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*Research Article*

# On the Positivity and Zero Crossings of Solutions of Stochastic Volterra Integrodifferential Equations

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We consider the zero crossings and positive solutions of scalar nonlinear stochastic Volterra integrodifferential equations of Itô type. In the equations considered, the diffusion coefficient is linear and depends on the current state, and the drift term is a convolution integral which is in some sense mean reverting towards the zero equilibrium. The state dependent restoring force in the integral can be nonlinear. In broad terms, we show that when the restoring force is of linear or lower order in the neighbourhood of the equilibrium, or if the kernel decays more slowly than a critical noise-dependent rate, then there is a zero crossing almost surely. On the other hand, if the kernel decays more rapidly than this critical rate, and the restoring force is globally superlinear, then there is a positive probability that the solution remains of one sign for all time, given a sufficiently small initial condition. Moreover, the probability that the solution remains of one sign tends to unity as the initial condition tends to zero.

## 1. Introduction

Deterministic and stochastic delay differential equations are widely used to model systems in ecology, economics, engineering, and physics [1–10].

Very often in deterministic systems, interest focusses on solutions of such equations which are oscillatory, as these could plausibly reflect cyclic motion of a system around an equilibrium. Over the last thirty years, an extensive theory of oscillatory solutions of deterministic equations has developed. Numerous papers and several monographs illustrate the extent of research [4, 11–13]; further, we would like to draw attention to the recent survey paper [14]. However, the effect that random perturbations of Itô type might have on the existence—creation or destruction—of oscillatory solutions of delay differential equations seems, at present, to have received comparatively little attention.

In this paper we consider whether solutions of the stochastic Volterra convolution integrodifferential equation

$$dX(t) = -\int_0^t k(t-s)f(X(s))ds\,dt + \sigma X(t)dB(t), \quad t \geq 0,\ X(0) = \alpha > 0, \tag{1.1}$$

remain positive for all time, or hit or cross zero in a finite time. In (1.1), $B$ is a standard one-dimensional Brownian motion or Wiener process. It is assumed that the kernel $k$ is a nonnegative, continuous, and integrable function and that the continuous function $f$ obeys $xf(x) > 0$ for $x \neq 0$ and $f(0) = 0$. Regularity assumptions on $f$ and $k$ are required to guarantee the existence of solutions. The sign conditions on $f$ and $k$ are motivated by the underlying deterministic Volterra integrodifferential equation

$$x'(t) = -\int_0^t k(t-s)f(x(s))ds, \quad t \geq 0,\ x(0) = \alpha > 0. \tag{1.2}$$

These conditions on $f$ and $k$ ensure that zero is the unique steady-state solution of (1.2) and that solutions tend to revert towards the equilibrium (at least ab initio). If the strength of the mean reversion is sufficiently strong, or the kernel fades sufficiently slowly, solutions of (1.2) can hit zero in finite time. This phenomenon is referred to as a zero crossing. Results on the zero crossing of solutions of (1.2) include work by Gopalsamy and Lalli [15] and Györi and Ladas [16], and a significant literature exists for the zero crossings of such deterministic equations. However, less seems to be known in the stochastic case. Therefore, the question addressed in this paper is: how does a linear state-dependent, instantaneous and equilibrium preserving stochastic perturbation effect the zero crossing and positivity properties of solutions of (1.2)? We answer this question by proving three interrelated results.

First, we show that if $f$ is of linear or lower order in the neighbourhood of the zero equilibrium, then the solution of (1.1) has a zero crossing almost surely, provided that the kernel $k$ is not identically zero.

Second, we show that if $f(x)$ is of order $x^\gamma$ for $\gamma > 1$ as $x \to 0^+$ (i.e., in the neighbourhood of the equilibrium), and $f$ also obeys a global superlinear upper bound on $(0, \infty)$, then any solution of (1.1) which starts sufficiently close to the equilibrium will remain strictly positive with a probability arbitrarily close to unity. This result holds if the kernel $k$ decays more quickly than some critical exponential rate (which depends on the noise intensity $\sigma$). Therefore, if the restoring force is sufficiently weak close to the equilibrium (relative to the *linear* stochastic intensity), solutions will never change sign. Indeed, it is a fortiori shown that solutions can remain positive with arbitrarily high probability once the initial value is small enough.

Finally, if $k$ decays more slowly than the critical exponential rate, then all solutions of (1.1) will have zero crossings, regardless of how weakly the restoring function $f$ acts on the solution. Therefore, we conclude that solutions of (1.1) will remain positive only if (i) $k$ decays more quickly than some critical noise-intensity dependent rate *and* (ii) $f$ is superlinear (at least in the neighbourhood of the equilibrium).

It is interesting to observe that the change in sign of solutions is similar to that seen for the corresponding deterministic equations: at the first zero, the sample path of the solution is differentiable and the derivative is negative. This is notable because the sample path of the solution of (1.1) is not differentiable at any other point. Therefore "oscillation" is not a result of the lack of regularity in the sample path of the nondifferentiable Brownian motion $B$, but rather results from the fluctuation properties of its increments. The presence of delay is important as well: for a stochastic ordinary differential equation, the presence of noise does not induce an oscillation about the equilibrium, if it is a strong solution, see, for example, [17].

Although results in this paper are established for convolution equations, the elegant theory of zero crossings and oscillation for deterministic Volterra equations, which hinge on the existence of real zeros of the characteristic equation, is not employed here. See, for example, [15, 16]. This is largely because the effect of the stochastic perturbation dominates. Instead we employ ideas developed for stochastic functional differential equations with a single (and finite) delay in [18–21].

One motivation for this work is to establish that in the presence of uncertainty, mean reverting systems with delay tend to overshoot equilibrium levels, rather than to approach them monotonically, as appears more likely in the absence of stochastic shocks. This is postulated as a mechanism by which economic systems overshoot an equilibrium, in which the system experiences external stochastic shocks whose intensity depends on the state of the system. Therefore, overconfidence among economic agents, and their feedback behaviour based on the past history of the system, is likely to have a significant impact on the adjustment of the system towards, and overshooting of, its equilibrium, when the system is truly random. Examples of stochastic functional differential and difference equation models of financial markets in which agents use the past information of the system to determine their trading behaviour include [5, 22–24].

The paper is organised as follows. Mathematical preliminaries, including remarks on the existence and uniqueness of solutions of (1.1), are presented in Section 2. The main results of the paper are stated and discussed in Section 3. In Section 4, we show that solutions of (1.1) can be written as the product of the positive solution of a linear stochastic differential equation and the solution of a random Volterra integrodifferential equation. This Volterra equation has solution $y$ which has continuously differentiable paths and is of the form

$$y'(t) = -\int_0^t K(t,s)F(s,y(s))\,ds, \quad t \geq 0, \; y(0) = \alpha > 0, \tag{1.3}$$

where $K$ and $F$ inherit positivity properties from $k$ and $f$. Therefore, the zero crossings of the solution $X$ of (1.1) correspond to zero crossings of the solution $y$ of (1.3). The proofs of the main results are given in the final three sections of the paper.

## 2. Preliminaries

### *2.1. Notation*

In advance of stating and discussing our main results, we introduce some standard notation. We denote the maximum of the real numbers $x$ and $y$ by $x \vee y$ and the minimum of $x$ and $y$ by $x \wedge y$. Let $C(I;J)$ denote the space of continuous functions $f : I \to J$ where $I$ and $J$ are intervals contained in $\mathbb{R}$. Similarly, we let $C^1(I;J)$ denote the space of differentiable functions

$f : I \to J$, where $f' \in C(I; J)$. We denote by $L^1(0,\infty)$ the space of Lebesgue integrable functions $f : [0,\infty) \to \mathbb{R}$ such that

$$\int_0^\infty |f(s)| ds < +\infty. \tag{2.1}$$

If $A$ is an event we denote its complement by $\overline{A}$. We frequently use the standard abbreviations a.s. to stand for almost sure, and a.a. to stand for almost all.

### *2.2. Existence of Solutions of the Stochastic Equation*

Let us fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}(t))_{t\geq 0}$ satisfying the usual conditions and let $(B(t) : t \geq 0)$ be a standard one-dimensional Brownian motion on this space. Let $\sigma$ be a real positive constant. Suppose that

$$k \in C([0,\infty); [0,\infty)), \quad k \in L^1(0,\infty). \tag{2.2}$$

Suppose also that

$$f \in C(\mathbb{R}; \mathbb{R}), \quad xf(x) > 0 \quad x \neq 0, \quad f(0) = 0. \tag{2.3}$$

Let $\alpha > 0$. We consider the stochastic Volterra equation

$$dX(t) = -\int_0^t k(t-s) f(X(s)) ds\, dt + \sigma X(t) dB(t), \quad t \geq 0, \ X(0) = \alpha. \tag{2.4}$$

Let $n \in \mathbb{N}$. Suppose, in addition to (2.3), that $f$ is locally Lipschitz continuous. This means the following.

For every $n \in \mathbb{N}$ there exists $K_n > 0$ such that

$$|f(x) - f(y)| \leq K_n |x - y| \quad \forall x, y \in \mathbb{R} \text{ for which } |x| \vee |y| \leq n. \tag{2.5}$$

Then there is a unique continuous $\mathcal{F}^B$-adapted process $X$ which obeys

$$X(t \wedge t_n) = \alpha - \int_0^{t\wedge t_n} \int_0^s k(s-u) f(X(u)) du\, ds + \int_0^{t\wedge t_n} \sigma X(s) dB(s), \quad t \geq 0, \ \text{a.s,} \tag{2.6}$$

where $t_n = \inf\{t \geq 0 : |X(t)| = n\}$. Suppose in addition that $f$ is globally linearly bounded. More precisely, this means that $f$ also obeys the following:

$$\text{There exists } L_1 \geq 0 \text{ such that } |f(x)| \leq L_1(1 + |x|), \quad x \in \mathbb{R}. \tag{2.7}$$

If $k$ obeys (2.2) and $f$ obeys (2.5) and (2.7), then there exists a unique continuous $\mathcal{F}^B$-adapted process $X$ which obeys

$$X(t) = \alpha - \int_0^t \int_0^s k(s-u) f(X(u))du\, ds + \int_0^t \sigma X(s) dB(s), \quad t \geq 0, \text{ a.s.} \tag{2.8}$$

See, for example, Berger and Mizel [25, Theorem 2E]. In this situation, we say that (2.4) has a unique strong solution. Throughout the paper, we will assume that (2.4) has a unique strong solution but will not necessarily impose conditions (2.5) or (2.7) on $f$ in order to guarantee this. Hereinafter we will often refer to the solution rather than the strong solution of (2.4). We denote the almost sure event on which (2.8) holds by $\Omega^*$. For each $\omega \in \Omega^*$ we denote by $X(t,\omega)$ the value of $X$ at time $t$. We denote by $X(\omega)$ the *realisation* (or *sample path*) $X(\omega) = \{X(t,\omega) : t \geq 0\}$.

### 2.3. Zero Crossing and Positivity of Solutions

Let $X$ be the solution of (2.4), where $\alpha > 0$. For each $\omega \in \Omega^*$, the stopping time $\tau(\alpha)$ is defined by

$$\tau(\alpha,\omega) = \inf\{t > 0 : X(t,\omega) = 0\}. \tag{2.9}$$

We interpret $\tau(\alpha,\omega) = +\infty$ in the case when $\{t > 0 : X(t,\omega) = 0\}$ is the empty set. We say that the sample path $X(\omega)$ has a *zero crossing* if $\tau(\alpha,\omega) < +\infty$. Define

$$A_\alpha = \left\{\omega \in \Omega^* : X(\omega) \text{ is positive on } [0,\infty)\right\} = \{\omega \in \Omega^* : \tau(\alpha,\omega) = +\infty\}. \tag{2.10}$$

## 3. Statement and Discussion of Main Results

Before stating our main results on the solutions of (2.4), we discuss the significance of the hypotheses on $f$ and $k$. We motivate these by considering the deterministic Volterra equation corresponding to (2.4). This deterministic equation can be constructed by setting $\sigma = 0$, resulting in

$$x'(t) = -\int_0^t k(t-s) f(x(s))ds, \quad t \geq 0, \; x(0) = \alpha. \tag{3.1}$$

Conditions (2.2), (2.5), and (2.7) ensure that (3.1) possesses a unique continuous global solution. Clearly, in the case when $\alpha = 0$ the hypothesis (2.3) ensures that $x(t) = 0$ for all $t \geq 0$ is the unique steady-state solution. We also notice that there are no other steady-state solutions $K$ because (2.3) implies that $f(K) \neq 0$ for $K \neq 0$. The fact that the intensity of the stochastic perturbation is zero if and only if the solution is at the steady-state solution of (3.1) means that this stochastic perturbation *preserves the unique equilibrium solution* of the deterministic equation (3.1) indeed if $X(0) = 0$, then $X(t) = 0$ for all $t \geq 0$ a.s. Moreover, the stochastic perturbation does not produce more point equilibria.

The fact that $k$ is nonnegative and $f(x)$ is positive when $x$ is greater than the equilibrium solution of (3.1) means that the solution $x$ of (3.1) is initially attracted towards the equilibrium, because $x'(t) \leq 0$ for all $t \geq 0$, provided $x(t) > 0$ for all $t \geq 0$. The question then arises: does the solution ever reach the zero equilibrium solution in finite time? If so, does it overshoot and become negative (this is referred to as a *zero crossing*), or hit zero and remain there indefinitely thereafter. The paper addresses these questions for the solutions of the stochastic equation (2.4).

Let $\alpha > 0$. Our first result demonstrates that the solutions of (2.4) has a zero crossing for a.a. sample paths in the case when $f$ has at least linear-order leading behaviour at the equilibrium, and when $k$ is not identically zero. More precisely we request that $f$ obeys the following:

$$\text{There exists } L_3 > 0 \text{ such that } \liminf_{x \to 0^+} \frac{f(x)}{x} = L_3, \tag{3.2}$$

and that $k(t) \not\equiv 0$. Moreover, it transpires that $X$ not only hits the zero level, but even assumes negative values. Furthermore, although the sample path of solutions of (2.4) is not differentiable at time $t(\omega)$, provided that $X(t(\omega), \omega) \neq 0$, it is nonetheless differentiable at $\tau(\alpha)$, and the zero level is crossed because this first zero $\tau(\alpha)$ of $X$ is a simple zero of $X$.

**Theorem 3.1.** *Suppose that $k$ obeys* (2.2)*, $f$ obeys* (2.3) *and* (3.2)*, and that $k(t) \not\equiv 0$. Let $X$ be the unique strong solution of* (2.4)*. If $\tau(\alpha)$ is defined by* (2.9)*, then for any $\alpha > 0$*

$$\mathbb{P}[\tau(\alpha) < +\infty] = 1. \tag{3.3}$$

*Moreover, $X$ is differentiable at $\tau(\alpha)$ and $X'(\tau(\alpha)) < 0$.*

See [20] for related comments concerning the zero set of the solution of a stochastic delay differential equation with a single fixed delay. An immediate and interesting corollary of Theorem 3.1 concerns the linear stochastic Volterra equation

$$dX(t) = -\int_0^t k(t-s)X(s)ds\,dt + \sigma X(t)dB(t), \quad t \geq 0. \tag{3.4}$$

Under assumption (2.2), it follows that there is a unique strong solution of this equation (see, e.g., [25]).

**Theorem 3.2.** *Suppose that $k$ obeys* (2.2) *and $k(t) \not\equiv 0$. Let $X$ be the unique strong solution of* (3.4)*. If $\tau(\alpha)$ is defined by* (2.9)*, then for any $\alpha > 0$ one has*

$$\mathbb{P}[\tau(\alpha) < +\infty] = 1. \tag{3.5}$$

*Moreover, $X$ is differentiable at $\tau(\alpha)$ and $X'(\tau(\alpha)) < 0$.*

The proof of these results is a consequence of Lemma 5.1 below. This lemma is inspired by a result of Staikos and Stavroulakis [26, Theorem 2], which applies to linear nonautonomous delay-differential equations. See also [13, Theorem 2.1.3]. This theorem has been employed in [18–20] to demonstrate the existence of a.s. oscillatory solutions of stochastic delay differential equations with a single delay. In each of [18–20] the analysis of the large fluctuations of integral functionals of increments of standard Brownian motion plays an important role in verifying the deterministic oscillation criterion. Similarly, the proofs of Theorems 3.1 and 3.2 in this work hinge on an analysis of increments of the standard Brownian motion $B$.

It is interesting to compare Theorem 3.2 with known results on the zero crossings of the corresponding deterministic linear Volterra integrodifferential equation

$$x'(t) = -\int_0^t k(t-s)x(s)ds, \quad t \geq 0, x(0) = 1 \tag{3.6}$$

in the case when $k$ obeys (2.2) and $k$ is nontrivial. It has been shown (see, e.g., [16]) that (3.6) has zero crossings if and only if the characteristic equation of (3.6)

$$\lambda + \int_0^\infty k(s)e^{-\lambda s}ds = 0, \quad \lambda \in \mathbb{C} \tag{3.7}$$

has no real solutions. However, solutions of (3.4) have zero crossings for a.a. sample paths provided that $k$ is nontrivial. Therefore, the presence of the noise term tends to induce crossing of the equilibrium, even when this is absent in the underlying deterministic equation. On the other hand, if the solution of (3.6) possesses zero crossings, then so does that of (3.4). Therefore, the presence of a stochastic term tends to induce oscillatory behaviour in the solution.

Theorems 3.1 and 3.2 show that positive solutions are impossible if $f$ is of linear, or lower order, leading behaviour at zero. It is reasonable therefore to ask whether positive solutions can ever persist in the presence of a stochastic perturbation. To this end, we now consider the case when $f$ does not necessarily have linear-order leading behaviour at zero. We assume not only that $f$ is weakly nonlinear close to zero, but also that it obeys the following:

$$\text{There exists } \gamma > 1 \text{ and } L_2 > 0 \text{ such that } f(x) \leq L_2 x^\gamma \quad \forall x \geq 0. \tag{3.8}$$

In addition, we suppose that $k$ decays more quickly than $t \mapsto e^{-\sigma^2 t/2}$ as $t \to \infty$ in the sense that

$$\text{There exists } \epsilon > 0 \text{ such that } \int_0^\infty e^{(\sigma^2/2+\epsilon)t}k(t)dt < +\infty. \tag{3.9}$$

Under these conditions, the next result states that (2.4) can possess positive solutions with positive probability, provided that the initial condition is sufficiently small. Moreover, the probability that the solution remains positive for all time approaches unity as the positive initial condition tends to zero.

**Theorem 3.3.** *Suppose that $k$ obeys* (2.2) *and* (3.9). *Suppose also that $f$ obeys* (2.3) *and* (3.8). *Let $X$ be the unique strong solution of* (2.4). *If $A_\alpha$ is defined by* (2.10), *then there exists $\alpha_* > 0$ such that $\mathbb{P}[A_\alpha] > 0$ for all $\alpha \leq \alpha_*$. Moreover*

$$\lim_{\alpha \to 0^+} \mathbb{P}[A_\alpha] = 1. \tag{3.10}$$

The result and proof are inspired by [19, Theorem 4.4], which applies to the stochastic delay differential equation

$$dX(t) = -f(X(t-\tau))dt + \sigma X(t)dB(t), \tag{3.11}$$

where $f$ obeys (2.3) and (3.8). Under these conditions, similar conclusions to those of Theorem 3.3 apply to the solutions of (3.11).

Theorems 3.1 and 3.3 show the importance of the linearity of $f$ local to zero in the presence or absence of zero crossings. However, it is natural to ask whether condition (3.9) in Theorem 3.3 is essential in allowing for positive solutions of (2.4), or whether it is merely a convenient condition which enables us to establish positivity in some cases. The next result shows that condition (3.9) is more or less essential if zero crossings are to be precluded with positive probability.

In order to show this, we consider a condition on the rate of decay of $k$ which is slightly stronger than the negation of condition (3.9). We assume that $k$ decays more slowly to zero than $t \mapsto e^{-\sigma^2 t/2}$ in the sense that

$$\text{There exists } \epsilon > 0 \text{ such that } \int_0^\infty e^{(\sigma^2/2-\epsilon)t} k(t)dt = +\infty. \tag{3.12}$$

Under this condition, solutions of (2.4) cross zero on almost all sample paths, irrespective of how weakly the nonlinear restoring function $f$ acts on the solution.

**Theorem 3.4.** *Suppose that $k$ obeys* (2.2) *and* (3.12). *Suppose also that $f$ obeys* (2.3). *Let $X$ be the unique continuous solution of* (2.4). *If $\tau(\alpha)$ is defined by* (2.9), *then for any $\alpha > 0$ one has*

$$\mathbb{P}[\tau(\alpha) < +\infty] = 1. \tag{3.13}$$

*Moreover, $X$ is differentiable at $\tau(\alpha)$ and $X'(\tau(\alpha)) < 0$.*

This result is interesting because, in the case when $f$ is in $C^1$ and $f'(0) = 0$, the linearisation of (2.4) that is the linear SDE given by

$$dY(t) = \sigma Y(t)dB(t), \quad t \geq 0, \; Y(0) = \alpha > 0, \tag{3.14}$$

has positive solutions with probability one. In a complete contrast however, (2.4) has zero crossings with probability one.

## 4. Reformulation in Terms of a Random Differential Equation

The results in the paper are often a consequence of a reformulation of (2.4) as a random differential equation with continuously differentiable sample paths. This approach has proved successful for studying the oscillation and positivity of solutions of stochastic delay differential equations in [18–20].

Define $\varphi = \{\varphi(t) : t \geq 0\}$ by

$$\varphi(t) = e^{\sigma B(t) - \sigma^2 t/2}, \quad t \geq 0. \tag{4.1}$$

Then $\varphi$ is a strictly positive process (i.e., $\varphi(t) > 0$ for all $t \geq 0$ a.s.) which obeys the stochastic differential equation

$$d\varphi(t) = \sigma\varphi(t)dB(t), \quad t \geq 0, \ \varphi(0) = 1. \tag{4.2}$$

Then $\varphi^{-1}(t) > 0$ for all $t \geq 0$ a.s. and by Itô's lemma, $\varphi^{-1}(t)$ obeys the stochastic differential equation

$$d\varphi^{-1}(t) = \sigma^2\varphi^{-1}(t)dt - \sigma\varphi^{-1}(t)dB(t). \tag{4.3}$$

By (stochastic) integration by parts it follows that

$$\begin{aligned} d\left(\frac{X(t)}{\varphi(t)}\right) &= X(t)d\varphi^{-1}(t) + \varphi(t)^{-1} \ dX(t) + \sigma X(t)\left(-\sigma\varphi^{-1}(t)\right)dt \\ &= -\varphi(t)^{-1}\int_0^t k(t-s)f(X(s))ds\,dt. \end{aligned} \tag{4.4}$$

Therefore, as $X(0) = \alpha$, we have

$$\frac{X(t)}{\varphi(t)} = \alpha - \int_0^t \varphi(s)^{-1}\int_0^s k(s-u)f(X(u))du\,ds, \quad t \geq 0. \tag{4.5}$$

Since $X$ has continuous sample paths, it follows from (2.3) and (2.2) that each realisation of

$$t \longmapsto \varphi(t)^{-1}\int_0^t k(t-s)f(X(s))ds \tag{4.6}$$

is continuous. Therefore we have that each realisation of the process $y = \{y(t) : t \geq 0\}$ defined by

$$y(t) = \frac{X(t)}{\varphi(t)}, \quad t \geq 0 \tag{4.7}$$

is in $C^1((0,\infty);\mathbb{R})$ and by (4.5) we have

$$y'(t) = -\varphi(t)^{-1}\int_0^t k(t-s)f\big(\varphi(s)y(s)\big)ds, \quad t \geq 0, \ y(0) = \alpha > 0. \tag{4.8}$$

It is convenient here to record another fact concerning $\varphi$: on account of the Strong Law of Large Numbers for standard Brownian motion, it follows that

$$\lim_{t\to\infty}\frac{1}{t}\log\varphi(t) = -\frac{\sigma^2}{2}, \quad \text{a.s.} \tag{4.9}$$

and therefore we have that $\varphi(t) \to 0$ as $t \to \infty$ a.s.

## 5. Proof of Theorem 3.1

### 5.1. Supporting Lemmas

We start by developing a criterion independent of the solution of (2.4), but which depends on $\varphi$ given by (4.1), which ensures that the solution of (2.4) exhibits a zero crossing a.s.

**Lemma 5.1.** *Let $\sigma > 0$. Suppose that $k$ obeys* (2.2) *and that $f$ obeys* (2.3) *and* (3.2)*. Suppose that $\varphi$ is defined by* (4.1)*, and $\kappa$ by*

$$\kappa(t,s) = \varphi(t)^{-1}k(t-s), \quad 0 \le s \le t. \tag{5.1}$$

*Suppose that there exists $\tau > 0$ such that*

$$\limsup_{t\to\infty}\int_{t-2\tau}^{t-\tau}\int_{t-\tau}^{t}\kappa(s,u)\varphi(u)ds\,du > \frac{1}{L_3}. \tag{5.2}$$

*Let $\alpha > 0$. If $X$ is the unique strong solution of* (2.4)*, and $\tau(\alpha)$ is defined by* (2.9)*, then*

$$\mathbb{P}[\tau(\alpha) < +\infty] = 1. \tag{5.3}$$

*Moreover, $X$ is differentiable at $\tau(\alpha)$ and $X'(\tau(\alpha)) < 0$.*

*Proof.* By (4.8) and the definition of $\kappa$ in (5.1) we have

$$y'(t) = -\int_0^t \kappa(t,s)f(\varphi(s)y(s))ds, \quad t \ge 0. \tag{5.4}$$

Note that $A_\alpha = \{\omega : y(t,\omega) > 0 \text{ for all } t \ge 0\}$. Suppose that $\mathbb{P}[A_\alpha] > 0$. Let $\omega \in A_\alpha$. Note that $y'(t,\omega) \le 0$ for all $t \ge 0$. Therefore $y(t,\omega)$ tends to a nonnegative limit as $t \to \infty$. Since

$\varphi(t,\omega) \to 0$ as $t \to \infty$, we have that $X(t,\omega) \to 0$ as $t \to \infty$. We temporarily suppress the dependence on $\omega$. Since $y(t) > 0$, for any $t \geq 2\tau$, by (5.4) we have

$$\begin{aligned} y(t-\tau) &= \int_{t-\tau}^{t}\int_{0}^{s}\kappa(s,u)f(\varphi(u)y(u))du\,ds + y(t) \\ &\geq \int_{t-\tau}^{t}\int_{0}^{s}\kappa(s,u)\varphi(u)y(u)\frac{f(X(u))}{X(u)}du\,ds \\ &= \int_{0}^{t}\int_{u\vee(t-\tau)}^{t}\kappa(s,u)\varphi(u)ds\cdot\frac{f(X(u))}{X(u)}y(u)du \\ &\geq \int_{t-2\tau}^{t-\tau}\int_{u\vee(t-\tau)}^{t}\kappa(s,u)\varphi(u)ds\cdot\frac{f(X(u))}{X(u)}y(u)du \\ &\geq \inf_{u\in[t-2\tau,t-\tau]}\frac{f(X(u))}{X(u)}\int_{t-2\tau}^{t-\tau}\int_{u\vee(t-\tau)}^{t}\kappa(s,u)\varphi(u)ds\cdot y(u)du. \end{aligned} \tag{5.5}$$

Now, because $y$ is nonincreasing we have

$$\begin{aligned} y(t-\tau) &\geq \inf_{u\geq t-2\tau}\frac{f(X(u))}{X(u)}\int_{t-2\tau}^{t-\tau}\int_{u\vee(t-\tau)}^{t}\kappa(s,u)\varphi(u)ds\,du\cdot y(t-\tau) \\ &= \inf_{u\geq t-2\tau}\frac{f(X(u))}{X(u)}\int_{t-2\tau}^{t-\tau}\int_{t-\tau}^{t}\kappa(s,u)\varphi(u)ds\,du\cdot y(t-\tau). \end{aligned} \tag{5.6}$$

Since $y(t-\tau) > 0$ for all $t \geq 2\tau$ we have that

$$\inf_{u\geq t-2\tau}\frac{f(X(u))}{X(u)}\int_{t-2\tau}^{t-\tau}\int_{t-\tau}^{t}\kappa(s,u)\varphi(u)ds\,du \leq 1, \quad t\geq 2\tau. \tag{5.7}$$

Since $X(t) \to 0$ as $t \to \infty$ and $\liminf_{x\to 0^+} f(x)/x = L_3$ we have

$$\lim_{t\to\infty}\inf_{u\geq t-2\tau}\frac{f(X(u))}{X(u)} = L_3. \tag{5.8}$$

Therefore

$$\limsup_{t\to\infty}\int_{t-2\tau}^{t-\tau}\int_{t-\tau}^{t}\kappa(s,u)\varphi(u)ds\,du \leq \frac{1}{L_3}, \tag{5.9}$$

which contradicts (5.2). Therefore we have that $\mathbb{P}[\tau(\alpha) < +\infty] = 1$, as required.

Now $\overline{A_\alpha} = \{\omega \in \Omega^* :$ there exists $t' = t'(\omega) > 0$ such that $X(t'(\omega),\omega) = 0\} = \{\omega \in \Omega^* : \tau(\alpha,\omega) < +\infty\}$, and this event is almost sure. Fix $\omega \in \overline{A_\alpha}$. Since $y(t) = 0$ if and only if $X(t) = 0$, by (4.7) we have from (2.9) that $y(\tau(\alpha,\omega),\omega) = 0$ and that $y(t,\omega) > 0$ for all $t \in [0,\tau(\alpha,\omega))$. By (4.8) we have

$$y'(\tau(\alpha)) = -\varphi^{-1}(\tau(\alpha))\int_0^{\tau(\alpha)} k(\tau(\alpha)-s)f(\varphi(s)y(s))ds. \tag{5.10}$$

Since (2.3) implies that $f(\varphi(s)y(s)) > 0$ for all $s \in [0,\tau(\alpha))$, and $k$ obeys (2.2) we have that $y'(\tau(\alpha)) \le 0$. Suppose that $y'(\tau(\alpha)) = 0$. Since $\varphi(\tau(\alpha)) > 0$, we must have that $k(\tau(\alpha)-s) = 0$ for all $s \in (0,\tau(\alpha))$ or $k(t) = 0$ for $t \in (0,\tau(\alpha))$. Therefore we have that $y'(t) = 0$ for all $t \in (0,\tau(\alpha))$. Hence $y(\tau(\alpha)) = y(0) = \alpha > 0$, which contradicts the fact that $y(\tau(\alpha)) = 0$. Therefore we must have $y'(\tau(\alpha)) < 0$. This implies that there exists $t'(\omega) > \tau(\alpha,\omega)$ such that $y(t'(\omega),\omega) < 0$ and therefore we have that $X(t'(\omega),\omega) < 0$. Therefore for each $\omega$ in the a.s. event $\overline{A_\alpha}$, there exists a $t'(\omega) > 0$ such that $X(t'(\omega)) < 0$.

We now show that X is differentiable at $\tau(\alpha)$ and that $X'(t(\alpha)) < 0$. Let $t \neq \tau(\alpha)$. Then we have

$$\begin{aligned}\frac{X(t,\omega)-X(\tau(\alpha,\omega),\omega)}{t-\tau(\alpha,\omega)} &= \frac{y(t,\omega)\varphi(t,\omega)-y(\tau(\alpha,\omega),\omega)\varphi(t,\omega)}{t-\tau(\alpha,\omega)}\\ &= \varphi(t,\omega)\frac{y(t,\omega)-y(\tau(\alpha,\omega),\omega)}{t-\tau(\alpha,\omega)}.\end{aligned} \tag{5.11}$$

Now taking the limit as $t \to \tau(\alpha)$ on the righthand side we have

$$\lim_{t\to\tau(\alpha)}\varphi(t,\omega)\frac{y(t,\omega)-y(\tau(\alpha,\omega),\omega)}{t-\tau(\alpha,\omega)} = \varphi(\tau(\alpha,\omega),\omega)y'(\tau(\alpha,\omega),\omega) < 0. \tag{5.12}$$

Therefore we have

$$\lim_{t\to\tau(\alpha)}\frac{X(t,\omega)-X(\tau(\alpha,\omega),\omega)}{t-\tau(\alpha,\omega)} = \varphi(\tau(\alpha,\omega),\omega)y'(\tau(\alpha,\omega),\omega) < 0, \tag{5.13}$$

so $X'(\tau(\alpha,\omega),\omega)$ is well defined and indeed $X'(\tau(\alpha,\omega),\omega) < 0$. □

The next result develops a condition which depends only on the increments of $B$ and the kernel $k$ which implies condition (5.2).

**Lemma 5.2.** *Let $\sigma > 0$. If $\kappa$ is defined by* (5.1) *and $\varphi$ by* (4.1)*, and there exists $\tau > 0$ such that*

$$\limsup_{t\to\infty}\int_0^\tau\int_0^\tau e^{-\sigma(B(t+u)-B(t-w))}k(u+w)du\,dw = +\infty, \quad a.s. \tag{5.14}$$

*then* (5.2) *holds.*

*Proof.* Define for $t \ge 2\tau$

$$A_1(t) = \int_{t-2\tau}^{t-\tau}\int_{t-\tau}^{t}\kappa(s,u)\varphi(u)ds\,du = \int_{t-2\tau}^{t-\tau}\int_{t-\tau}^{t}\varphi(s)^{-1}k(s-u)\varphi(u)ds\,du. \tag{5.15}$$

Hence as $\varphi$ is given by (4.1) we have

$$\begin{aligned} A_1(t) &= \int_{t-2\tau}^{t-\tau}\int_{t-\tau}^{t} e^{-\sigma(B(s)-B(u))}e^{\sigma^2/2(s-u)}k(s-u)ds\,du \\ &= \int_{t-2\tau}^{t-\tau}\int_{t-u-\tau}^{t-u} e^{-\sigma(B(v+u)-B(u))}e^{\sigma^2 v/2}\,k(v)dv\,du \\ &= \int_0^{\tau}\int_w^{w+\tau} e^{-\sigma(B(v+t-\tau-w)-B(t-\tau-w))}e^{\sigma^2 v/2}k(v)dv\,dw \\ &\geq \int_0^{\tau}\int_w^{w+\tau} e^{-\sigma(B(v+t-\tau-w)-B(t-\tau-w))}k(v)dv\,dw. \end{aligned} \tag{5.16}$$

Therefore if (5.14) holds, then

$$\begin{aligned} \limsup_{t\to\infty} A_1(t) &\geq \limsup_{t\to\infty}\int_0^{\tau}\int_w^{w+\tau} e^{-\sigma(B(v+t-\tau-w)-B(t-\tau-w))}k(v)dv\,dw \\ &= \limsup_{t\to\infty}\int_0^{\tau}\int_w^{w+\tau} e^{-\sigma(B(v+t-w)-B(t-w))}k(v)dv\,dw \\ &= \limsup_{t\to\infty}\int_0^{\tau}\int_0^{\tau} e^{-\sigma(B(u+t)-B(t-w))}k(u+w)du\,dw \\ &= \infty, \end{aligned} \tag{5.17}$$

which implies (5.2). □

**Lemma 5.3.** *Suppose that* $\sigma > 0$. *Suppose that* $k$ *obeys* (2.2) *and* $k(t)\not\equiv 0$. *Then there exists* $\tau > 0$ *such that* (5.14) *holds.*

*Proof.* If $k(t)\not\equiv 0$ and $k(t) \geq 0$ it follows that there is a $t_0 \geq 0$ such that $k(t_0) =: 2k_0 > 0$. Since $k$ is continuous on $[0,\infty)$ there exists $\delta > 0$ such that $|k(t) - k(t_0)| \leq k_0$ for all $t \in [t_0, t_0+\delta]$. Therefore we have $k(t) \geq k_0$ for all $t \in [t_0+\delta/2, t_0+\delta] =: [\theta_1,\theta_2]$. Hence

$$\text{There exists } k_0 > 0 \text{ and } 0 < \theta_1 < \theta_2 \text{ such that } k(t) \geq k_0 \quad \forall t \in [\theta_1,\theta_2]. \tag{5.18}$$

Let $\tau = \theta_2$. Then $\theta_1 \leq \tau$, $\theta_2 \leq \tau$. Define

$$A_2(t) = \int_0^{\tau}\int_0^{\tau} e^{-\sigma(B(u+t)-B(t-w))}k(u+w)du\,dw, \quad t \geq \tau. \tag{5.19}$$

Equation (5.14) is equivalent to show that $\limsup_{t\to\infty} A_2(t) = \infty$ a.s. Clearly by (5.18) we have

$$A_2(t) \geq k_0\int_0^{\theta_2}\int_{(\theta_1-w)\vee 0}^{\theta_2-w} e^{-\sigma(B(u+t)-B(t-w))}du\,dw. \tag{5.20}$$

Now $[\theta_1/2, \theta_2/2]^2 \subset \{(w,u) : 0 \le w \le \theta_2, u \ge 0, \theta_1 \le w+u \le \theta_2\}$, so we have

$$A_2(t) \ge k_0 \int_{\theta_1/2}^{\theta_2/2} \int_{\theta_1/2}^{\theta_2/2} e^{-\sigma(B(u+t)-B(t-w))} du\, dw =: A(t). \tag{5.21}$$

Hence (5.14) follows if we can show that $\limsup_{t\to\infty} A(t) = \infty$ a.s. Define $A_n = A(n\theta_2)/k_0$ for $n \ge 1$ so that it suffices to prove that $\limsup_{n\to\infty} A_n = \infty$ a.s. Note that

$$A_n = \int_{\theta_1/2}^{\theta_2/2} \int_{\theta_1/2}^{\theta_2/2} e^{-\sigma(B(u+n\theta_2)-B(n\theta_2-w))} du\, dw. \tag{5.22}$$

Now, we note that each $A_n$ is a functional of increments of the standard Brownian motion $B$ over the interval $[n\theta_2 - \theta_2/2, n\theta_2 + \theta_2/2]$. Therefore as $(n+1)\theta_2 - \theta_2/2 = n\theta_2 + \theta_2/2$, it follows that the intervals on which the increments of $B$ are considered for $A_n$ and $A_{n+1}$ are nonoverlapping. Since the increments of $B$ are independent, it follows that $(A_n)_{n\ge 1}$ is a sequence of independent random variables. Hence by the Borel-Cantelli lemma, we are done if we can show that

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n > \beta] = +\infty, \quad \forall \beta > 0. \tag{5.23}$$

Note that $\widetilde{B} = -B$ is a standard Brownian motion. Then

$$A_n = \int_{\theta_1/2}^{\theta_2/2} \int_{\theta_1/2}^{\theta_2/2} e^{\sigma(\widetilde{B}(u+n\theta_2)-\widetilde{B}(n\theta_2-w))} du\, dw. \tag{5.24}$$

Define for $\eta \in \mathbb{R}$ the event

$$C_\eta(n) = \left\{ \omega : \min_{u\in[\theta_1/2,\theta_2/2]} \widetilde{B}(n\theta_2 + u, \omega) - \max_{w\in[\theta_1/2,\theta_2/2]} \widetilde{B}(n\theta_2 - w, \omega) \ge \eta \right\}. \tag{5.25}$$

If $\omega \in C_\eta(n)$, then $\widetilde{B}(n\theta_2 + u, \omega) - \widetilde{B}(n\theta_2 - w, \omega) \ge \eta$ for all $(u,w) \in [\theta_1/2, \theta_2/2]^2$. Therefore $\omega \in C_\eta(n)$ implies

$$A_n(\omega) = \int_{\theta_1/2}^{\theta_2/2} \int_{\theta_1/2}^{\theta_2/2} e^{\sigma(\widetilde{B}(n\theta_2+u,\omega)-\widetilde{B}(n\theta_2-w,\omega))} du\, dw \ge \left(\frac{\theta_2}{2} - \frac{\theta_1}{2}\right)^2 e^{\sigma\eta}. \tag{5.26}$$

Thus

$$\mathbb{P}\left[A_n \ge \left(\frac{\theta_2}{2} - \frac{\theta_1}{2}\right)^2 e^{\sigma\eta}\right] \ge \mathbb{P}[C_\eta(n)], \quad n \ge 1,\ \eta \in \mathbb{R}. \tag{5.27}$$

Now

$$\begin{aligned}
&\min_{u\in[\theta_1/2,\theta_2/2]} \widetilde{B}(n\theta_2+u) - \max_{w\in[\theta_1/2,\theta_2/2]} \widetilde{B}(n\theta_2-w) \\
&= \min_{u\in[\theta_1/2,\theta_2/2]} \left\{ \widetilde{B}(n\theta_2+u) - \widetilde{B}\left(n\theta_2+\frac{\theta_1}{2}\right)\right\} \\
&\quad + \left\{ \widetilde{B}\left(n\theta_2+\frac{\theta_1}{2}\right) - \widetilde{B}\left(n\theta_2-\frac{\theta_1}{2}\right)\right\} \\
&\quad - \max_{w\in[\theta_1/2,\theta_2/2]} \left\{ \widetilde{B}(n\theta_2-w) - \widetilde{B}\left(n\theta_2-\frac{\theta_1}{2}\right)\right\} \\
&= \min_{u\in[n\theta_2+\theta_1/2,n\theta_2+\theta_2/2]} \left\{ \widetilde{B}(u) - \widetilde{B}\left(n\theta_2+\frac{\theta_1}{2}\right)\right\} \\
&\quad + \left\{ \widetilde{B}\left(n\theta_2+\frac{\theta_1}{2}\right) - \widetilde{B}\left(n\theta_2-\frac{\theta_1}{2}\right)\right\} \\
&\quad + \min_{w\in[n\theta_2-\theta_2/2,n\theta_2-\theta_1/2]} \left\{ \widetilde{B}\left(n\theta_2-\frac{\theta_1}{2}\right) - \widetilde{B}(w)\right\} \\
&=: W_1(n)+W_2(n)+W_3(n).
\end{aligned} \tag{5.28}$$

Since $\theta_1 > 0$, each of $W_1(n)$, $W_2(n)$, and $W_3(n)$ is well defined and independent random variables. Hence

$$\begin{aligned}
\mathbb{P}\left[C_\eta(n)\right] &= \mathbb{P}\left[W_1(n)+W_2(n)+W_3(n)\geq \eta\right] \\
&\geq \mathbb{P}\left[W_1(n)\geq 0, W_2(n)\geq \eta, W_3(n)\geq 0\right] \\
&= \mathbb{P}[W_1(n)\geq 0]\cdot \mathbb{P}\left[W_2(n)\geq \eta\right]\cdot \mathbb{P}[W_3(n)\geq 0].
\end{aligned} \tag{5.29}$$

Now we note that $\widetilde{B}_1(u) := \widetilde{B}(u+n\theta_2+\theta_1/2) - \widetilde{B}(n\theta_2+\theta_1/2)$ for $u\in[0,\theta_2/2-\theta_1/2]$ is a standard Brownian motion, so we have

$$\begin{aligned}
\mathbb{P}[W_1(n)\geq 0] &= \mathbb{P}\left[\inf_{u\in[n\theta_2+\theta_1/2,n\theta_2+\theta_2/2]} \widetilde{B}(u) - \widetilde{B}\left(n\theta_2+\frac{\theta_1}{2}\right)\geq 0\right] \\
&= \mathbb{P}\left[\inf_{u\in[0,\theta_2/2-\theta_1/2]} \widetilde{B}_1(u)\geq 0\right] = 1.
\end{aligned} \tag{5.30}$$

Next, if we define $\widetilde{B}_2(u) := \widetilde{B}(n\theta_2 - \theta_1/2) - \widetilde{B}(u)$ for $u \in [0, \theta_2/2 - \theta_1/2]$, then $\widetilde{B}_2$ is another standard Brownian motion. Therefore we have

$$\begin{aligned}\mathbb{P}[W_3(n) \geq 0] &= \mathbb{P}\left[\inf_{w\in[n\theta_2-\theta_2/2, n\theta_2-\theta_1/2]}\left\{\widetilde{B}\left(n\theta_2 - \frac{\theta_1}{2}\right) - \widetilde{B}(w)\right\} \geq 0\right] \\ &= \mathbb{P}\left[\inf_{u\in[0,\theta_2/2-\theta_1/2]}\left\{\widetilde{B}\left(n\theta_2 - \frac{\theta_1}{2}\right) - \widetilde{B}\left(n\theta_2 - \frac{\theta_1}{2} - u\right)\right\} \geq 0\right] \\ &= \mathbb{P}\left[\inf_{u\in[0,\theta_2/2-\theta_1/2]} \widetilde{B}_2(u) \geq 0\right] = 1.\end{aligned} \tag{5.31}$$

Therefore if $Z$ is a standard normal random variable and $\Phi$ is the distribution function of $Z$, we have

$$\begin{aligned}\mathbb{P}[C_\eta(n)] &= \mathbb{P}[W_1(n) + W_2(n) + W_3(n) \geq \eta] \\ &\geq \mathbb{P}[W_2(n) \geq \eta] \\ &= \mathbb{P}\left[\widetilde{B}\left(n\theta_2 + \frac{\theta_1}{2}\right) - \widetilde{B}\left(n\theta_2 - \frac{\theta_1}{2}\right) \geq \eta\right] \\ &= \mathbb{P}[\theta_1 Z \geq \eta] = 1 - \Phi\left(\frac{\eta}{\theta_1}\right).\end{aligned} \tag{5.32}$$

Hence by (5.27) we have

$$\mathbb{P}\left[A_n \geq \left(\frac{\theta_2}{2} - \frac{\theta_1}{2}\right)^2 e^{\sigma\eta}\right] \geq 1 - \Phi\left(\frac{\eta}{\theta_1}\right), \quad \forall \eta \in \mathbb{R},\ n \geq 1. \tag{5.33}$$

Let $\beta > 0$ and define $\eta \in \mathbb{R}$ by

$$\eta = \frac{1}{\sigma}\log\left(\frac{4\beta}{(\theta_2 - \theta_1)^2}\right). \tag{5.34}$$

Then $\beta = (\theta_2 - \theta_1)^2 e^{\sigma\eta}/4$. Hence

$$\mathbb{P}[A_n \geq \beta] \geq 1 - \Phi\left(\frac{1}{\sigma\theta_1}\log\left(\frac{4\beta}{(\theta_2 - \theta_1)^2}\right)\right) =: c(\beta) > 0, \quad \forall \beta > 0,\ n \geq 1. \tag{5.35}$$

This implies that (5.23) holds, and therefore that $\limsup_{n\to\infty} A_n = \infty$ a.s., from which it has already been shown that the lemma follows. □

### *5.2. Proof of Theorem 3.1*

The proof of Theorem 3.1 is now an immediate consequence of the last three lemmas. By Lemma 5.3 it follows that (5.14) holds. By Lemma 5.2 it therefore follows that (5.2) holds.

Hence by Lemma 5.1 it follows that $\mathbb{P}[\tau(\alpha) < +\infty] = 1$ for any $\alpha > 0$ and that $X'(\tau(\alpha))$ exists and is negative. Since these are the desired conclusions of Theorem 3.1, the proof is complete.

## 6. Proof of Theorem 3.3

We start by proving a technical lemma.

**Lemma 6.1.** *Let $\sigma > 0$, $\gamma > 1$, and suppose that $\varphi$ is given by* (4.1). *Suppose that $k$ obeys* (2.2) *and* (3.9). *Define*

$$I = \int_0^\infty \int_0^t \varphi(t)^{-1} k(t-s)\varphi^\gamma(s)\,ds\,dt. \tag{6.1}$$

*Then $I < +\infty$ a.s.*

*Proof.* By the definition of $\varphi$ we have

$$I = \int_0^\infty \int_0^t k(t-s)e^{\sigma\gamma B(s)-\sigma B(t)}e^{\sigma^2 t/2-\sigma^2\gamma s/2}\,ds\,dt. \tag{6.2}$$

By the Strong Law of Large Numbers for standard Brownian motion (see, e.g., Karatzas and Shreve [27]), there exists an almost sure event $\Omega_1$ such that

$$\Omega_1 = \left\{\omega : \frac{\lim_{t\to\infty} B(t,\omega)}{t} = 0\right\}. \tag{6.3}$$

Therefore for each $\omega \in \Omega_1$ and for every $\varepsilon > 0$ there exists a finite $T(\omega,\varepsilon) > 0$ such that

$$|B(t,\omega)| \le \varepsilon t, \quad t \ge T(\omega,\varepsilon). \tag{6.4}$$

Define

$$\begin{aligned}
I_1 &= \int_0^{T(\omega,\varepsilon)} \int_0^t k(t-s)e^{\sigma\gamma B(s)-\sigma B(t)}e^{\sigma^2 t/2-\sigma^2\gamma s/2}\,ds\,dt,\\
I_2 &= \int_{T(\omega,\varepsilon)}^\infty \int_{T(\omega,\varepsilon)}^t k(t-s)e^{\sigma\gamma B(s)-\sigma B(t)}e^{\sigma^2 t/2-\sigma^2\gamma s/2}\,ds\,dt,\\
I_3 &= \int_{T(\omega,\varepsilon)}^\infty \int_0^{T(\omega,\varepsilon)} k(t-s)e^{\sigma\gamma B(s)-\sigma B(t)}e^{\sigma^2 t/2-\sigma^2\gamma s/2}\,ds\,dt.
\end{aligned} \tag{6.5}$$

Then $I = I_1 + I_2 + I_3$. The continuity of the integrand and finiteness of $T(\omega,\varepsilon) > 0$ ensures that $I_1 < +\infty$. Consider now $I_3$. Suppose that $\varepsilon > 0$ is so small that $|\sigma|\varepsilon < \epsilon$ where $\epsilon > 0$ is defined by (3.9) (note the distinction between the constant $\epsilon$ defined by (3.9) and the small parameter $\varepsilon > 0$). Define $\beta_2 = \sigma^2/2 + |\sigma|\epsilon$. By (3.9) we therefore have

$$\int_0^\infty k(s)e^{(\sigma^2/2+|\sigma|\epsilon)s}ds = \int_0^\infty k(s)e^{\beta_2 s}ds < +\infty. \tag{6.6}$$

Then by (6.4) we have

$$\begin{aligned} I_3 &\le \max_{0\le s\le T(\omega,\varepsilon)} e^{\sigma\gamma B(s)-\sigma^2\gamma s/2} \cdot \int_{T(\omega,\varepsilon)}^\infty \int_0^{T(\omega,\varepsilon)} k(t-s)e^{-\sigma B(t)}e^{\sigma^2 t/2}ds\,dt \\ &\le \max_{0\le s\le T(\omega,\varepsilon)} \varphi^\gamma(s,\omega) \cdot \int_{T(\omega,\varepsilon)}^\infty \int_0^{T(\omega,\varepsilon)} k(t-s)e^{\beta_2(t-s)}e^{\beta_2 s}ds\,dt \\ &\le e^{\beta_2 T(\omega,\varepsilon)} \max_{0\le s\le T(\omega,\varepsilon)} \varphi^\gamma(s,\omega) \cdot \int_{T(\omega,\varepsilon)}^\infty \int_0^{T(\omega,\varepsilon)} k(t-s)e^{\beta_2(t-s)}ds\,dt. \end{aligned} \tag{6.7}$$

Now by the nonnegativity of the integrand and Fubini's theorem we have

$$\begin{aligned} &\int_{T(\omega,\varepsilon)}^\infty \int_0^{T(\omega,\varepsilon)} k(t-s)e^{\beta_2(t-s)}ds\,dt \\ &\quad= \int_{T(\omega,\varepsilon)}^\infty \int_{t-T(\omega,\varepsilon)}^{t} k(u)e^{\beta_2 u}du\,dt \\ &\quad= \int_0^\infty \int_{u\vee T(\varepsilon,\omega)}^{u+T(\varepsilon,\omega)} dt\cdot k(u)e^{\beta_2 u}du \\ &\quad= \int_0^{T(\varepsilon,\omega)} uk(u)e^{\beta_2 u}du + T(\varepsilon,\omega)\int_{T(\varepsilon,\omega)}^\infty k(u)e^{\beta_2 u}du. \end{aligned} \tag{6.8}$$

By (6.6) we have

$$\int_{T(\omega,\varepsilon)}^\infty \int_0^{T(\omega,\varepsilon)} k(t-s)e^{\beta_2(t-s)}ds\,dt < +\infty, \tag{6.9}$$

which implies that $I_3 < +\infty$.

Finally we show that $I_2 < +\infty$, a.s. Suppose that $\varepsilon > 0$ is so small that $\varepsilon < |\sigma|/2$ and $\varepsilon < (\gamma-1)|\sigma|/[2(\gamma+1)]$. Define $\beta_1 = \gamma(\sigma^2/2 - |\sigma|\varepsilon)$. Then $\beta_1 > 0$. Hence by (6.4) we have

$$\begin{aligned} I_2 &= \int_{T(\omega,\varepsilon)}^{\infty}\int_{T(\omega,\varepsilon)}^{t} k(t-s)e^{\sigma\gamma B(s)-\sigma B(t)}e^{\sigma^2 t/2-\sigma^2\gamma s/2}ds\,dt \\ &\le \int_{T(\omega,\varepsilon)}^{\infty}\int_{T(\omega,\varepsilon)}^{t} k(t-s)e^{|\sigma|\gamma\varepsilon s+|\sigma|\varepsilon t}e^{\sigma^2 t/2-\sigma^2\gamma s/2}ds\,dt \\ &= \int_{T(\omega,\varepsilon)}^{\infty}\int_{T(\omega,\varepsilon)}^{t} k(t-s)e^{-\beta_1 s}e^{\beta_2 t}ds\,dt. \end{aligned} \tag{6.10}$$

Now $\beta_1 - \beta_2 = (\gamma-1)\sigma^2/2 - (\gamma+1)|\sigma|\varepsilon > 0$ since $\varepsilon < (\gamma-1)|\sigma|/[2(\gamma+1)]$. Hence

$$\begin{aligned} &\int_{T(\omega,\varepsilon)}^{\infty}\int_{T(\omega,\varepsilon)}^{t} k(t-s)e^{-\beta_1 s}e^{\beta_2 t}ds\,dt \\ &\quad= \int_{T(\omega,\varepsilon)}^{\infty}\int_{T(\omega,\varepsilon)}^{t} k(t-s)e^{\beta_1(t-s)}ds\cdot e^{(\beta_2-\beta_1)t}dt \\ &\quad= \int_{T(\omega,\varepsilon)}^{\infty}\int_{0}^{t-T(\omega,\varepsilon)} k(u)e^{\beta_1 u}du\cdot e^{(\beta_2-\beta_1)t}dt \\ &\quad= \int_{0}^{\infty}\int_{T(\omega,\varepsilon)+u}^{\infty} e^{-(\beta_1-\beta_2)t}dt\cdot k(u)e^{\beta_1 u}du \\ &\quad= \frac{e^{-(\beta_1-\beta_2)T(\omega,\varepsilon)}}{\beta_1-\beta_2}\int_0^{\infty} e^{\beta_2 u}k(u)du. \end{aligned} \tag{6.11}$$

Therefore by (6.6) we have that

$$\int_{T(\omega,\varepsilon)}^{\infty}\int_{T(\omega,\varepsilon)}^{t} k(t-s)e^{-\beta_1 s}e^{\beta_2 t}ds\,dt < +\infty, \tag{6.12}$$

and so

$$I_2 \le \int_{T(\omega,\varepsilon)}^{\infty}\int_{T(\omega,\varepsilon)}^{t} k(t-s)e^{-\beta_1 s}e^{\beta_2 t}ds\,dt < +\infty. \tag{6.13}$$

Hence $I = I_1 + I_2 + I_3 < +\infty$, as required. □

### *6.1. Proof of Theorem 3.3*

Let $\tau(\alpha)$ be defined by (2.9). Let $\omega \in \overline{A_\alpha}$. Let $y$ be given by (4.7). Then for all $t \in [0, \tau(\alpha))$ we have $0 < y(t,\omega) \le \alpha$. Note also that $y(\tau(\alpha)) = 0$. Therefore we have $0 < X(t,\omega) \le \alpha\varphi(t)$ for all $t \in [0, \tau(\alpha))$. Therefore by (3.8) it follows that

$$f(X(t,\omega)) \le L_2 X^\gamma(t,\omega) \le L_2 \alpha^\gamma \varphi^\gamma(t), \quad t \in [0, \tau(\alpha)]. \tag{6.14}$$

By (4.8) we have

$$0 = y(\tau(\alpha)) = \alpha - \int_0^{\tau(\alpha)} \int_0^t \varphi(t)^{-1} k(t-s) f(X(s)) ds\, dt. \tag{6.15}$$

Hence by (6.14) and the nonnegativity of $k$ and $\varphi$ we have

$$\begin{aligned} \alpha &= \int_0^{\tau(\alpha)} \int_0^t \varphi(t)^{-1} k(t-s) f(X(s)) ds\, dt \\ &\le L_2 \alpha^\gamma \int_0^{\tau(\alpha)} \int_0^t \varphi(t)^{-1} k(t-s) \varphi^\gamma(s) ds\, dt \\ &\le L_2 \alpha^\gamma \int_0^{\infty} \int_0^t \varphi(t)^{-1} k(t-s) \varphi^\gamma(s) ds\, dt. \end{aligned} \tag{6.16}$$

Therefore as $I$ is defined by (6.1) we have

$$I(\omega) \ge \frac{1}{L_2} \alpha^{1-\gamma}, \quad \text{for each } \omega \in \overline{\mathrm{A}_\alpha}. \tag{6.17}$$

Therefore

$$1 - \mathbb{P}[A_\alpha] = \mathbb{P}\left[\overline{A_\alpha}\right] \le \mathbb{P}\left[I \ge \frac{\alpha^{1-\gamma}}{L_2}\right]. \tag{6.18}$$

By Lemma 6.1 it follows that $\mathbb{P}[I < +\infty] = 1$. Therefore as $\gamma > 1$, by taking limits on both sides of (6.18), we obtain

$$1 - \liminf_{\alpha \to 0^+} \mathbb{P}[A_\alpha] = \limsup_{\alpha \to 0^+} \{1 - \mathbb{P}[A_\alpha]\} \le \limsup_{\alpha \to 0^+} \mathbb{P}\left[I \ge \frac{\alpha^{1-\gamma}}{L_2}\right] = 0. \tag{6.19}$$

Therefore we have $\liminf_{\alpha \to 0^+} \mathbb{P}[A_\alpha] \ge 1$. On the other hand, because we evidently have $\limsup_{\alpha \to 0^+} \mathbb{P}[A_\alpha] \le 1$, it follows that $\mathbb{P}[A_\alpha] \to 1$ as $\alpha \to 0^+$.

On the other hand, by (6.18) we have $\mathbb{P}[A_\alpha] \geq \mathbb{P}[I < \alpha^{1-\gamma}/L_2]$. Therefore as $I \in [0,\infty)$ a.s. it follows that there is an $\alpha_* > 0$ such that $\mathbb{P}[I < \alpha_*^{1-\gamma}/L_2] > 0$. Now suppose $\alpha \leq \alpha_*$. Then as $\gamma > 1$ we have

$$\mathbb{P}\left[I < \frac{\alpha^{1-\gamma}}{L_2}\right] \geq \mathbb{P}\left[I < \frac{\alpha_*^{1-\gamma}}{L_2}\right] > 0, \tag{6.20}$$

which implies $\mathbb{P}[A_\alpha] > 0$ for all $\alpha \leq \alpha_*$, proving the result.

## 7. Proof of Theorem 3.4

Let $A_\alpha$ be defined by (2.10). We suppose that $\mathbb{P}[A_\alpha] > 0$. Define also

$$h(t) = \int_0^t k(t-s)f(X(s))ds, \quad t \geq 0. \tag{7.1}$$

Then by (4.5) we have

$$X(t) = \varphi(t)\left(\alpha - \int_0^t \varphi(s)^{-1}h(s)ds\right), \quad t \geq 0. \tag{7.2}$$

Fix $\omega \in A_\alpha$. Since $h(t,\omega) \geq 0$ for all $t \geq 0$, we have that $X(t,\omega) \leq \alpha\varphi(t,\omega)$ for all $t \geq 0$. Also, as $\varphi(t) > 0$ for all $t \geq 0$ and $X(t,\omega) > 0$ we have

$$\int_0^t \varphi(s,\omega)^{-1}h(s,\omega)ds \leq \alpha, \quad \forall t \geq 0. \tag{7.3}$$

Hence

$$\int_0^\infty \varphi(s,\omega)^{-1}h(s,\omega)ds \leq \alpha, \quad \text{for each } \omega \in \mathrm{A}_\alpha. \tag{7.4}$$

Let $\Omega_1$ be the event defined in (6.3). Now define $C_\alpha = A_\alpha \cap \Omega_1$. Then $\mathbb{P}[C_\alpha] = \mathbb{P}[A_\alpha] > 0$. Let $\varepsilon > 0$ be so small that $|\sigma|^2/2 - |\sigma|\varepsilon \geq \sigma^2/2 - \epsilon > 0$, where $\epsilon > 0$ is defined by (3.12) (as in the proof of Lemma 6.1 above, note the distinction between the constant $\epsilon$ defined by (3.12) and the small parameter $\varepsilon > 0$). Then for every $\varepsilon > 0$ so chosen and $\omega \in C_\alpha$ there exists $T(\omega,\varepsilon) > 0$ such that $B$ obeys (6.4). Then as $\omega \in C_\alpha$ we have $X(t,\omega) > 0$ for all $t \in [0, T(\omega,\varepsilon)]$ and so by (2.3) we have that $f(X(t,\omega)) > 0$ for all $t \in [0, T(\omega,\varepsilon)]$. Define

$$F_\varepsilon(\omega) = \min_{u\in[0,T(\omega,\varepsilon)]} f(X(u,\omega)) > 0. \tag{7.5}$$

Let $s \geq T(\varepsilon,\omega)$. Then

$$h(s,\omega) \geq \int_0^{T(\varepsilon,\omega)} k(s-u) f(X(u,\omega))du \geq F_\varepsilon(\omega)\int_0^{T(\varepsilon,\omega)} k(s-u)du. \tag{7.6}$$

Therefore for all $\omega \in C_\alpha$, by using (6.4) and (4.1), we obtain

$$\begin{aligned}
&\int_0^\infty h(s,\omega)\varphi(s,\omega)^{-1}ds \\
&\quad \geq \int_{T(\varepsilon,\omega)}^\infty h(s,\omega)\varphi(s,\omega)^{-1}ds \\
&\quad = \int_{T(\varepsilon,\omega)}^\infty h(s,\omega)e^{\sigma^2/2s-\sigma B(s,\omega)}ds \\
&\quad \geq F_\varepsilon(\omega)\int_{T(\varepsilon,\omega)}^\infty \int_0^{T(\varepsilon,\omega)} k(s-u)du \cdot e^{\sigma^2/2s-\sigma B(s,\omega)}ds \\
&\quad \geq F_\varepsilon(\omega)\int_{T(\varepsilon,\omega)}^\infty \int_0^{T(\varepsilon,\omega)} k(s-u)du \cdot e^{(\sigma^2/2-|\sigma|\varepsilon)s}ds.
\end{aligned} \tag{7.7}$$

Therefore

$$\int_0^\infty h(s,\omega)\varphi(s,\omega)^{-1}ds \geq F_\varepsilon(\omega)\int_{T(\varepsilon,\omega)}^\infty \int_0^{T(\varepsilon,\omega)} k(s-u)du \cdot e^{(\sigma^2/2-\epsilon)s}ds. \tag{7.8}$$

Define $\beta_3 = \sigma^2/2 - \epsilon$. Then $\beta_3 > 0$ and by (3.12) we have

$$\int_0^\infty k(s)e^{\beta_3 s}ds = +\infty. \tag{7.9}$$

Then by the nonnegativity of the integrand and Fubini's theorem we have

$$\begin{aligned}
&\int_{T(\varepsilon,\omega)}^\infty \int_0^{T(\varepsilon,\omega)} k(s-u)du \cdot e^{\beta_3 s}ds \\
&\quad = \int_{T(\varepsilon,\omega)}^\infty \int_0^{T(\varepsilon,\omega)} k(s-u)e^{\beta_3(s-u)}e^{\beta_3 u}du\, ds
\end{aligned}$$

$$\geq \int_{T(\varepsilon,\omega)}^{\infty} \int_{0}^{T(\varepsilon,\omega)} k(s-u) e^{\beta_3 (s-u)} du\, ds$$
$$= \int_{T(\varepsilon,\omega)}^{\infty} \int_{s-T(\varepsilon,\omega)}^{s} k(v) e^{\beta_3 v} dv\, ds$$
$$= \int_{0}^{\infty} \int_{v \vee T(\varepsilon,\omega)}^{v+T(\varepsilon,\omega)} ds \cdot k(v) e^{\beta_3 v} dv$$
$$= \int_{0}^{T(\varepsilon,\omega)} v k(v) e^{\beta_3 v} dv + T(\varepsilon,\omega) \int_{T(\varepsilon,\omega)}^{\infty} k(v) e^{\beta_3 v} dv. \tag{7.10}$$

Therefore by (7.9), for each $\omega \in C_\alpha$ we have

$$\int_{T(\varepsilon,\omega)}^{\infty} \int_{0}^{T(\varepsilon,\omega)} k(s-u) du \cdot e^{\beta_3 s} ds = +\infty. \tag{7.11}$$

Therefore by (7.11) and (7.8) we have

$$\int_{0}^{\infty} h(s,\omega) \varphi(s,\omega)^{-1} ds = +\infty, \quad \text{for each } \omega \in C_\alpha. \tag{7.12}$$

Recall that $C_\alpha$ is an a.s. subset of $A_\alpha$ so that (7.4) implies

$$\int_{0}^{\infty} h(s,\omega) \varphi(s,\omega)^{-1} ds \leq \alpha, \quad \text{for each } \omega \in C_\alpha, \tag{7.13}$$

which contradicts (7.12). Therefore we have that $\mathbb{P}[A_\alpha] = 0$, or $\mathbb{P}[\overline{A_\alpha}] = 1$. Therefore we have that $\mathbb{P}[\tau(\alpha) < +\infty] = 1$. The proof that for each $\omega$ in the a.s. event $\{\tau(\alpha) < +\infty\}$, there exists a $t'(\omega) > 0$ such that $X(t'(\omega)) < 0$, that $X$ is differentiable at $\tau(\alpha)$, and that $X'(t(\alpha)) < 0$, is identical to that in the proof of Lemma 5.1. The proof of the theorem is therefore complete.

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