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Research Article

Sign-Changing Solutions for Nonlinear Elliptic Problems Depending on Parameters

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The study of multiple solutions for quasilinear elliptic problems under Dirichlet or nonlinear Neumann type boundary conditions has received much attention over the last decades. The main goal of this paper is to present multiple solutions results for elliptic inclusions of Clarke's gradient type under Dirichlet boundary condition involving the p-Laplacian which, in general, depend on two parameters. Assuming different structure and smoothness assumptions on the nonlinearities generating the multivalued term, we prove the existence of multiple constant-sign and sign-changing (nodal) solutions for parameters specified in terms of the Fučik spectrum of the p-Laplacian. Our approach will be based on truncation techniques and comparison principles (sub-supersolution method) for elliptic inclusions combined with variational and topological arguments for, in general, nonsmooth functionals, such as, critical point theory, Mountain Pass Theorem, Second Deformation Lemma, and the variational characterization of the "beginning"of the Fučik spectrum of the p-Laplacian. In particular, the existence of extremal constant-sign solutions and their variational characterization as global (resp., local) minima of the associated energy functional will play a key-role in the proof of sign-changing solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < +\infty$, denote the usual Sobolev spaces with their dual spaces V^* and V_0^*, respectively. We consider the following nonlinear multi-valued elliptic boundary value problem under Dirichlet boundary condition: find $u \in V_0 \setminus \{0\}$ and parameters $a \in \mathbb{R}$, $b \in \mathbb{R}$ such that

$$-\Delta_p u \in \partial j(\cdot, u, a, b) \quad \text{in } V_0^*, \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, and $s \mapsto \partial j(x,s,a,b)$ denotes Clarke's generalized gradient of some locally Lipschitz function $s \mapsto j(x,s,a,b)$ which depends on $x \in \Omega$ and the parameters $a,\ b$. For $a = b =: \lambda$ problem (1.1) reduces to

$$-\Delta_p u \in \partial j(\cdot, u, \lambda) \quad \text{in } V_0^*, \tag{1.2}$$

which may be considered as a nonlinear and nonsmooth eigenvalue problem. We are going to study the existence of multiple solutions of (1.1) for two different classes of j which are in some sense complementary. Our presentation is based on and extends the authors' recent results obtained in [1–3]. For the first class of j we let $a = b = \lambda$ and assume the following structure of j:

$$j(x,s,\lambda) = \int_0^s f(x,t,\lambda)\, dt, \tag{1.3}$$

where $f : \Omega \times \mathbb{R} \times (0,\overline{\lambda}) \to \mathbb{R}$ is such that $f(\cdot,\cdot,\lambda) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Problem (1.1) reduces then to the following nonlinear eigenvalue problem:

$$u \in V_0 \setminus \{0\} : -\Delta_p u = f(\cdot, u, \lambda) \quad \text{in } V_0^*, \tag{1.4}$$

which will be considered in Section 2 when the parameter λ is small enough.

The second class of j has the following structure:

$$j(x,s,a,b) = \frac{a}{p}(s^+)^p + \frac{b}{p}(s^-)^p + G(x,s), \tag{1.5}$$

where $s^+ = \max\{s,0\}$ and $s^- = \max\{-s,0\}$ is the positive and negative part of s, respectively, and $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be the primitive of a measurable function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ that is merely bounded on bounded sets; that is, $g \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R})$ and G is given by

$$G(x,s) := \int_0^s g(x,t)\, dt. \tag{1.6}$$

Problem (1.1) reduces then to the following parameter-dependent multi-valued elliptic problem:

$$u \in V_0 \setminus \{0\} : -\Delta_p u \in a(u^+)^{p-1} - b(u^-)^{p-1} - \partial G(x,u) \quad \text{in } V_0^*, \tag{1.7}$$

which will be studied in Section 3 for parameters a and b large enough. Note that $s \mapsto \partial G(x,s)$ stands for the generalized Clarke's gradient of the locally Lipschitz function $s \mapsto G(x,s)$. Obviously, if $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $x \mapsto g(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$ and $s \mapsto g(x,s)$ is continuous in $\mathbb{R}$ for a.a. $x \in \Omega$, then $\partial G(x,s) = \{g(x,s)\}$ is single-valued, and thus problem (1.7) reduces to the following nonlinear elliptic problem depending on parameters a and b: find $u \in V_0 \setminus \{0\}$ and constants $a \in \mathbb{R}$, $b \in \mathbb{R}$ such that

$$-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} - g(x,u) \quad \text{in } V_0^*. \tag{1.8}$$

Multiple solution results for (1.8) were obtained by the authors in [4]. Furthermore, note that

$$|u|^{p-2}u = |u|^{p-2}(u^+ - u^-) = (u^+)^{p-1} - (u^-)^{p-1}. \tag{1.9}$$

Therefore, if one assumes, in addition, $a = b =: \lambda$, then (1.8) reduces to the nonlinear elliptic eigenvalue problem: find $u \in V_0 \setminus \{0\}$ and a constant $\lambda \in \mathbb{R}$ such that

$$-\Delta_p u = \lambda |u|^{p-2}u - g(x,u) \quad \text{in } V_0^*. \tag{1.10}$$

In a recent paper (see [5]) the authors considered the eigenvalue problem (1.10) for a Carathéodory function g. Combining the method of sub-supersolution with variational techniques and assuming certain growth conditions of $s \mapsto g(x,s)$ at infinity and at zero the authors were able to prove the existence of at least three nontrivial solutions including one that changes sign. The results in [5] improve among others recent results obtained in [6]. For $a = b =: \lambda$, (1.7) reduces to the corresponding multivalued eigenvalue problem: find $u \in V_0 \setminus \{0\}$ and a constant $\lambda \in \mathbb{R}$ such that

$$-\Delta_p u \in \lambda |u|^{p-2}u - \partial G(x,u) \quad \text{in } V_0^*. \tag{1.11}$$

The existence of multiple solutions for (1.11) has been shown recently in [7] where techniques for single-valued problems developed in [5] and hemivariational methods applied in [8] have been used. Multiplicity results for (1.11) have been obtained also in [9].

The existence of multiple solutions for semilinear and quasilinear elliptic problems has been studied by a number of authors, for example, [10–24]. All these papers deal with nonlinearities $(x,s) \mapsto g(x,s)$ that are sufficiently smooth.

2. Problem (1.4) for λ being Small

The aim of this section is to provide an existence result of multiple solutions for all values of the parameter λ in an interval $(0,\lambda_0)$, with $\lambda_0 > 0$, guaranteeing that for any such λ there exist at least three nontrivial solutions of problem (1.4), two of them having opposite constant sign and the third one being sign-changing (or nodal). More precisely, we demonstrate that under suitable assumptions there exist a smallest positive solution, a greatest negative solution, and a sign-changing solution between them, whereas the notions *smallest* and *greatest* refer to the underlying natural partial ordering of functions. This continues the works of Jin [25] (where $p = 2$ and $f(x,s,\lambda)$ is Hölder continuous with respect to $(x,s) \in \overline{\Omega} \times \mathbb{R}$ for every fixed λ) and of Motreanu-Motreanu-Papageogiou [26]. In these cited works one obtains three nontrivial solutions, two of which being of opposite constant sign, but without knowing that the third one changes sign. Here we derive the new information of having, in addition, a sign-changing solution by strengthening the unilateral condition for the right-hand side of the equation in

(1.4) at zero. Furthermore, under additional hypotheses, we demonstrate that one can obtain two sign-changing solutions.

2.1. Hypotheses and Example

Let $L^q(\Omega)_+, 1 \le q \le +\infty$, denote the positive cone of $L^q(\Omega)$ given by

$$L^q(\Omega)_+ = \{v \in L^q(\Omega) : v(x) \ge 0 \text{ for a.a. } x \in \Omega\}. \tag{2.1}$$

We impose the following hypotheses on the nonlinearity $f(x,s,\lambda)$ in problem (1.4).

H(f) $f : \Omega \times \mathbb{R} \times (0, \overline{\lambda}) \to \mathbb{R}$, with $\overline{\lambda} > 0$, is a function such that $f(x,0,\lambda) = 0$ for a.a. $x \in \Omega$, whenever $\lambda \in (0, \overline{\lambda})$, and one has the following.

(i) For all $\lambda \in (0, \overline{\lambda})$, $f(\cdot,\cdot,\lambda)$ is Carathéodory (i.e., $f(\cdot, s, \lambda)$ is measurable for all $s \in \mathbb{R}$ and $f(x,\cdot,\lambda)$ is continuous for almost all $x \in \Omega$).

(ii) There are constants $c > 0$, $r > p - 1$, and functions $a(\cdot,\lambda) \in L^\infty(\Omega)_+$ ($\lambda \in (0,\overline{\lambda})$) with $\|a(\cdot,\lambda)\|_\infty \to 0$ as $\lambda \downarrow 0$ such that

$$\left|f(x,s,\lambda)\right| \le a(x,\lambda) + c|s|^r \quad \text{for a.a. } x \in \Omega \ \forall (s,\lambda) \in \mathbb{R} \times \left(0,\overline{\lambda}\right). \tag{2.2}$$

(iii) For all $\lambda \in (0, \overline{\lambda})$ there exist constants $\mu_0 = \mu_0(\lambda) > \lambda_2$, $\nu_0 = \nu_0(\lambda) > \mu_0$ and a set $\Omega_\lambda \subset \Omega$ with $\Omega \setminus \Omega_\lambda$ of Lebesgue measure zero such that

$$\mu_0 < \liminf_{s \to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \le \limsup_{s \to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \le \nu_0 \tag{2.3}$$

uniformly with respect to $x \in \Omega_\lambda$.

In H(f)(iii), λ_2 denotes the second eigenvalue of $(-\Delta_p, V_0)$. As mentioned in the Introduction, the strengthening with respect to [26] (see also [25]) of the unilateral condition for the right-hand side f in (1.4), which enables us to obtain, in addition, sign-changing solutions, consists in adding the part involving the limit superior in H(f)(iii).

Let us provide an example where all the assumptions formulated in H(f) are fulfilled.

Example 2.1. For the sake of simplicity we drop the x dependence for the function f in the right-hand side of (1.4). The function $f : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ given by

$$f(s,\lambda) = \lambda \arctan\left(\frac{\lambda + \lambda_2}{\lambda}|s|^{p-2}s\right) + c|s|^{r-1}s \quad \forall (s,\lambda) \in \mathbb{R} \times (0,+\infty), \tag{2.4}$$

with $c > 0$ and $r > p - 1$, satisfies hypotheses H(f). Next we give an example of function $f : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ verifying assumptions H(f) which is generally not odd with respect to s:

$$f(s,\lambda) = \begin{cases} \lambda \arctan\left(a_1 \dfrac{\lambda + \lambda_2}{\lambda} |s|^{p-2} s \right) + c_1 |s|^{r_1 - 1} s & \text{if } s \leq 0, \\ \lambda \arctan\left(a_2 \dfrac{\lambda + \lambda_2}{\lambda} s^{p-1} \right) + c_2 s^{r_2} & \text{if } s > 0, \end{cases} \tag{2.5}$$

with $\lambda > 0$, $a_1 \geq 1$, $a_2 \geq 1$, $c_1 > 0$, $c_2 > 0$, $r_1 > p - 1$, $r_2 > p - 1$.

2.2. Constant-Sign Solutions

The operator $-\Delta_p : V_0 \to V_0^*$ is maximal monotone and coercive; therefore there exists a unique solution $e \in V_0$ of the Dirichlet problem

$$e \in V_0 : -\Delta_p e = 1 \quad \text{in } V_0^*. \tag{2.6}$$

With $s^- = \max\{-s, 0\}$ for $s \in \mathbb{R}$, and using $-e^- \in V_0$ as a test function, we see that

$$\|\nabla e^-\|_p^p = \left\langle -\Delta_p e, -e^- \right\rangle = -\int_\Omega e^-(x)dx \leq 0, \tag{2.7}$$

which implies that $e \geq 0$. From the nonlinear regularity theory (cf., e.g., [27, Theorem 1.5.6]) we have $e \in C_0^1(\overline{\Omega})$. Then from the nonlinear strong maximum principle (see [28]) we infer that $e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Here $\operatorname{int}(C_0^1(\overline{\Omega})_+)$ denotes the interior of the positive cone $C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0, \forall x \in \Omega\}$ in the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u(x) = 0,\ \forall x \in \partial\Omega\}$, given by

$$\operatorname{int}\left(C_0^1\left(\overline{\Omega}\right)_+\right) = \left\{ u \in C_0^1\left(\overline{\Omega}\right) : u(x) > 0,\ \forall x \in \Omega, \text{and } \frac{\partial u}{\partial n}(x) < 0,\ \forall x \in \partial\Omega \right\}, \tag{2.8}$$

where $n = n(x)$ is the outer unit normal at $x \in \partial\Omega$.

Lemma 2.2. *Let the data c, r, and $a(\cdot, \lambda)$ be as in H(f)(ii). Then for every constant $\theta > 0$ there is $\lambda_0 \in (0, \overline{\lambda})$ with the property that if $\lambda \in (0, \lambda_0)$, one can choose $\xi_0 = \xi_0(\lambda) \in (0, \theta)$ such that*

$$c(\xi_0 \|e\|_\infty)^r + \|a(\cdot, \lambda)\|_\infty < \xi_0^{p-1}. \tag{2.9}$$

Proof. On the contrary there would exist a constant $\theta > 0$ and a sequence $\lambda_n \downarrow 0$ as $n \to \infty$ such that

$$c(\xi \|e\|_\infty)^r + \|a(\cdot, \lambda_n)\|_\infty \geq \xi^{p-1} \quad \forall n \in \mathbb{N},\ \xi \in (0, \theta). \tag{2.10}$$

Letting $n \to \infty$ we get $c\|e\|_\infty^r \xi^{r-p+1} \geq 1$ for all $\xi \in (0,\theta)$ because we have $\|a(\cdot,\lambda)\|_\infty \to 0$ as $\lambda \downarrow 0$. Since $r > p-1$, a contradiction is achieved as $\xi \downarrow 0$. Therefore (2.9) holds true. □

We denote by λ_1 the first eigenvalue of $(-\Delta_p, V_0)$ and by φ_1 the eigenfunction of $(-\Delta_p, V_0)$ corresponding to λ_1 satisfying

$$\varphi_1 \in \operatorname{int}\left(C_0^1\left(\overline{\Omega}\right)_+\right), \quad \|\varphi_1\|_p = 1. \tag{2.11}$$

Lemma 2.3. *Assume $H(f)$(i) and (ii) and the following weaker form of hypothesis $H(f)$(iii): for all $\lambda \in (0,\overline{\lambda})$ there exist $\mu_0 = \mu_0(\lambda) > \lambda_1$ and $\Omega_\lambda \subset \Omega$ with $\Omega \setminus \Omega_\lambda$ of Lebesgue measure zero such that*

$$\mu_0 < \liminf_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \tag{2.12}$$

uniformly with respect to $x \in \Omega_\lambda$.

Fix a constant $\theta > 0$ and consider the corresponding number $\lambda_0 \in (0,\overline{\lambda})$ obtained in Lemma 2.2. Then for any $\lambda \in (0,\lambda_0)$ the function $\overline{u} = \xi_0 e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, with $\xi_0 \in (0,\theta)$ given by Lemma 2.2, is a supersolution for problem (1.4), and the function $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is a subsolution of problem (1.4) provided that the number $\varepsilon > 0$ is sufficiently small.

Proof. For a fixed $\lambda \in (0,\lambda_0)$, from (2.9) and H$(f)$(ii) we derive

$$-\Delta_p \overline{u} = \xi_0^{p-1} > \|a(\cdot,\lambda)\|_\infty + c\|\overline{u}\|_\infty^r \geq f(\cdot,\overline{u}(\cdot),\lambda), \tag{2.13}$$

which says that $\overline{u} = \xi_0 e$ is a supersolution for problem (1.4).

On the other hand, by hypothesis we can find $\mu = \mu(\lambda) > \lambda_1$ and $\delta = \delta(\lambda) > 0$ such that

$$\mu < \frac{f(x,s,\lambda)}{|s|^{p-2}s} \quad \text{for a.a. } x \in \Omega\ \forall 0 < |s| \leq \delta. \tag{2.14}$$

Choose $\varepsilon \in (0, \delta/\|\varphi_1\|_\infty)$. Then by (2.14) we have

$$-\Delta_p(\varepsilon\varphi_1) = \lambda_1 \varepsilon^{p-1}\varphi_1^{p-1} < \mu\varepsilon^{p-1}\varphi_1^{p-1} < f(x,\varepsilon\varphi_1(x),\lambda) \quad \text{for a.a. } x \in \Omega, \tag{2.15}$$

which ensures that $\underline{u} = \varepsilon\varphi_1$ is a subsolution of problem (1.4). □

The following result which asserts the existence of two solutions of problem (1.4) having opposite constant sign and being extremal plays an important role in the proof of the existence of sign-changing solutions.

Theorem 2.4. *Assume $H(f)$(i) and (ii) and the following weaker form of $H(f)$(iii): for all $\lambda \in (0,\overline{\lambda})$ there exist constants $\mu_0 = \mu_0(\lambda) > \lambda_1$, $\nu_0 = \nu_0(\lambda) > \mu_0$ and a set $\Omega_\lambda \subset \Omega$ with $\Omega \setminus \Omega_\lambda$ of Lebesgue measure zero such that*

$$\mu_0 < \liminf_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \leq \limsup_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \leq \nu_0 \tag{2.16}$$

uniformly with respect to $x \in \Omega_\lambda$. *Then for all* $b > 0$ *there exists a number* $\lambda_0 \in (0, \overline{\lambda})$ *with the property that if* $\lambda \in (0, \lambda_0)$, *then there is a constant* $\xi_0 = \xi_0(\lambda) \in (0, b/\|e\|_\infty)$ *such that problem* (1.4) *has a least positive solution* $u_+ = u_+(\lambda) \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *in the order interval* $[0, \xi_0 e]$ *and a greatest negative solution* $u_- = u_-(\lambda) \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *in the order interval* $[-\xi_0 e, 0]$.

Proof. Since the proof of the existence of the greatest negative solution follows the same lines, we only provide the arguments for the existence of the least positive solution.

Applying Lemma 2.3 for $\theta = b/\|e\|_\infty$ we find $\lambda_0 \in (0, \overline{\lambda})$ as therein. Fix $\lambda \in (0, \lambda_0)$. Lemma 2.3 ensures that $\overline{u} = \xi_0 e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is a supersolution for problem (1.4), with $\xi_0 \in (0, b/\|e\|_\infty)$ given by Lemma 2.2, and $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is a subsolution for problem (1.4) if $\varepsilon > 0$ is small enough. Passing eventually to a smaller $\varepsilon > 0$, we may assume that $\varepsilon\varphi_1 \le \xi_0 e$. Then by the method of sub-supersolution we know that in the order interval $[\varepsilon\varphi_1, \xi_0 e]$ there is a least (i.e., smallest) solution $u_\varepsilon = u_\varepsilon(\lambda) \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ of problem (1.4) (see [29]).

We thus obtain that for every positive integer n sufficiently large there is a least solution $u_n \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ of problem (1.4) in the order interval $[(1/n)\varphi_1, \xi_0 e]$. Clearly, we have

$$u_n \downarrow u_+ \quad \text{pointwise}, \tag{2.17}$$

with some function $u_+ : \Omega \to \mathbb{R}$ satisfying $0 \le u_+ \le \xi_0 e$. First we claim that

$$u_+ \text{ is a solution of problem (1.4).} \tag{2.18}$$

Taking into account that u_n solves (1.4), and the fact that u_n belongs to the order interval $[0, \xi_0 e]$, from H(f)(ii) we see that

$$\|\nabla u_n\|_p^p = \int_\Omega f(x, u_n(x), \lambda) u_n(x) dx \le \int_\Omega \left(a(x,\lambda) + c\xi_0^r e(x)^r\right) \xi_0 e(x) dx, \tag{2.19}$$

which implies the boundedness of the sequence (u_n) in V_0. Then due to (2.17) we have that $u_+ \in V_0$ as well as

$$u_n \rightharpoonup u_+ \quad \text{in } V_0, \qquad u_n \longrightarrow u_+ \quad \text{in } L^p(\Omega) \text{ and a.e. in } \Omega. \tag{2.20}$$

Since u_n solves problem (1.4), one has

$$\langle -\Delta_p u_n, \varphi \rangle = \int_\Omega f(x, u_n(x), \lambda) \varphi(x) dx, \quad \forall \varphi \in V_0. \tag{2.21}$$

Setting $\varphi = u_n - u_+$ in (2.21) gives

$$\langle -\Delta_p u_n, u_n - u_+ \rangle = \int_\Omega f(x, u_n(x), \lambda)(u_n(x) - u_+(x)) dx. \tag{2.22}$$

As already noticed that the sequence $(f(\cdot, u_n(\cdot), \lambda)$ is uniformly bounded on Ω, so (2.20) and (2.22) yield

$$\lim_{n\to\infty} \langle -\Delta_p u_n, u_n - u_+ \rangle = 0. \tag{2.23}$$

The S_+-property of $-\Delta_p$ on V_0 implies

$$u_n \longrightarrow u_+ \quad \text{in } V_0 \text{ as } n \longrightarrow \infty. \tag{2.24}$$

The strong convergence in (2.24) and Lebesgue's dominated convergence theorem permit to pass to the limit in (2.21) that results in (2.18).

By (2.18) and the nonlinear regularity theory (cf., e.g., Theorem 1.5.6 in [27]) it turns out $u_+ \in C_0^1(\overline{\Omega})$. The choice of ξ_0 guarantees that

$$0 \le u_+(x) \le \xi_0 e(x) \le b \quad \text{for a.e. } x \in \Omega. \tag{2.25}$$

Thus, from (2.18), assumptions H(f)(ii) and (iii), and the boundedness of u_+, we get

$$-\Delta_p u_+(x) = f(x, u_+(x), \lambda) \ge -\hat{c} u_+(x)^{p-1} \quad \text{for a.a. } x \in \Omega, \tag{2.26}$$

with a constant $\hat{c} > 0$. Applying the nonlinear strong maximum principle (cf. [28]) we conclude that either $u_+ = 0$ or $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

We claim that

$$u_+ \in \operatorname{int}\left(C_0^1\left(\overline{\Omega}\right)_+\right). \tag{2.27}$$

Assume on the contrary that $u_+ = 0$. Then (2.17) becomes

$$u_n(x) \downarrow 0 \quad \forall x \in \Omega. \tag{2.28}$$

Since $u_n \ge (1/n)\varphi_1$, we may consider

$$\tilde{u}_n = \frac{u_n}{\|\nabla u_n\|_p} \quad \forall n. \tag{2.29}$$

Along a relabelled subsequence we may suppose

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in } V_0, \qquad \tilde{u}_n \longrightarrow \tilde{u} \quad \text{in } L^p(\Omega) \text{ and a.e. in } \Omega \tag{2.30}$$

for some $\tilde{u} \in V_0$. Moreover, one can find a function $w \in L^p(\Omega)_+$ such that $|\tilde{u}_n(x)| \le w(x)$ for almost all $x \in \Omega$. Relation (2.21) reads

$$\langle -\Delta_p \tilde{u}_n, \varphi \rangle = \int_\Omega \frac{f(x, u_n(x), \lambda)}{\|\nabla u_n\|_p^{p-1}} \varphi \, dx, \quad \forall \varphi \in V_0. \tag{2.31}$$

Setting $\varphi = \tilde{u}_n - \tilde{u}$ leads to

$$\langle -\Delta_p \tilde{u}_n, \tilde{u}_n - \tilde{u} \rangle = \int_\Omega \frac{f(x, u_n(x), \lambda)}{\|\nabla u_n\|_p^{p-1}} (\tilde{u}_n - \tilde{u}) dx. \tag{2.32}$$

By H(f)(iii) we know that there exist constants $c_0 = c_0(\lambda) > \lambda_1$ and $\alpha = \alpha(\lambda) > 0$ such that

$$|f(x, s, \lambda)| \leq c_0 |s|^{p-1} \quad \text{for a.a. } x \in \Omega, \ \forall |s| < \alpha, \tag{2.33}$$

while H(f)(ii) entails

$$|f(x, s, \lambda)| \leq a(x, \lambda) + c|s|^r \leq \left(\frac{\|a(\cdot, \lambda)\|_\infty}{\alpha^r} + c \right) |s|^r \tag{2.34}$$

for a.a. $x \in \Omega$ and for all $|s| \geq \alpha$. Combining the two estimates gives

$$|f(x, s, \lambda)| \leq c_0 |s|^{p-1} + c_1 |s|^r \quad \text{for a.a. } x \in \Omega, \ \forall s \in \mathbb{R} \tag{2.35}$$

with a constant $c_1 = c_1(\lambda) > 0$. Since $u_n \in [(1/n)\varphi_1, \xi_0 e]$, $r > p-1$ and (2.35) holds, there exists a constant $C > 0$ such that

$$\frac{|f(x, u_n(x), \lambda)|}{u_n(x)^{p-1}} \leq C \quad \text{for a.a. } x \in \Omega, \ \forall n. \tag{2.36}$$

We see from (2.36) that

$$\frac{|f(x, u_n(x), \lambda)|}{\|\nabla u_n\|_p^{p-1}} |\tilde{u}_n(x) - \tilde{u}(x)| \leq C w(x)^{p-1} (w(x) + |\tilde{u}(x)|) \quad \text{for a.a. } x \in \Omega. \tag{2.37}$$

Then, because the right-hand side of the above inequality is in $L^1(\Omega)$, by means of (2.30) and (2.36) we can apply Lebesgue's dominated convergence theorem to get

$$\lim_{n \to \infty} \int_\Omega \frac{f(x, u_n(x), \lambda)}{\|\nabla u_n\|_p^{p-1}} (\tilde{u}_n - \tilde{u}) \ dx = 0. \tag{2.38}$$

Consequently, from (2.32) we obtain

$$\lim_{n \to \infty} \langle -\Delta_p \tilde{u}_n, \tilde{u}_n - \tilde{u} \rangle = 0. \tag{2.39}$$

The S_+-property of $-\Delta_p$ on V_0 implies

$$\tilde{u}_n \longrightarrow \tilde{u} \quad \text{in } V_0 \text{ as } n \longrightarrow \infty. \tag{2.40}$$

On the basis of (2.31) and (2.40) it follows

$$\langle -\Delta_p \tilde{u}, \varphi \rangle = \lim_{n\to\infty} \int_\Omega \frac{f(x, u_n(x), \lambda)}{\|\nabla u_n\|_p^{p-1}} \varphi \, dx, \quad \forall \varphi \in V_0. \tag{2.41}$$

Notice from (2.36) that

$$\frac{|f(x, u_n(x), \lambda)|}{\|\nabla u_n\|_p^{p-1}} |\varphi(x)| \le C w(x)^{p-1} |\varphi(x)| \tag{2.42}$$

for a.a. $x \in \Omega$ and for all $\varphi \in V_0$. We are thus allowed to apply Fatou's lemma which in conjunction with (2.28), (2.30), and (2.16) ensures

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega \frac{f(x, u_n(x), \lambda)}{\|\nabla u_n\|_p^{p-1}} \varphi(x) dx &= \lim_{n\to\infty} \int_\Omega \frac{f(x, u_n(x), \lambda)}{u_n(x)^{p-1}} \tilde{u}_n(x)^{p-1} \varphi(x) \, dx \\ &\ge \int_\Omega \liminf_{n\to\infty} \left(\frac{f(x, u_n(x), \lambda)}{u_n(x)^{p-1}} \tilde{u}_n(x)^{p-1} \varphi(x) \right) dx \\ &\ge \mu_0 \int_\Omega \tilde{u}(x)^{p-1} \varphi(x) dx \end{aligned} \tag{2.43}$$

for all $\varphi \in V_{0,+} := V_0 \cap L^p(\Omega)_+$. Thanks to (2.41) we obtain

$$\langle -\Delta_p \tilde{u}, \varphi \rangle \ge \mu_0 \int_\Omega \tilde{u}(x)^{p-1} \varphi(x) dx, \quad \forall \varphi \in V_{0,+}. \tag{2.44}$$

Owing to (2.42) we may once again use Fatou's lemma; so according to (2.28), (2.30), and the last part of (2.16), we find

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega \frac{f(x, u_n(x), \lambda)}{\|\nabla u_n\|_p^{p-1}} \varphi(x) dx &= \lim_{n\to\infty} \int_\Omega \frac{f(x, u_n(x), \lambda)}{u_n(x)^{p-1}} \tilde{u}_n(x)^{p-1} \varphi(x) dx \\ &\le \int_\Omega \limsup_{n\to\infty} \left(\frac{f(x, u_n(x), \lambda)}{u_n(x)^{p-1}} \tilde{u}_n(x)^{p-1} \varphi(x) \right) dx \\ &\le \nu_0 \int_\Omega \tilde{u}(x)^{p-1} \varphi(x) dx \end{aligned} \tag{2.45}$$

for all $\varphi \in V_{0,+}$. Then (2.41) ensures

$$\langle -\Delta_p \tilde{u}, \varphi \rangle \le \nu_0 \int_\Omega \tilde{u}(x)^{p-1} \varphi(x) dx, \quad \forall \varphi \in V_{0,+}. \tag{2.46}$$

Combining (2.44) and (2.46) results in

$$\mu_0 \widetilde{u}^{p-1} \leq -\Delta_p \widetilde{u} \leq \nu_0 \widetilde{u}^{p-1} \quad \text{a.e. in } \Omega, \tag{2.47}$$

which guarantees to have $\widetilde{u} \in L^\infty(\Omega)$ (see [27, Theorem 1.5.5]). Since by (2.47) we know that $\Delta_p \widetilde{u} \in L^\infty(\Omega)$, we are in a position to address Theorem 1.5.6 in [27], which provides $\widetilde{u} \in C^{1,\beta}(\overline{\Omega})$ with some $\beta \in (0,1)$. This regularity up to the boundary and the fact that $\widetilde{u} \geq 0$ a.e. in Ω and (2.47) enable us to refer to the strong maximum principle (see Theorem 5 of Vázquez [28]). Recalling that $\widetilde{u}$ does not vanish identically on Ω (because $\|\nabla \widetilde{u}\|_p = 1$) we deduce that $\widetilde{u}(x) > 0$ for all $x \in \Omega$ and $(\partial \widetilde{u}/\partial n)(x) < 0$ for all $x \in \partial\Omega$ which amounts to saying $\widetilde{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Consequently, there exist constants $k_0 > 0$ and $k_1 > 0$ such that

$$k_0 \varphi_1 \leq \widetilde{u} < k_1 \varphi_1 \quad \text{a.e. in } \Omega. \tag{2.48}$$

Following [30] let us denote

$$I(u,v) = \left\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} \right\rangle - \left\langle -\Delta_p v, \frac{u^p - v^p}{v^{p-1}} \right\rangle \tag{2.49}$$

whenever $(u,v) \in D_I$, where

$$D_I = \left\{ (w_1, w_2) \in (V_0)^2 : \ w_i \geq 0, \ \frac{w_i}{w_j} \in L^\infty(\Omega) \text{ for } i,j \in \{1,2\} \right\}. \tag{2.50}$$

Relation (2.48) justifies that $(k_1\varphi_1, \widetilde{u}) \in D_I$. Then Proposition 1 of Anane [30] implies $I(k_1\varphi_1, \widetilde{u}) \geq 0$. On the other hand a direct computation based on (2.48) and (2.47) shows

$$\begin{aligned} I(k_1\varphi_1, \widetilde{u}) &= \left\langle -\Delta_p (k_1\varphi_1), \frac{(k_1\varphi_1)^p - \widetilde{u}^p}{(k_1\varphi_1)^{p-1}} \right\rangle - \left\langle -\Delta_p \widetilde{u}, \frac{(k_1\varphi_1)^p - \widetilde{u}^p}{\widetilde{u}^{p-1}} \right\rangle \\ &\leq (\lambda_1 - \mu_0) \int_\Omega \left((k_1\varphi_1)^p - \widetilde{u}^p \right) dx < 0. \end{aligned} \tag{2.51}$$

This contradiction proves that the claim in (2.27) holds true.

In view of (2.18) it remains to establish that u_+ is the smallest positive solution of problem (1.4) in the interval $[0, \overline{u}]$. Let $u \in V_0$ be a positive solution to (1.4) in $[0, \overline{u}]$. Since $u \in L^\infty(\Omega)$, then (1.4) and H(f)(ii) allow to deduce that $-\Delta_p u \in L^\infty(\Omega)$. Using Theorem 1.5.6 of [27] leads to $u \in C_0^1(\overline{\Omega})$. Then, as u is a solution to (1.4) and $u \in [0, \overline{u}]$, with $\|\overline{u}\|_\infty < b$, by means of hypotheses H(f)(ii) and (iii), we are able to apply the strong maximum principle. So we get $u \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, hence $u \in [(1/n)\varphi_1, \overline{u}]$ for n sufficiently large. The fact that u_n is the least solution of (1.4) in $[(1/n)\varphi_1, \overline{u}]$ ensures $u_n \leq u$. Taking into account (2.17), we obtain $u_+ \leq u$. This completes the proof. □

2.3. Sign-Changing Solution

The main result of this section is as follows.

Theorem 2.5. *Under hypotheses $H(f)$, for all $b > 0$, there exists a number $\lambda_0 \in (0, \overline{\lambda})$ with the property that if $\lambda \in (0, \lambda_0)$, then problem* (1.4) *has a (positive) solution $u_+ = u_+(\lambda) \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, a (negative) solution $u_- = u_-(\lambda) \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$, and a nontrivial sign-changing solution $u_0 = u_0(\lambda) \in C_0^1(\overline{\Omega})$ satisfying $\|u_+\|_\infty < b$, $\|u_-\|_\infty < b$, $\|u_0\|_\infty < b$.*

Proof. Let $b > 0$. Consider the positive number λ_0 given by Theorem 2.4 and fix $\lambda \in (0, \lambda_0)$. Let $u_+ \in \operatorname{int}\,(C_0^1(\overline{\Omega})_+)$ and $u_- \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ be the two extremal solutions determined in Theorem 2.4. We introduce on $\Omega \times \mathbb{R}$ the truncation functions

$$\tau_+(x,s) = \begin{cases} 0 & \text{if } s \le 0, \\ s & \text{if } 0 < s < u_+(x), \\ u_+(x) & \text{if } s \ge u_+(x), \end{cases}$$
$$\tau_-(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x), \\ s & \text{if } u_-(x) < s < 0, \\ 0 & \text{if } s \ge 0, \end{cases} \tag{2.52}$$
$$\tau_0(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x), \\ s & \text{if } u_-(x) < s < u_+(x), \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}$$

and then define the following associated functionals:

$$E_+(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} f(x, \tau_+(x,s), \lambda)ds\; dx, \quad \forall u \in V_0,$$
$$E_-(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} f(x, \tau_-(x,s), \lambda)ds\; dx, \quad \forall u \in V_0, \tag{2.53}$$
$$E_0(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} f(x, \tau_0(x,s), \lambda)ds\; dx, \quad \forall u \in V_0.$$

It is clear that $E_+, E_-, E_0 \in C^1(V_0)$.

We observe that if v is a critical point of E_+, then

$$\left\langle -\Delta_p\, v + \Delta_p\, u_+, (v - u_+)^+ \right\rangle = \int_\Omega \big(f(x, \tau_+(x, v(x)), \lambda) - f(x, u_+(x), \lambda)\big)(v - u_+)^+ dx = 0 \tag{2.54}$$

which implies $v \le u_+$. Similarly, it follows that $v \ge 0$. This leads to

$$v \text{ is a critical point of } E_+ \Longrightarrow 0 \le v(x) \le u_+(x) \quad \text{for a.a. } x \in \Omega. \tag{2.55}$$

Since the function E_+ is coercive and weakly lower semicontinuous, there exists a global minimizer $z_+ \in V_0$ of it. Using (2.14), it is seen that

$$E_+(z_+) = \inf_{V_0} E_+ < 0, \tag{2.56}$$

and so $z_+ \ne 0$. Relation (2.55) shows that z_+ is a nontrivial solution of problem (1.4) belonging to the order interval $[0, u_+]$. Via assumptions H(f)(ii) and (iii) and the boundedness of z_+, we may apply the strong maximum principle which ensures $z_+ > 0$ on Ω. In view of the minimality property of u_+ as stated in Theorem 2.4, it follows that $z_+ = u_+$. In fact, u_+ is the unique global minimizer of E_+.

Since $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, there exists a neighborhood $\mathcal{U}$ of u_+ in the space $C_0^1(\overline{\Omega})$ such that $\mathcal{U} \subset C_0^1(\overline{\Omega})_+$. Therefore $E_0 = E_+$ on $\mathcal{U}$, which guarantees that u_+ is a local minimizer of E_0 on $C_0^1(\overline{\Omega})$. It results that u_+ is also a local minimizer of E_0 on the space V_0 (see [27], pages 655-656). Employing the functional E_- and proceeding as in the case of u_+, we establish that u_- is a local minimizer of E_0 on V_0.

As in the case of (2.55), we verify that every critical point of E_0 belongs to the set $\{u \in V_0 : u_-(x) \le u(x) \le u_+(x) \text{ a.e. } x \in \Omega\}$, which implies that every critical point of E_0 is a solution to problem (1.4). The functional E_0 is coercive, weakly lower semicontinuous, with $\inf_{V_0} E_0 < 0$. Thus E_0 has a global minimizer $y_0 \in V_0$ with $y_0 \ne 0$. The above properties ensure that y_0 is a nontrivial solution of problem (1.4) belonging to the order interval $[u_-, u_+]$. Assume $y_0 \ne u_+$ and $y_0 \ne u_-$. We claim that y_0 changes sign. Indeed, if not, y_0 would have constant sign, for instance $y_0 \ge 0$ a.e. on Ω. Using assumptions H(f)(ii) and (iii) and the boundedness of y_0, we may apply the strong maximum principle which leads to $y_0 > 0$ on Ω. This is impossible because it contradicts the minimality property of the solution u_+ as given by Theorem 2.4. According to the claim, we obtain the conclusion of the theorem setting $u_0 = y_0$.

Thus, the proof reduces to consider the cases $y_0 = u_+$ or $y_0 = u_-$. To make a choice, suppose $y_0 = u_+$. We may also admit that u_- is a strict local minimizer of E_0. This is true since on the contrary we would find (infinitely many) critical points x_0 of E_0 belonging to the order interval $[u_-, u_+]$ which are different from 0, u_-, u_+, and if x_0 does not change sign, taking into account the strong maximum principle, the extremality properties of the solutions u_-, u_+ given in Theorem 2.4 will be contradicted. A straightforward argument allows then to find $\rho \in (0, \|u_+ - u_-\|)$ such that

$$E_0(u_+) \le E_0(u_-) < \inf\{E_0(u) : u \in \partial B_\rho(u_-)\}, \tag{2.57}$$

where $\partial B_\rho(u_-) = \{u \in V_0 : \|u - u_-\| = \rho\}$. Relation (2.57) in conjunction with the Palais-Smale condition (which holds for E_0 due to its coercivity) enables us to apply the mountain pass theorem to the functional E_0 (see, e.g., [31]). In this way we get $u_0 \in V_0$ satisfying $E'_0(u_0) = 0$ and

$$\inf\{E_0(u) : u \in \partial B_\rho(u_-)\} \le E_0(u_0) = \inf_{\gamma\in\Gamma} \max_{t\in[-1,1]} E_0(\gamma(t)), \tag{2.58}$$

where

$$\Gamma = \{\gamma \in C([-1,1], V_0) : \ \gamma(-1) = u_-, \ \gamma(1) = u_+\}. \tag{2.59}$$

We infer from (2.57) and (2.58) that $u_0 \neq u_-$ and $u_0 \neq u_+$.

The next step in the proof is to show that

$$E_0(u_0) < 0. \tag{2.60}$$

By the equality in (2.58), it suffices to produce a path $\hat{\gamma} \in \Gamma$ such that

$$E_0(\hat{\gamma}(t)) < 0 \quad \forall t \in [-1,1]. \tag{2.61}$$

Let $S = V_0 \cap \partial B_1^{L^p(\Omega)}$, where $\partial B_1^{L^p(\Omega)} = \{u \in L^p(\Omega) : \|u\|_p = 1\}$, and $S_C = S \cap C_0^1(\overline{\Omega})$ be endowed with the topologies induced by V_0 and $C_0^1(\overline{\Omega})$, respectively. We set

$$\Gamma_{0,C} = \{\gamma \in C([-1,1], S_C) : \ \gamma(-1) = -\varphi_1, \ \gamma(1) = \varphi_1\}. \tag{2.62}$$

Making use of the first inequality in assumption H(f) (iii), we fix numbers $\mu > \lambda_2$ and $\delta > 0$ such that (2.14) holds, and then let $\rho_0 \in (0, \mu - \lambda_2)$. We recall the following variational expression for λ_2 given by Cuesta et al. [32]:

$$\lambda_2 = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([-1,1])} \|\nabla u\|_p^p, \tag{2.63}$$

where

$$\Gamma_0 = \{\gamma \in C([-1,1], S) : \ \gamma(-1) = -\varphi_1, \ \gamma(1) = \varphi_1\}. \tag{2.64}$$

By (2.63) there exists $\gamma \in \Gamma_0$ such that

$$\max_{t\in[-1,1]} \|\nabla\gamma(t)\|_p^p < \lambda_2 + \frac{\rho_0}{2}. \tag{2.65}$$

Choose some number r with $0 < r \le (\lambda_2+\rho_0)^{1/p} - (\lambda_2+\rho_0/2)^{1/p}$. The density of S_C in S implies that $\Gamma_{0,C}$ is dense in Γ_0; so there is $\gamma_0 \in \Gamma_{0,C}$ satisfying

$$\max_{t\in[-1,1]} \left\| \nabla\gamma(t) - \nabla\gamma_0(t) \right\|_p < r. \tag{2.66}$$

Then the choice of r establishes

$$\max_{t\in[-1,1]} \left\| \nabla\gamma_0(t) \right\|_p^p < \lambda_2 + \rho_0. \tag{2.67}$$

The boundedness of the set $\gamma_0([-1,1])(\overline{\Omega})$ in $\mathbb{R}$ ensures the existence of some $\varepsilon_1 > 0$ such that

$$\varepsilon_1 |u(x)| \le \delta \quad \forall x \in \Omega \ \forall u \in \gamma_0([-1,1]). \tag{2.68}$$

Since $u_+, -u_- \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ (see Theorem 2.4), for every $u \in \gamma_0([-1,1])$ and any bounded neighborhood V_u of u in $C_0^1(\overline{\Omega})$ there exist positive numbers h_u and j_u such that

$$u_+ - \frac{1}{h}v \in \operatorname{int}\Big(C_0^1\big(\overline{\Omega}\big)_+\Big), \qquad -u_- + \frac{1}{j}v \in \operatorname{int}\Big(C_0^1\big(\overline{\Omega}\big)_+\Big), \tag{2.69}$$

whenever $h \ge h_u$, $j \ge j_u$, and $v \in V_u$. This fact and the compactness of $\gamma_0([-1,1])$ in $C_0^1(\overline{\Omega})$ allow to determine a number $\varepsilon_0 > 0$ for which one has

$$u_-(x) \le \varepsilon u(x) \le u_+(x) \quad \forall x \in \Omega, u \in \gamma_0([-1,1]),\ \varepsilon \in (0,\varepsilon_0). \tag{2.70}$$

We now focus on the continuous path $\varepsilon\gamma_0$ in $C_0^1(\overline{\Omega})$ joining $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$ with a fixed constant ε satisfying $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$. By (2.70), (2.67), (2.68), (2.14) with $\mu > \lambda_2$, and taking into account the choice of ρ_0 as well as $\gamma_0([-1,1]) \subset \partial B_1^{L^p(\Omega)}$ we obtain

$$\begin{aligned} E_0(\varepsilon\gamma_0(t)) &= \frac{\varepsilon^p}{p} \ \left\|\nabla\gamma_0(t)\right\|_p^p - \int_\Omega \int_0^{\varepsilon\gamma_0(t)(x)} f(x,\tau_0(x,s),\lambda) ds\ dx \\ &= \frac{\varepsilon^p}{p} \ \left\|\nabla\gamma_0(t)\right\|_p^p - \int_\Omega \int_0^{\varepsilon\gamma_0(t)(x)} f(x,s,\lambda) ds\ dx \\ &\le \frac{\varepsilon^p}{p} \ (\lambda_2 + \rho_0 - \mu) < 0 \quad \forall t \in [-1,1]. \end{aligned} \tag{2.71}$$

At this point we apply the second deformation lemma (see, e.g., [27, page 366]) to the C^1 functional $E_+ : V_0 \to \mathbb{R}$. Towards this let us denote

$$\begin{gathered} c_+ = c_+(\lambda) = E_+(\varepsilon\varphi_1), \qquad m_+ = m_+(\lambda) = E_+(u_+), \\ E_+^{c_+} = \{u \in V_0\ :\ E_+(u) \le c_+\}. \end{gathered} \tag{2.72}$$

It was already shown that u_+ is the unique global minimizer of E_+, and so we have $m_+ < c_+$. Taking into account (2.55), E_+ has no critical values in the interval $(m_+, c_+]$ (for, otherwise, the minimality of the positive solution u_+ of (1.4) would be contradicted). Using also that the functional E_+ satisfies the Palais-Smale condition (because it is coercive), the second deformation lemma can be applied to E_+ yielding a continuous mapping $\eta \in C([0,1] \times E_+^{c_+}, E_+^{c_+})$ such that $\eta(0,u) = u$ and $\eta(1,u) = u_+$ for all $u \in E_+^{c_+}$, as well as $E_+(\eta(t,u)) \leq E_+(u)$ whenever $t \in [0,1]$ and $u \in E_+^{c_+}$. Introducing $\gamma_+ : [0,1] \to V_0$ by

$$\gamma_+(t) := \left(\eta(t, \varepsilon\varphi_1)\right)^+ := \max\{\eta(t, \varepsilon\varphi_1), 0\} \tag{2.73}$$

for all $t \in [0,1]$, it is seen that γ_+ is a continuous path in V_0 joining $\varepsilon\varphi_1$ and u_+. (Note the mapping $w \mapsto w^+$ is continuous from V_0 into itself.) The properties of the deformation η imply

$$E_0(\gamma_+(t)) = E_+(\gamma_+(t)) \leq E_+(\eta(t, \varepsilon\varphi_1)) = E_+(\varepsilon\varphi_1) = E_0(\varepsilon\varphi_1) < 0 \tag{2.74}$$

for all $t \in [0,1]$. Similarly, applying the second deformation lemma to the functional E_-, we construct a continuous path $\gamma_- : [0,1] \to V_0$ joining u_- and $-\varepsilon\varphi_1$ such that

$$E_0(\gamma_-(t)) < 0 \quad \forall t \in [0,1]. \tag{2.75}$$

The union of the curves γ_-, $\varepsilon\gamma_0$, and γ_+ gives rise to a path $\widehat{\gamma} \in \Gamma$. We see from (2.75), (2.71), and (2.74) that (2.61) is satisfied. Hence (2.60) holds, and so $u_0 \neq 0$. Recalling that the critical points of E_0 are in the order interval $\{u \in V_0 \ : \ u_-(x) \leq u(x) \leq u_+(x) \text{ a.e. } x \in \Omega\}$ we derive that u_0 is a nontrivial solution of (1.4) distinct from u_- and u_+, with $u_- \leq u_0 \leq u_+$. By the nonlinear regularity theory we have that $u_0 \in C_0^1(\overline{\Omega})$. The extremality properties of the constant sign solutions u_- and u_+ as described in Theorem 2.4 force u_0 to be sign-changing. This completes the proof. □

2.4. Two Sign-Changing Solutions

The goal of this section is to show that under hypotheses stronger than those in Theorem 2.5, problem (1.4) possesses at least two sign-changing solutions.

The new hypotheses on the nonlinearity $f(x,s,\lambda)$ in problem (1.4) are the following.

H$'(f)$ $f : \overline{\Omega} \times \mathbb{R} \times (0, \overline{\lambda}) \to \mathbb{R}$, with $\overline{\lambda} > 0$, is a function such that $f(x,0,\lambda) = 0$ for a.a. $x \in \Omega$, whenever $\lambda \in (0, \overline{\lambda})$.

(i) For all $\lambda \in (0, \overline{\lambda})$, $f(\cdot,\cdot,\lambda) \in C(\overline{\Omega} \times \mathbb{R})$.

(ii) There are constants $c > 0$, $r \in (p-1, p^*-1)$, and functions $a(\cdot,\lambda) \in L^\infty(\Omega)_+$ ($\lambda \in (0, \overline{\lambda})$) with $\|a(\cdot,\lambda)\|_\infty \to 0$ as $\lambda \downarrow 0$ such that

$$\left|f(x,s,\lambda)\right| \leq a(x,\lambda) + c|s|^r \quad \text{for a.a. } x \in \Omega \ \forall (s,\lambda) \in \mathbb{R} \times \left(0, \overline{\lambda}\right). \tag{2.76}$$

(iii) For all $\lambda \in (0, \overline{\lambda})$ there exist constants $\mu_0 = \mu_0(\lambda) > \lambda_2$, $\nu_0 = \nu_0(\lambda) > \mu_0$ and a set $\Omega_\lambda \subset \Omega$ with $\Omega \setminus \Omega_\lambda$ of Lebesgue measure zero such that

$$\mu_0 < \liminf_{s \to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \leq \limsup_{s \to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \leq \nu_0 \tag{2.77}$$

uniformly with respect to $x \in \Omega_\lambda$.

(iv) There exist constants $b_- < 0 < b_+$ such that for all $\lambda \in (0, \overline{\lambda})$ we have

$$\begin{gathered} f(x, b_-, \lambda) = 0 = f(x, b_+, \lambda) \quad \forall x \in \Omega, \\ f(x,s,\lambda) < 0 \quad \forall x \in \Omega, \text{all } s \in (\mathrm{b}_-, 0), \\ f(x,s,\lambda) > 0 \quad \forall x \in \Omega, \text{all } s \in (0, \mathrm{b}_+). \end{gathered} \tag{2.78}$$

(v) For every $\lambda \in (0, \overline{\lambda})$, there exist $M = M(\lambda) > 0$ and $\mu = \mu(\lambda) > p$ such that

$$0 < \mu F(x,s,\lambda) \leq f(x,s,\lambda)s \quad \forall x \in \Omega, \text{ all } |\mathrm{s}| \geq M. \tag{2.79}$$

We notice that hypotheses H′(f) are stronger than H(f). In particular, for every $\lambda \in (0, \overline{\lambda})$, we added the Ambrosetti-Rabinowitz condition for $f(\cdot,\cdot,\lambda)$ (see hypothesis H′(f)(v)).

We state now the main result of this section, which produces two sign-changing solutions for problem (1.4).

Theorem 2.6. *Assume that hypotheses H′(f) are fulfilled. Then there exists a number $\lambda_0 \in (0, \overline{\lambda})$ with the property that if $\lambda \in (0, \lambda_0)$, then problem* (1.4) *has a minimal (positive) solution $u_+ = u_+(\lambda) \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, a maximal (negative) solution $u_- = u_-(\lambda) \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$, and two nontrivial sign-changing solutions $u_0 = u_0(\lambda)$, $w_0 = w_0(\lambda) \in C_0^1(\overline{\Omega})$ satisfying $\|u_+\|_\infty < b$, $\|u_-\|_\infty < b$, $u_- \leq u_0 \leq u_+$ a.e. in Ω (so $\|u_0\|_\infty < b$) and $\|w_0\|_\infty \geq b$, where $b := \min\{b_+, |b_-|\}$.*

Proof. Since hypotheses H′(f) are stronger than H(f), we can apply Theorem 2.5 with $b = \min\{b_+, |b_-|\}$, which ensures the existence of a number $\lambda_0 \in (0, \overline{\lambda})$ such that for every $\lambda \in (0, \lambda_0)$, problem (1.4) possesses a positive solution $u_+ = u_+(\lambda) \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, a negative solution $u_- = u_-(\lambda) \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$, and a sign-changing solution $u_0 = u_0(\lambda) \in C_0^1(\overline{\Omega})$ with $-b < u_- \leq u_0 \leq u_+ < b$. The proof of Theorem 2.5 shows that u_+ and u_- can be chosen to be the minimal positive solution and the maximal negative solution, respectively.

On the other hand, hypotheses H′(f) enable us to apply Theorem 1.1 of Bartsch et al. [33]. It follows that there exists a sign-changing solution $w_0 = w_0(\lambda) \in C_0^1(\overline{\Omega})$ (by the nonlinear regularity theory) with $\max_{\overline{\Omega}} w_0 \geq b_+$ and $\min_{\overline{\Omega}} w_0 \leq b_-$. Therefore, we have $\|w_0\|_\infty \geq b$, which shows that the sign-changing solutions u_0 and w_0 are different. This completes the proof. □

Remark 2.7. In fact, under hypotheses H′(f), for $\lambda \in (0, \lambda_0)$, problem (1.4) admits at least six nontrivial solutions: two positive solutions, two negative solutions, and two sign-changing solutions, as seen in Theorem 5 in [34].

3. Problem (1.7) for Parameters a and b being Large

The main goal of this section is to provide a detailed multiplicity analysis of the nonsmooth elliptic problem (1.7) in dependence of the two parameters a and b. Conditions in terms of the Fučik spectrum are formulated that ensure the existence of sign-changing solutions. As for the precise formulation of this result we recall the Fučik spectrum, see, for example, [13].

The set Σ_p of those points $(\mu_1, \mu_2) \in \mathbb{R}^2$ for which the problem

$$u \in V_0 : -\Delta_p \ u = \mu_1 \ (u^+)^{p-1} - \mu_2 \ (u^-)^{p-1} \quad \text{in } V_0^* \tag{3.1}$$

has a nontrivial solution is called the Fučik spectrum of the negative p-Laplacian on Ω. Hence, Σ_p clearly contains the two lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ with λ_1 being the first Dirichlet eigenvalue of $-\Delta_p$. In addition, the spectrum $\sigma(-\Delta_p)$ of the negative p-Laplacian has an unbounded sequence of variational eigenvalues (λ_l), $l \in \mathbb{N}$, satisfying a standard min-max characterization, and Σ_p contains the corresponding sequence of points (λ_l, λ_l), $l \in \mathbb{N}$. A first nontrivial curve $\mathcal{C}$ in Σ_p through (λ_2, λ_2) asymptotic to $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ at infinity was constructed and variationally characterized by a mountain-pass procedure by Cuesta et al. [32] (see Figure 1), which implies the existence of a continuous path in $\{u \in V_0 : I^{(a,b)}(u) < 0, \ \|u\|_p = 1\}$ joining $-\varphi_1$ and φ_1 provided (a, b) is above the curve $\mathcal{C}$. Here the functional $I^{(a,b)}$ on V_0 is given by

$$I^{(a,b)}(u) = \int_\Omega \left(|\nabla u|^p - a(u^+)^p - b(u^-)^p\right) dx. \tag{3.2}$$

The hypothesis on the parameters a and b that will finally ensure the existence of sign-changing solutions is as follows.

(H) Let $(a, b) \in \mathbb{R}_+^2$ be above the curve $\mathcal{C}$ of the Fučik spectrum constructed in [32]; see Figure 1.

3.1. Hypotheses, Definitions, and Preliminaries

We impose the following hypotheses on the nonlinearity $g : \Omega \times \mathbb{R} \to \mathbb{R}$ whose primitive is G of problem (1.7)

(g1) $(x, s) \mapsto g(x, s)$ is measurable in each variable separately.

(g2) There exists $c > 0$, and $q \in [p, p^*)$ such that

$$\left|g(x, s)\right| \le c\left(1 + |s|^{q-1}\right) \tag{3.3}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$, where p^* denotes the critical Sobolev exponent which is $p^* = Np/(N - p)$ if $p < N$, and $p^* = +\infty$ if $p \ge N$.

(g3) One has

$$\lim_{s \to 0} \frac{g(x, s)}{|s|^{p-2} s} = 0 \quad \text{uniformly with respect to a.a. } x \in \Omega. \tag{3.4}$$

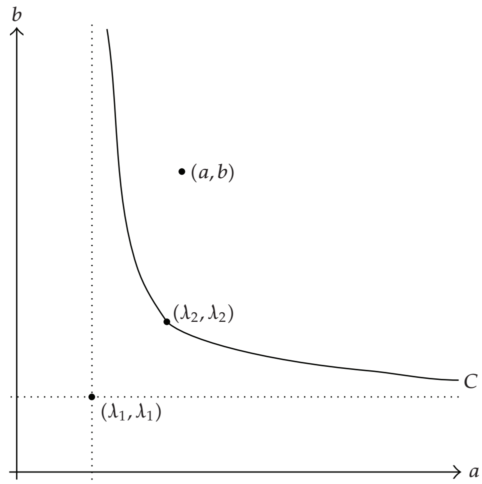

Figure 1: Fučik Spectrum.

(g4) One has

$$\lim_{|s|\to\infty}\frac{g(x,s)}{|s|^{p-2}s}=+\infty \quad \text{uniformly with respect to a.a. } x\in\Omega. \tag{3.5}$$

In view of assumptions (g1) and (g2) the function $s\mapsto G(x,s)$ is locally Lipschitz and the functional $\mathcal{G}: L^q(\Omega)\to\mathbb{R}$ defined by

$$\mathcal{G}(u):=\int_\Omega G(x,u(x))dx, \quad u\in L^q(\Omega) \tag{3.6}$$

is well defined and locally Lipschitz continuous as well. The generalized gradients $\partial G(x,\cdot)$ and $\partial\mathcal{G}$ can be characterized as follows: Define for every $(x,t)\in\Omega\times\mathbb{R}$,

$$g_1(x,t):=\lim_{\delta\to 0^+}\operatorname{ess}\inf_{|\tau-t|<\delta} g(x,\tau), \quad g_2(x,t):=\lim_{\delta\to 0^+}\operatorname{ess}\sup_{|\tau-t|<\delta} g(x,\tau). \tag{3.7}$$

Proposition 1.7 in [35] ensures that

$$\partial G(x,\xi)=\left[g_1(x,\xi),g_2(x,\xi)\right], \tag{3.8}$$

while Theorem 4.5.19 of [36] implies

$$\partial\mathcal{G}(u)\subseteq\left\{w\in L^{q'}(\Omega):\ g_1(x,u(x))\le w(x)\le g_2(x,u(x)) \text{ a.e. in } \Omega\right\} \tag{3.9}$$

with $q\prime:=q/(q-1)$. The next result is an immediate consequence of [37, Proposition 2.1.5].

Lemma 3.1. *Suppose $u_n \to u$ in V_0, $w_n \rightharpoonup w$ in $L^{q'}(\Omega)$, and $w_n \in \partial\mathcal{G}(u_n)$ for all $n \in \mathbb{N}$. Then $w \in \partial\mathcal{G}(u)$.*

Definition 3.2. A function $u \in V_0$ is called a solution of (1.7) if there is an $\eta \in L^{q'}(\Omega)$ such that

$$\eta(x) \in \partial G(x, u(x)) \quad \text{for a.a. } x \in \Omega,$$
$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\, dx + \int_\Omega \left(\eta - a(u^+)^{p-1} + b(u^-)^{p-1}\right)\varphi\, dx = 0, \quad \forall\varphi \in V_0. \tag{3.10}$$

Remark 3.3. Due to assumption (g3) we have $g_1(x,0) \le 0 \le g_2(x,0)$ for almost all $x \in \Omega$. Hence, in view of (3.8), problem (1.7) always possesses the trivial solution.

Definition 3.4. A function $\underline{u} \in V := W^{1,p}(\Omega)$ is called a subsolution of (1.7) if $\underline{u}|_{\partial\Omega} \le 0$, and if there is an $\underline{\eta} \in L^{q'}(\Omega)$ such that

$$\underline{\eta}(x) \in \partial G(x, \underline{u}(x)) \quad \text{for a.a. } x \in \Omega,$$
$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla\varphi\, dx + \int_\Omega \left(\underline{\eta} - a(\underline{u}^+)^{p-1} + b(\underline{u}^-)^{p-1}\right)\varphi\, dx \le 0, \quad \forall\varphi \in V_0 \cap L^p(\Omega)_+. \tag{3.11}$$

Similarly, we define a supersolution as follows.

Definition 3.5. A function $\overline{u} \in V$ is called a supersolution of (1.7) if $\overline{u}|_{\partial\Omega} \ge 0$, and if there is an $\overline{\eta} \in L^{q'}(\Omega)$ such that

$$\overline{\eta}(x) \in \partial G(x, \overline{u}(x)) \quad \text{for a.a. } x \in \Omega,$$
$$\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \nabla\varphi\, dx + \int_\Omega \left(\overline{\eta} - a(\overline{u}^+)^{p-1} + b(\overline{u}^-)^{p-1}\right)\varphi\, dx \ge 0, \quad \forall\varphi \in V_0 \cap L^p(\Omega)_+. \tag{3.12}$$

Lemma 3.6. *Let e be the uniquely defined solution of (2.6). If $a > \lambda_1$, then there exists a constant $\alpha_a > 0$ such that for any $b \in \mathbb{R}$ the function $\alpha_a e$ is a positive supersolution of problem (1.7).*

Proof. Let $a > \lambda_1$. By (g4) there is $s_a > 0$ such that

$$\frac{g(x,s)}{s^{p-1}} > a \quad \text{for a.a. } x \in \Omega\ \forall s > \mathrm{s_a}, \tag{3.13}$$

and by (g2) we get

$$\left|g(x,s) - as^{p-1}\right| \le \left|g(x,s)\right| + as^{p-1} \le c_a, \quad \text{for a.a. } x \in \Omega\ \forall s \in [0, \mathrm{s_a}], \tag{3.14}$$

which implies

$$g(x,s) \ge as^{p-1} - c_a \quad \text{for a.a. } x \in \Omega\ \forall s \ge 0, \tag{3.15}$$

and thus in view of the definition of g_2 we obtain

$$g_2(x,s) \geq as^{p-1} - c_a \quad \text{for a.a. } x \in \Omega \ \forall s \geq 0, \tag{3.16}$$

Let $\overline{u} = \alpha_a e$, where α_a is a positive constant to be specified. Then we get

$$-\Delta_p \overline{u} - a\left(\overline{u}^+\right)^{p-1} + b\left(\overline{u}^-\right)^{p-1} + g_2(x,\overline{u}) = \alpha_a^{p-1} - a\alpha_a^{p-1}e^{p-1} + g_2(x,\alpha_a e) \geq \alpha_a^{p-1} - c_a, \tag{3.17}$$

which shows that for $\alpha_a := c_a^{1/(p-1)}$ the function $\alpha_a e$ is in fact a supersolution of (1.7) with $\overline{\eta}(x) = g_2(x,\overline{u}(x))$. □

In a similar way the following lemma on the existence of a negative subsolution can be proved.

Lemma 3.7. *Let e be the uniquely defined solution of* (2.6)*. If $b > \lambda_1$, then there exists a constant $\beta_b > 0$ such that for any $a \in \mathbb{R}$ the function $-\beta_b e$ is a negative subsolution of problem* (1.7)*.*

In the next lemma we demonstrate that small constant multiples of φ_1 may be sub- and supersolutions of (1.7). More precisely we have the following result.

Lemma 3.8. *Let φ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of $(-\Delta_p, V_0)$. If $a > \lambda_1$, then for $\varepsilon > 0$ sufficiently small and any $b \in \mathbb{R}$ the function $\varepsilon\varphi_1$ is a positive subsolution of problem* (1.7)*. If $b > \lambda_1$, then for $\varepsilon > 0$ sufficiently small and any $a \in \mathbb{R}$ the function $-\varepsilon\varphi_1$ is a negative supersolution of problem* (1.7)*.*

Proof. By (g3) there is a constant $\delta_a > 0$ such that

$$\frac{\left|g(x,s)\right|}{|s|^{p-1}} < a - \lambda_1 \quad \text{for a.a. } x \in \Omega \ \forall 0 < |s| \leq \delta_a, \tag{3.18}$$

which implies

$$\left|g(x,s)\right| \leq (a-\lambda_1)s^{p-1} \quad \text{for a.a. } x \in \Omega,\ \forall s : 0 \leq s \leq \delta_a. \tag{3.19}$$

Define $\underline{u} = \varepsilon\varphi_1$ with $\varepsilon > 0$. Applying (3.19) and the definition of g_1 we get

$$\begin{aligned} -\Delta_p \underline{u} - a\left(\underline{u}^+\right)^{p-1} + b\left(\underline{u}^-\right)^{p-1} + g_1(x,\underline{u}) &= \lambda_1(\varepsilon\varphi_1)^{p-1} - a(\varepsilon\varphi_1)^{p-1} + g_1(x,\varepsilon\varphi_1) \\ &\leq (\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} + (a-\lambda_1)(\varepsilon\varphi_1)^{p-1} = 0 \end{aligned} \tag{3.20}$$

provided $0 \leq \varepsilon\varphi_1 \leq \delta_a$. The latter can be satisfied by choosing ε sufficiently small such that $\varepsilon \in (0, \delta_a/\|\varphi_1\|_\infty)$, where $\|\varphi_1\|_\infty$ stands for the supremum-norm of φ_1. This proves that $\varepsilon\varphi_1$ is a subsolution if $\varepsilon \in (0, \delta_a/\|\varphi_1\|_\infty)$. In a similar way one can show that for ε sufficiently small the function $-\varepsilon\varphi_1$ is a negative supersolution. □

Applying a recently obtained comparison result that holds for even more general elliptic inclusions (see [38, Theorem 4.1, Corollary 4.1] we immediately obtain the following theorem.

Theorem 3.9. *Let hypotheses (g1)-(g2) be satisfied and assume the existence of a subsolution $\underline{u}$ and supersolution $\overline{u}$ of* (1.7) *such that $\underline{u} \le \overline{u}$. Then there exist extremal solutions of* (1.7) *within* $[\underline{u}, \overline{u}]$.

3.2. Extremal Constant-Sign Solutions and Their Variational Characterization

Combining the results of Lemmas 3.6, 3.7, and 3.8 and Theorem 3.9 we immediately deduce the existence of nontrivial positive solutions of problem (1.7) provided the parameter a satisfies $a > \lambda_1$ that and the existence of negative solutions of problem (1.7) provided that the parameter b satisfies $b > \lambda_1$. Our main goal of this section is to show that problem (1.7) has a smallest positive solution $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and a greatest negative solution $u_- \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$. More precisely the following result will be shown.

Theorem 3.10. *Let hypotheses (g1)–(g4) be fulfilled. For every $a > \lambda_1$ and $b \in \mathbb{R}$ there exists a smallest positive solution $u_+ = u_+(a) \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ of* (1.7) *within the order interval* $[0, \alpha_a e]$ *with the constant $\alpha_a > 0$ as in Lemma 3.6. For every $b > \lambda_1$ and $a \in \mathbb{R}$ there exists a greatest negative solution $u_- = u_-(b) \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ of* (1.7) *within the order interval* $[-\beta_b\ e, 0]$ *with the constant $\beta_b > 0$ as in Lemma 3.7.*

Proof. Let $a > \lambda_1$. Lemmas 3.6 and 3.8 ensure that $\overline{u} = \alpha_a e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is a supersolution of problem (1.7) and $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is a subsolution of problem (1.7) provided that $\varepsilon > 0$ is sufficiently small. We may choose $\varepsilon > 0$ such that, in addition, $\varepsilon\varphi_1 \le \alpha_a e$. Thus by Theorem 3.9 there exists a smallest and a greatest solution of (1.7) within the ordered interval $[\varepsilon\varphi_1, \alpha_a e]$. Let us denote the smallest solution by u_ε. Moreover, the nonlinear regularity theory for the p-Laplacian (cf., e.g., [27, Theorem 1.5.6]) and Vázquez's strong maximum principle [28] ensure that $u_\varepsilon \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Thus for every positive integer n sufficiently large there is a smallest solution $u_n \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ of problem (1.7) within $[(1/n)\varphi_1, \alpha_a e]$. In this way we inductively construct a sequence (u_n) of smallest solutions which is monotone decreasing; that is, we have

$$u_n \downarrow u_+ \quad \text{pointwise} \tag{3.21}$$

with some function $u_+ : \Omega \to \mathbb{R}$ satisfying $0 \le u_+ \le \alpha_a e$.

Claim 1. u_+ is a solution of problem (1.7).

As $u_n \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and u_n are solutions of (1.7) we have

$$\|\nabla u_n\|_p^p = \int_\Omega (a u_n(x)^p - \eta_n(x)) u_n(x) dx, \tag{3.22}$$

where $\eta_n \in L^{q'}(\Omega)$ and $\eta_n(x) \in \partial G(x, u_n(x))$ for almost all $x \in \Omega$. Since $u_n \in [(1/n)\varphi_1, \alpha_a e]$, the last equation together with (g2) implies that the sequence (u_n) is bounded in V_0. Taking into account (3.21) we obtain that $u_+ \in V_0$ and

$$u_n \rightharpoonup u_+ \quad \text{in } V_0, \qquad u_n \longrightarrow u_+ \quad \text{in } L^p(\Omega) \text{ and a.e. in } \Omega. \tag{3.23}$$

The solution u_n of (1.7) satisfies

$$\langle -\Delta_p u_n, \varphi \rangle = \int_\Omega \left(a u_n^{p-1} - \eta_n \right) \varphi \, dx, \quad \forall \varphi \in V_0, \tag{3.24}$$

which yields with $\varphi = u_n - u_+$ in (3.24) the equation

$$\langle -\Delta_p u_n, u_n - u_+ \rangle = \int_\Omega \left(a u_n^{p-1} - \eta_n \right) (u_n - u_+) dx. \tag{3.25}$$

Using the convergence properties (3.23) of (u_n) and (g2) as well as the uniform boundedness of the sequence (u_n), we get by applying Lebesgue's dominated convergence theorem

$$\lim_{n \to \infty} \langle -\Delta_p u_n, u_n - u_+ \rangle = 0, \tag{3.26}$$

which by the S_+-property of $-\Delta_p$ on V_0 implies

$$u_n \longrightarrow u_+ \quad \text{in } V_0 \text{ as } n \longrightarrow \infty. \tag{3.27}$$

Since u_n are uniformly bounded, from (g2) we see that there exists a constant $c > 0$ such that

$$|\eta_n(x)| \le c \quad \text{a.e. in } \Omega, \ \forall n \in \mathbb{N}, \tag{3.28}$$

and thus we get (for some subsequence if necessary) $\eta_n \rightharpoonup \eta_+$ in $L^{q'}(\Omega)$. By the strong convergence (3.27), Lemma 3.1 can be applied to show that $\eta_+(x) \in \partial G(x, u_+(x))$ for almost every $x \in \Omega$. Passing to the limit in (3.24) (for some subsequence if necessary) proves Claim 1.

As u_+ belongs, in particular, to $L^\infty(\Omega)$, Claim 1 and Assumption (g2) implies $\Delta_p u_+ \in L^\infty(\Omega)$. The nonlinear regularity theory (cf., e.g., Theorem 1.5.6 in [27]) ensures that $u_+ \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$, so $u_+ \in C_0^1(\overline{\Omega})$. In view of (g2) (g3) a constant $\tilde{c}_a > 0$ can be found such that

$$|g(x,s)| \le \tilde{c}_a s^{p-1} \quad \text{for a.a. } x \in \Omega \ \forall 0 \le \mathrm{s} \le \alpha_{\mathrm{a}} \|\mathrm{e}\|_\infty, \tag{3.29}$$

which yields in conjunction with Claim 1 that

$$\Delta_p u_+ \le \tilde{c}_a u_+^{p-1}. \tag{3.30}$$

We now apply Vázquez's strong maximum principle [28] where in its statement the function β is chosen as $\beta(s) = \tilde{c}_a s^{p-1}$ for all $s > 0$, which is possible because $\int_{0^+} (1/(s\beta(s))^{1/p}) ds = +\infty$.

This result guarantees that if $u_+ \neq 0$, then $u_+ > 0$ in Ω and $\partial u_+/\partial n < 0$ on $\partial\Omega$ which means that $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Claim 2. $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Suppose that Claim 2 does not hold. Then by Vázquez's strong maximum principle we must have $u_+ = 0$, and thus the sequence (u_n) satisfies

$$u_n(x) \downarrow 0 \quad \forall x \in \Omega. \tag{3.31}$$

Setting

$$\widetilde{u}_n = \frac{u_n}{\|\nabla u_n\|_p} \quad \forall n, \tag{3.32}$$

we may suppose that along a relabelled subsequence one has

$$\widetilde{u}_n \rightharpoonup \widetilde{u} \quad \text{in } V_0, \qquad \widetilde{u}_n \longrightarrow \widetilde{u} \quad \text{in } L^p(\Omega) \text{ and a.e in } \Omega \tag{3.33}$$

with some $\widetilde{u} \in V_0$, and there is a function $w \in L^p(\Omega)_+$ such that

$$|\widetilde{u}_n(x)| \leq w(x) \quad \text{for almost all } x \in \Omega. \tag{3.34}$$

Since u_n are positive solutions of (1.7), we get for $\widetilde{u}_n$ the following variational equation:

$$\langle -\Delta_p \widetilde{u}_n, \varphi \rangle = a\int_\Omega \widetilde{u}_n^{p-1}\varphi\, dx - \int_\Omega \frac{\eta_n}{u_n^{p-1}}\widetilde{u}_n^{p-1}\varphi\, dx, \quad \forall \varphi \in V_0. \tag{3.35}$$

With the special test function $\varphi = \widetilde{u}_n - \widetilde{u}$ in (3.35) we obtain

$$\langle -\Delta_p \widetilde{u}_n, \widetilde{u}_n - \widetilde{u} \rangle = a\int_\Omega \widetilde{u}_n^{p-1}(\widetilde{u}_n - \widetilde{u})dx - \int_\Omega \frac{\eta_n}{u_n^{p-1}}\widetilde{u}_n^{p-1}(\widetilde{u}_n - \widetilde{u})dx. \tag{3.36}$$

From (3.29) and (3.34) we get the estimate

$$\frac{|\eta_n(x)|}{u_n(x)^{p-1}}\widetilde{u}_n^{p-1}(x)|\widetilde{u}_n(x) - \widetilde{u}(x)| \leq \widetilde{c}_a w(x)^{p-1}(w(x) + |\widetilde{u}(x)|) \quad \text{for a.a. } x \in \Omega. \tag{3.37}$$

As the right-hand side of the last inequality is in $L^1(\Omega)$, we may apply Lebesgue's dominated convergence theorem, which in conjunction with (3.33) yields

$$\lim_{n\to\infty}\int_\Omega \frac{\eta_n}{u_n^{p-1}}\tilde{u}_n^{p-1}(\tilde{u}_n-\tilde{u})dx = 0. \tag{3.38}$$

From (3.33) and (3.36) we conclude

$$\lim_{n\to\infty}\langle -\Delta_p\tilde{u}_n, \tilde{u}_n-\tilde{u}\rangle = 0, \tag{3.39}$$

which in view of the S_+-property of $-\Delta_p$ on V_0 results in

$$\tilde{u}_n \longrightarrow \tilde{u} \quad \text{in } V_0 \text{ as } n\longrightarrow\infty, \tag{3.40}$$

and therefore, in particular, $\|\nabla\tilde{u}\|_p = 1$. Taking into account (g3), (3.31), and (3.40), we may pass to the limit in (3.35) which results in

$$\langle -\Delta_p\tilde{u}, \varphi\rangle = a\int_\Omega \tilde{u}^{p-1}\varphi\, dx, \quad \forall\varphi\in V_0. \tag{3.41}$$

As $\tilde{u}\neq 0$, relation (3.41) expresses the fact that $\tilde{u} \geq 0$ is an eigenfunction of $(-\Delta_p, V_0)$ corresponding to the eigenvalue a. As $a > \lambda_1$, this is impossible according to Anane [30], because $\tilde{u}$ must change sign. This contradiction proves that Claim 2 holds true. Note that unlike in the proof of Theorem 2.4, here the contradiction is achieved by the sign-changing property of eigenfunctions belonging to eigenvalues bigger than λ_1.

Claim 3. $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is the smallest positive solution of (1.7) in $[0, \alpha_a e]$.

We already know that $u_+ \in [0, \alpha_a e]$. Assume that $u \in V_0$ is any positive solution of (1.7) belonging to $[0, \alpha_a e]$. Since $u \in L^\infty(\Omega)$, then by (1.7) and (g3) we deduce $\Delta_p u \in L^\infty(\Omega)$. Using [27, Theorem 1.56] we derive $u \in C_0^1(\overline{\Omega})$, and applying Vázquez's strong maximum principle [28] we infer $u \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, which yields $u \in [(1/n)\varphi_1, \alpha_a e]$ for n sufficiently large. This in conjunction with the fact that u_n is the least solution of (1.7) in $[(1/n)\varphi_1, \alpha_a e]$ ensures $u_n \leq u$ if n is large enough. In view of (3.21), we obtain $u_+ \leq u$, which proves Claim 3.

The proof of the existence of the greatest negative solution $u_- = u_-(b) \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ of (1.7) within the ordered interval $[-\beta_b e, 0]$ can be done in a similar way. This completes the proof of the theorem. □

Under hypotheses (g1)–(g4), Theorem 3.10 ensures the existence of extremal positive and negative solutions of (1.7) for all $a > \lambda_1$ and $b > \lambda_1$ denoted by $u_+ = u_+(a) \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $u_- = u_-(b) \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$, respectively. In what follows we are going to characterize these extremal solutions as global (local) minimizers of certain locally Lipschitz functionals that are generated by truncation procedures. To this end let us introduce truncation functions related to the extremal solutions u_+ and u_- as follows:

$$\tau_+(x,s) = \begin{cases} 0 & \text{if } s < 0, \\ s & \text{if } 0 \le s \le u_+(x), \\ u_+(x) & \text{if } s > u_+(x), \end{cases}$$
$$\tau_-(x,s) = \begin{cases} u_-(x) & \text{if } s < u_-(x), \\ s & \text{if } u_-(x) \le s \le 0, \\ 0 & \text{if } s > 0, \end{cases} \tag{3.42}$$
$$\tau_0(x,s) = \begin{cases} u_-(x) & \text{if } s < u_-(x), \\ s & \text{if } u_-(x) \le s \le u_+(x), \\ u_+(x) & \text{if } s > u_+(x). \end{cases}$$

The truncations $\tau_+, \tau_-, \tau_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous, uniformly bounded, and Lipschitzian with respect to s. The extremal positive and negative solutions u_+ and u_- of (1.7), respectively, ensured by Theorem 3.10 satisfy

$$-\Delta_p u_+ = a(u_+)^{p-1} - \eta_+, \quad -\Delta_p u_- = -b(u_-)^{p-1} - \eta_- \quad \text{in } V_0^*, \tag{3.43}$$

where $\eta_+, \eta_- \in L^{q'}(\Omega)$ and

$$\eta_+(x) \in \partial G(x, u_+(x)), \quad \eta_-(x) \in \partial G(x, u_-(x)) \tag{3.44}$$

for a.a. $x \in \Omega$. By means of η_+, η_- we introduce the following truncations of the nonlinearity $g : \Omega \times \mathbb{R} \to \mathbb{R}$:

$$g_+(x,s) := \begin{cases} 0 & \text{if } s < 0, \\ g(x,s) & \text{for } 0 \le s \le u_+(x), \\ \eta_+(x) & \text{when } s > u_+(x), \end{cases}$$
$$g_-(x,s) := \begin{cases} \eta_-(x) & \text{if } s < u_-(x), \\ g(x,s) & \text{for } u_-(x) \le s \le 0, \\ 0 & \text{when } s > 0, \end{cases} \tag{3.45}$$
$$g_0(x,s) := \begin{cases} \eta_-(x) & \text{if } s < u_-(x), \\ g(x,s) & \text{for } u_-(x) \le s \le u_+(x), \\ \eta_+(x) & \text{when } s > u_+(x) \end{cases}$$

and define functionals $E_+,\ E_-,\ E_0$ by

$$\begin{aligned}E_+(u) &= \frac{1}{p}\ \|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} \Big(a\tau_+(x,s)^{p-1} - g_+(x,s)\Big)ds\ dx,\\ E_-(u) &= \frac{1}{p}\ \|\nabla u\|_p^p + \int_\Omega \int_0^{u(x)} \Big(b|\tau_-(x,s)|^{p-1} + g_-(x,s)\Big)ds\ dx,\\ E_0(u) &= \frac{1}{p}\ \|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} \Big(a\tau_+(x,s)^{p-1} - b|\tau_-(x,s)|^{p-1} - g_0(x,s)\Big)ds\ dx.\end{aligned} \tag{3.46}$$

Due to (g2) the functionals $E_+, E_-, E_0 : V_0 \to \mathbb{R}$ are locally Lipschitz continuous. Moreover, in view of the truncations involved these functionals are bounded below, coercive, and weakly lower semicontinuous such that their global minimizers exist. The following lemma provides a characterization of the critical points of these functionals.

Lemma 3.11. *Let u_+ and u_- be the extremal constant-sign solutions of* (1.7)*. Then the following holds.*

(i) *A critical point $v \in V_0$ of E_+ is a (nonnegative) solution of* (1.7) *satisfying $0 \le v \le u_+$.*

(ii) *A critical point $w \in V_0$ of E_- is a (nonpositive) solution of* (1.7) *satisfying $u_- \le w \le 0$.*

(iii) *A critical point $z \in V_0$ of E_0 is a solution of* (1.7) *satisfying $u_- \le z \le u_+$.*

Proof. To prove (i) let v be a critical point of E_+, that is, $0 \in \partial E_+(v)$, which results in

$$v \in V_0 : -\Delta_p v = a\tau_+(x,v)^{p-1} - w \quad \text{in } V_0^* \tag{3.47}$$

for some $w \in L^{q'}(\Omega)$ and such that $w(x) \in \partial G_+(x, v(x))$ almost everywhere in Ω, with

$$G_+(x,\xi) := \int_0^\xi g_+(x,t)dt, \quad (x,\xi) \in \Omega \times \mathbb{R}. \tag{3.48}$$

Let us show that $v \le u_+$ holds. As u_+ is a positive solution of (1.7), it satisfies the first equation in (3.43), and by subtracting that equation from (3.47) and applying the special test function $\varphi = (v-u_+)^+$ we get

$$\begin{aligned}&\int_\Omega \Big(|\nabla v|^{p-2}\nabla v - |\nabla u_+|^{p-2}\nabla u_+\Big)\nabla (v-u_+)^+ dx\\ &\quad= \int_\Omega \Big[a\Big(\tau_+(x,v)^{p-1} - u_+^{p-1}\Big) - (w-\eta_+)\Big](v-u_+)^+ dx.\end{aligned} \tag{3.49}$$

By the definition of the truncations introduced above we have $\tau_+(x,v)^{p-1}(x) - u_+^{p-1}(x) = 0$ and $w(x) - \eta_+(x) = 0$ for a.a. $x \in \{v > u_+\}$, and thus the right-hand side of (3.49) is zero which leads to

$$\int_\Omega \Big(|\nabla v|^{p-2}\nabla v - |\nabla u_+|^{p-2}\nabla u_+\Big)\ \nabla (v-u_+)^+ dx = 0, \tag{3.50}$$

and hence it follows $\nabla(v-u_+)^+ = 0$. Because $(v-u_+)^+ \in V_0$, this implies $(v-u_+)^+ = 0$ which proves $v \le u_+$. To prove $0 \le v$ we test (3.47) with $\varphi = v^- = \max\{-v, 0\} \in V_0$ and get

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla v^- dx = \int_\Omega \Big(a\tau_+(x,v)^{p-1} - w\Big) v^- dx = 0, \tag{3.51}$$

which results in $\|\nabla v^-\|_p^p = 0$, and thus $v^- = 0$, that is, $0 \le v$. Thus the critical point v of E_+ which is a solution of the Dirichlet problem (3.47) satisfies $0 \le v \le u_+$, and therefore $\tau_+(x,v) = v$. Because $\partial G_+(x, v(x)) \subset \partial G(x, v(x))$, it follows $w \in \partial G(x, v(x))$, and therefore v must be a solution of (1.7). This proves (i). In just the same way one can prove also (ii) and (iii) which is omitted. □

The following lemma provides a variational characterization of the extremal constant-sign solutions u_+ and u_-.

Lemma 3.12. *Let $a > \lambda_1$ and $b > \lambda_1$. Then the extremal positive solution u_+ of* (1.7) *is the unique global minimizer of the functional E_+, and the extremal negative solution u_- of* (1.7) *is the unique global minimizer of the functional E_-. Both u_+ and u_- are local minimizers of E_0.*

Proof. The functional $E_+ : V_0 \to \mathbb{R}$ is bounded below, coercive, and weakly lower semicontinuous. Thus there exists a global minimizer $v_+ \in V_0$ of E_+, that is,

$$E_+(v_+) = \inf_{u \in V_0} E_+(u), \tag{3.52}$$

As v_+ is a critical point of E_+, so by Lemma 3.11 it is a nonnegative solution of (1.7) satisfying $0 \le v_+ \le u_+$. Since $a - \lambda_1 > 0$, there is a $\nu_a > 0$ such that $a - \lambda_1 - \nu_a > 0$. By (g3) we infer the existence of a $\tilde{\delta}_a > 0$ such that

$$\big|g(x,s)\big| \le (a - \lambda_1 - \nu_a)\ s^{p-1}, \quad \forall s : 0 < s \le \tilde{\delta}_a, \tag{3.53}$$

and thus for $\varepsilon > 0$ sufficiently small such that

$$\varepsilon\varphi_1 \le u_+, \quad \varepsilon\|\varphi_1\|_{L^\infty(\Omega)} \le \tilde{\delta}_a, \tag{3.54}$$

we obtain (note $\|\varphi_1\|_p = 1$)

$$\begin{aligned} E_+(v_+) \le E_+(\varepsilon\varphi_1) &= \frac{\lambda_1}{p}\varepsilon^p + \int_\Omega \int_0^{\varepsilon\varphi_1(x)} \Big(-as^{p-1} + g(x,s)\Big) ds\ dx \\ &\le \frac{\lambda_1}{p}\varepsilon^p + \int_\Omega \int_0^{\varepsilon\varphi_1(x)} (-\lambda_1 - \nu_a)\ s^{p-1} ds\ dx < 0. \end{aligned} \tag{3.55}$$

Hence it follows that $E_+(v_+) < 0$, and thus $v_+ \ne 0$. Applying nonlinear regularity theory for the p-Laplacian (cf., e.g., [27, Theorem 1.5.6]) and Vázquez's strong maximum principle, we see that $v_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. As u_+ is the smallest positive solution of (1.7) in $[0, \alpha_a e]$ and

$0 \le v_+ \le u_+$, it follows $v_+ = u_+$, which shows that the global minimizer v_+ must be unique and equal to u_+. By similar arguments one can show that the global minimizer v_- of E_- must be unique and $v_- = u_-$. It remains to prove that u_+ and u_- are local minimizers of E_0. Let us show this last assertion for u_+ only. By definition we have

$$E_+(u) = E_0(u) \quad \forall u \in V_0 \text{ with } u \ge 0. \tag{3.56}$$

Since u_+ is a global minimizer of E_+ and $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, it follows that u_+ is a local minimizer of E_0 with respect to the C^1 topology. Due to a result by Motreanu and Papageorgiou in [39, Proposition 4], we conclude that u_+ is also a local minimizer of E_0 with respect to the V_0 topology. This completes the proof of the lemma. □

Lemma 3.13. *The functional $E_0 : V_0 \to \mathbb{R}$ has a global minimizer v_0 which is a nontrivial solution of* (1.7) *satisfying $u_- \le v_0 \le u_+$.*

Proof. One easily verifies that $E_0 : V_0 \to \mathbb{R}$ is coercive and weakly lower semicontinuous, and thus a global minimizer v_0 exists which is a critical point of E_0. Apply Lemma 3.11(iii) and note that, for example, $E_0(u_+) = E_+(u_+) < 0$, which shows that $v_0 \ne 0$. □

3.3. Sign-Changing Solutions

Theorem 3.10 ensures the existence of a smallest positive solution $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ in $[0, \alpha_a e]$ and a greatest negative solution $u_- \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ of (1.7) in $[-\beta_b e, 0]$. The idea to show the existence of sign-changing solutions is to prove the existence of nontrivial solutions u_0 of (1.7) satisfying $u_- \le u_0 \le u_+$ with $u_0 \ne u_-$ and $u_0 \ne u_+$, which then must be sign-changing, because u_+ and u_- are the extremal constant-sign solutions.

Theorem 3.14. *Let hypotheses (g1)–(g4) and (H) be satisfied. Then problem* (1.7) *has a smallest positive solution $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ in $[0, \alpha_a e]$, a greatest negative solution $u_- \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ in $[-\beta_b e, 0]$, and a nontrivial sign-changing solution $u_0 \in C_0^1(\overline{\Omega})$ with $u_- \le u_0 \le u_+$.*

Proof. Clearly the existence of the extremal positive and negative solution u_+ and u_- follows from Theorem 3.10, because (H), in particular, implies that $a > \lambda_1$ and $b > \lambda_1$. As for the existence of a sign-changing solution we first note that by Lemma 3.13 it follows that the global minimizer v_0 of E_0 is a nontrivial solution of (1.7) satisfying $u_- \le v_0 \le u_+$. Therefore, if $v_0 \ne u_+$ and $v_0 \ne u_-$, then $v_0 = u_0$ must be a sign-changing solution as asserted, because u_- is the greatest negative and u_+ is the smallest positive solution of (1.7). Thus, we still need to prove the existence of sign-changing solutions in case that either $v_0 = u_-$ or $v_0 = u_+$.

Let us consider the case $v_0 = u_+$ only, since the case $v_0 = u_-$ can be treated quite similarly. By Lemma 3.12, u_- is a local minimizer of E_0. Without loss of generality we may even assume that u_- is a strict local minimizer of E_0, because on the contrary we would find (infinitely many) critical points z of E_0 that are sign-changing solutions thanks to $u_- \le z \le u_+$ and the extremality of the solutions u_-, u_+ obtained in Theorem 3.10 which proves the assertion.

Therefore, it remains to prove the existence of sign-changing solutions under the assumptions that the global minimizer v_0 of E_0 is equal to u_+, and u_- is a strict local minimizer of E_0. This implies the existence of $\rho \in (0, \|u_- - u_+\|)$ such that

$$E_0(u_+) \leq E_0(u_-) < \inf\{E_0(u) : u \in \partial B_\rho(u_-)\}, \tag{3.57}$$

where $\partial B_\rho(u_-) = \{u \in V_0 : \|u - u_-\| = \rho\}$. The functional E_0 satisfies the Palais-Smale condition, because it is bounded below, locally Lipschitz continuous, and coercive; see, for example, [40, Corollary 2.4]. Thus in view of (3.57) we may apply the nonsmooth version of Ambrosetti-Rabinowitz's Mountain-Pass Theorem (see, e.g., [41, Theorem 3.4]) which ensures the existence of a critical point $u_0 \in V_0$ satisfying $0 \in \partial E_0(u_0)$ and

$$\inf\{E_0(u) : u \in \partial B_\rho(u_-)\} \leq E_0(u_0) = \inf_{\gamma\in\Gamma} \max_{t\in[-1,1]} E_0(\gamma(t)), \tag{3.58}$$

where

$$\Gamma = \{\gamma \in C([-1,1], V_0) : \gamma(-1) = u_-, \ \gamma(1) = u_+\}. \tag{3.59}$$

It is clear from (3.57) and (3.58) that $u_0 \neq u_-$ and $u_0 \neq u_+$, and thus u_0 is a sign-changing solution provided $u_0 \neq 0$. To prove the latter we claim

$$E_0(u_0) < 0 \tag{3.60}$$

for which it suffices to construct a path $\hat{\gamma} \in \Gamma$ such that

$$E_0(\hat{\gamma}(t)) < 0 \quad \forall t \in [-1,1]. \tag{3.61}$$

The construction of such a path $\hat{\gamma}$ can be done by adopting an approach due to the authors in [3] and applying the Second Deformation Lemma for locally Lipschitz functionals as it can be found in [42, Theorem 2.10]. This completes the proof. □

Remark 3.15. The multiplicity results and the existence of sign-changing solutions obtained in this section generalize very recent results due to the authors obtained in [3, 4, 7]. Moreover, if the function $t \mapsto g(x,t)$ is continuous on $\mathbb{R}$ and $a = b = \lambda$, then $\partial G(x,\xi) = \{g(x,\xi)\}$, and problem (1.7) reduces to

$$u \in V_0 : \ -\Delta_p u = \lambda |u|^{p-2} u - g(x,u) \quad \text{in } V_0^*. \tag{3.62}$$

However, even in this setting the results obtained here are more general than obtained in [6, Theorem 3.9], because we do not assume that $g(x,t)t \geq 0$ for all $t \in \mathbb{R}$.

Remark 3.16. Theorem 3.14 improves also Corollary 3.2 of [8]. In fact, let $p = 2$, let $u \in V_0$ be a solution of (1.7) in case $a = b = \lambda$ and $g(x,t) \equiv g(t)$, $(x,t) \in \Omega \times \mathbb{R}$ with $\eta \in L^{q'}(\Omega)$ satisfying $\eta(x) \in \partial G(u(x))$. By definition of Clarke's gradient we have, for any $\varphi \in V_0$,

$$\eta(x)\varphi(x) \leq G^0(u(x);\varphi(x)) \quad \text{a.e. in } \Omega. \tag{3.63}$$

As u is a solution, the following holds: $u \in V_0$ and $(p = 2)$,

$$\int_\Omega \nabla u \nabla \varphi \, dx = \lambda \int_\Omega u\varphi \, dx - \int_\Omega \eta\varphi \, dx, \tag{3.64}$$

which yields

$$-\int_\Omega \nabla u \nabla \varphi \, dx + \lambda \int_\Omega u\varphi \, dx = \int_\Omega \eta\varphi \, dx \leq \int_\Omega G^0(u;\varphi)dx, \quad \forall \varphi \in V_0, \tag{3.65}$$

That is, u turns out to be a solution of the hemivariational inequality studied in [8]. Since the hypotheses of [8, Corollary 3.2] imply (g1)–(g4), the assertion follows.

Remark 3.17. Multiplicity results for a nonsmooth version of problem (1.4) in form of (1.2) can be established under structure conditions for Clarke's gradient ∂j similar to H(f).

Multiplicity and sign-changing solutions results have been obtained recently in [43] for the following nonlinear Neumann-type boundary value problem: find $u \in V \setminus \{0\}$ and parameters $a, b \in \mathbb{R}$ such that

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u) \quad \text{on } \partial\Omega. \end{aligned} \tag{3.66}$$

For problem (3.66) conditions on the parameters have been given in terms of the "Steklov-Fučik" spectrum to ensure multiplicity results.

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