1. \tag{2.41}$$

Consequently, one immediately derives from Theorem 2.6 the following useful corollary for the oscillation of (1.1).

Corollary 2.8. *Assume that there exist functions $\rho(t) \in C([t_0,\infty),(0,\infty))$ and $\varphi(t) \in C([t_0,\infty),\Re)$ satisfying*

$$\limsup_{t\to\infty}\frac{1}{t^{m-1}}\int_{t_0}^{t}\left[(t-s)^{m-1}Q(s)-\frac{\rho(s)r(s)\beta}{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}(t-s)^{m-3}(m-1)^2\right]ds\geq\varphi(u),\quad u\geq 0,$$
$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{g^{n-2}(u,a)g'(u,a)}{\rho(u)r(u)}\varphi_+^2(u)du=0,\quad \varphi_+(u)=\max_{u\geq t_0}\{\varphi(u),(0)\},\tag{2.42}$$

where $Q(t)$ is defined by (2.2)*, then* (1.1) *is oscillatory.*

Theorem 2.9. *Assume that the conditions of Theorem 2.1 and* (2.26) *hold, and*

$$\liminf_{t\to\infty}\frac{1}{H(t,u)}\int_{u}^{t}\left[H(t,s)Q(s)-\frac{h^2(t,s)\rho(s)r(s)\beta}{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}\right]ds\geq\varphi(u),\quad u\geq t_0,$$
$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{g^{n-2}(u,a)g'(u,a)}{\rho(u)r(u)}\varphi_+^2(u)du=\infty,\quad \varphi_+(u)=\max_{u\geq t_0}\{\varphi(u),0\},\tag{2.43}$$

then (1.1) *is oscillatory.*

Proof. Assume that there exists a nonoscillatory solution $x(t)$ of (1.1) on $[t_0,\infty)$, such that $x(t)\neq 0$ on $[t_0,\infty)$. Without loss of generality, assume that $x(t)>0$, $t\geq t_0$. Then, proceeding as in the proof of Theorem 2.1, for $t>u\geq t_1\geq t_0$, we have

$$\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)Q(s)-\frac{\beta h^2(t,s)\rho(s)r(s)}{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}\right]ds$$
$$\leq w(T)-\frac{1}{H(t,T)}\frac{(\beta-1)K_1M_\theta L}{4\beta}\int_{T}^{t}\frac{H(t,s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)}w^2(s)ds.\tag{2.44}$$

Let $t\to\infty$, and taking lower limits, we have

$$\liminf_{t\to\infty}\frac{1}{H(t,u)}\int_{u}^{t}\left[H(t,s)Q(s)-\frac{\beta h^2(t,s)\rho(s)r(s)}{K_1LM_\theta g^{n-2}(s,a)g'(s,a)}\right]ds$$
$$\leq w(u)-\limsup_{t\to\infty}\frac{1}{H(t,u)}\frac{(\beta-1)K_1M_\theta L}{4\beta}\int_{u}^{t}\frac{H(t,s)g^{n-2}(s,a)g'(s,a)}{\rho(s)r(s)}w^2(s)ds.\tag{2.45}$$

The following proof is similar to Theorem 2.6, so we omit the details. This completes the proof of Theorem 2.9. □

3. When $f(x)$ Is Not Monotone

In this section, we will deal with the oscillation for (1.1) under the assumptions (A_1)–(A_8) and the following assumption:

(A_{10}) $f(x)/x \geq K_2$ and $\psi(x) \leq L^{-1}$ for $x \neq 0$.

Theorem 3.1. *Let* (A_1)–(A_8) *and* (A_{10}) *hold. Equation* (1.1) *is oscillatory provided that* $\rho(t) \in C^1([t_0, \infty), R)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)Q_2(s) - \frac{h^2(t,s)\rho(s)r(s)\beta}{LM_\theta g^{n-2}(s,a)g'(s,a)} \right] ds = \infty, \tag{3.1}$$

where

$$Q_2(t) = \rho(t)K_2 \int_a^b p(t,\xi) f\left[1 - q(g(t,\xi))\right] d\sigma(\xi) - \frac{(\rho'(t))^2 r(t)}{LM_\theta g^{n-2}(t,a)g'(t,a)\rho(t)}, \tag{3.2}$$

then (1.1) *is oscillatory.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1). As in the proof of Theorem 2.1, there exists $t_1 \geqslant t_0$, such that (2.3), (2.4), and (2.7) hold. Thus, from (1.1) and (A_{10}), we get

$$\begin{aligned}
0 &= \left(r(t)\psi(x(t))Z^{(n-1)}(t)\right)' + \int_a^b p(t,\xi) f\left[x(g(t,\xi))\right] d\sigma(\xi) \\
&\geq \left(r(t)\psi(x(t))Z^{(n-1)}(t)\right)' + K_2 \int_a^b p(t,\xi) x\left[g(t,\xi)\right] d\sigma(\xi) \\
&= \left(r(t)\psi(x(t))Z^{(n-1)}(t)\right)' + K_2 \int_a^b p(t,\xi) \left\{ Z\left[g(t,\xi)\right] - q\left[g(t,\xi)\right] x\left[g(t,\xi) - \tau\right] \right\} d\sigma(\xi).
\end{aligned} \tag{3.3}$$

Noting that

$$Z\left[g(t,\xi)\right] \geq Z\left[g(t,\xi) - \tau\right] \geq x\left[g(t,\xi) - \tau\right]. \tag{3.4}$$

Thus, (3.3) implies that

$$\left(r(t)\psi(x(t))Z^{(n-1)}(t)\right)' + K_2 \int_a^b p(t,\xi) \left[1 - q(g(t,\xi))\right] Z\left[g(t,\xi)\right] d\sigma(\xi) \leq 0, \quad t \geq t_1. \tag{3.5}$$

From (2.10) and (3.5) we get

$$\left(r(t)\psi(x(t))z^{(n-1)}(t)\right)' + K_2 Z\left[g(t,a)\right] \int_a^b p(t,\xi) \left[1 - q(g(t,\xi))\right] d\sigma(\xi) \leq 0, \quad t \geq t_1. \tag{3.6}$$

Define

$$w(t) = \rho(t)\frac{r(t)\varphi(x(t))Z^{(n-1)}(t)}{Z[(g(t,a)/2)]}, \quad t \geqslant t_1. \tag{3.7}$$

Differentiating (3.7) and using (3.6), Lemma 1.1, and 1.3 we get

$$\begin{aligned}
w'(t) &\leq \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)\left[K_2\int_a^b p(t,\xi)\{1-q(g(t,\xi))\}d\sigma(\xi)\right] - \frac{M_\theta L g^{n-2}(t,a)g'(t,a)}{2r(t)\rho(t)}w^2(t) \\
&\leq -K_2\rho(t)\left[\int_a^b p(t,\xi)\{1-q(g(t,\xi))\}d\sigma(\xi)\right] + \frac{(\rho'(t))^2 r(t)}{M_\theta L g^{n-2}(t,a)g'(t,a)\rho(t)} \\
&\quad - \frac{M_\theta L g^{n-2}(t,a)g'(t,a)}{4\rho(t)r(t)}w^2(t) \\
&= -Q_2(t) - \frac{M_\theta L g^{n-2}(t,a)g'(t,a)}{4\rho(t)r(t)}w^2(t).
\end{aligned} \tag{3.8}$$

The rest proof is similar to that of Theorem 2.1 and hence is omitted. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Assume that the conditions of Theorem 2.1 and* (2.26) *hold; if there exists a function* $\varphi(t) \in C([t_0,\infty),\mathfrak{R})$ *satisfying*

$$\begin{gathered}
\limsup_{t\to\infty}\frac{1}{H(t,u)}\int_u^t\left[H(t,s)Q_2(s) - \frac{h^2(t,s)\rho(s)r(s)\beta}{LM_\theta g^{n-2}(s,a)g'(s,a)}\right]ds \geq \varphi(u), \quad u \geq t_0, \\
\limsup_{t\to\infty}\int_{t_0}^t\frac{g^{n-2}(u,a)g'(u,a)}{\rho(u)r(u)}\varphi_+^2(u)du = \infty, \quad \varphi_+(u) = \max_{u\geq t_0}\{\varphi(u),(0)\},
\end{gathered} \tag{3.9}$$

then (1.1) *is oscillatory.*

Theorem 3.3. *Let all assumptions of Theorem 2.6 be satisfied except that* lim sup *in condition Theorem 3.2 is replaced with* lim inf, *then* (1.1) *is oscillatory.*

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