Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2011, Article ID 268465, 10 pages
doi:10.1155/2011/268465

Research Article

Boundary Value Problems with Integral Gluing Conditions for Fractional-Order Mixed-Type Equation

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Received 18 May 2011; Accepted 11 October 2011

Academic Editor: Shaher M. Momani

Analogs of the Tricomi and the Gellerstedt problems with integral gluing conditions for mixed parabolic-hyperbolic equation with parameter have been considered. The considered mixed-type equation consists of fractional diffusion and telegraph equation. The Tricomi problem is equivalently reduced to the second-kind Volterra integral equation, which is uniquely solvable. The uniqueness of the Gellerstedt problem is proven by energy integrals' method and the existence by reducing it to the ordinary differential equations. The method of Green functions and properties of integral-differential operators have been used.

1. Introduction

Mathematical model of the movement of gas in a channel surrounded by a porous environment was described by parabolic-hyperbolic equation. This was done in the fundamental work of Gel'fand [1]. Modeling of heat transfer processes in composite environment with finite and infinite velocities leads to boundary value problems (BVPs) for parabolic-hyperbolic equations [2]. Omitting the huge amount of works devoted to studying these kinds of equations, we refer the readers to [3, 4].

We would like to note works [5–10], devoted to the studying of BVPs for parabolic-hyperbolic equations, involving fractional derivatives. In turn, applications of Fractional-order differential equations can be found in the monographs [11–15]. We also note some recent papers [16–18], related to the fractional diffusion and diffusion-wave equations.

BVP for parabolic-hyperbolic equations with integral gluing condition for the first time was investigated by Kapustin and Moiseev [19] and was generalized for this kind of equation, but with parameters, in the work [20]. Another motivation of the usage of integral gluing conditions comes from the appearance of them in heat exchange processes [21].

The consideration of equations with parameters was interesting because of the possibility of studying some multidimensional analogues of the main BVP via reducing them by Fourier transformation to the BVP for equations with parameters. On the other hand, consideration of equations with parameters will give possibility to study some spectral properties of BVPs for this kind of equations such as the existence of nontrivial solutions for corresponding homogeneous problem at some values of parameters [22].

2. Analog of the Tricomi Problem

Consider an equation

$$u_{xx} - D_{0y}^{\alpha H(x)+2H(-x)} u = \lambda u \tag{2.1}$$

in the domain $\Omega = \Omega_1 \cup AA_0 \cup \Omega_2$. Here $\Omega_1 = \{(x,y) : \ 0 < x < 1, \ 0 < y < 1\}$, Ω_2 is characteristic triangle with endpoints $A(0,0)$, $A_0(0,1)$, $C(-1/2, 1/2)$, $H(x)$ is Heaviside function,

$$D_{at}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{-\alpha+n-1} f(s) ds \tag{2.2}$$

is the αth Riemann-Liouville fractional-order derivative of a function f given on interval $[a,b]$, where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α, and $\Gamma(\cdot)$ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0. \tag{2.3}$$

For $\lambda > 0$ and $0 < \alpha \le 1$ given, we formulate the following problem called the analog of the Tricomi problem.

Problem AT

To find a solution of (2.1), which belongs to the class of functions

$$W_1 = \left\{u : D_{0y}^{\alpha-1} u \in C\left(\overline{\Omega_1}\right), \ u_{xx}, D_{0y}^{\alpha} u \in C(\Omega_1), \ u_x(0^{\pm}, y) \in H(0;1), u \in C\left(\overline{\Omega_2}\right) \cap C^2(\Omega_2)\right\}, \tag{2.4}$$

satisfying the initial condition

$$\lim_{y\to 0} y^{1-\alpha} u(x,y) = \omega(x), \quad 0 \le x \le 1 \tag{2.5}$$

together with the boundary conditions

$$\begin{gathered} u(-y/2, y/2) = \psi_1(y), \quad 0 \le y \le 1, \\ u(1,y) = \psi_2(y), \quad 0 \le y \le 1, \end{gathered} \tag{2.6}$$

and the gluing conditions

$$
\begin{gathered}
u(0^-,y)=\frac{1}{\Gamma(1-\alpha)}\int_0^y u(0^+,t)(y-t)^{-\alpha}dt,\quad 0<y\le 1,\\
\int_0^y u_x(0^-,t)J_0\left[\sqrt{\lambda}(y-t)\right]dt=\frac{1}{\Gamma(1-\alpha)}\int_0^y u_x(0^+,t)(y-t)^{-\alpha}dt,\quad 0<y<1.
\end{gathered}
\tag{2.7}
$$

Here $\omega(x)$, $\varphi_i(y)$ $(i=1,2)$ are given functions such as $\lim_{y\to 0}y^{1-\alpha}\varphi_1(y)=\omega(0)$.

Solution of the Cauchy problem for (2.1) in Ω_2 defined as

$$
\begin{aligned}
u(x,y)=\frac{1}{2}\Bigg\{&\tau^-(y+x)+\tau^-(y-x)+\int_{y-x}^{y+x}\nu^-(t)J_0\left[\sqrt{\lambda\left[(y-t)^2-x^2\right]}\right]dt\\
&+\lambda x\int_{y-x}^{y+x}\tau^-(t)\frac{J_1\left[\sqrt{\lambda\left[(y-t)^2-x^2\right]}\right]}{\sqrt{\lambda\left[(y-t)^2-x^2\right]}}dt\Bigg\},
\end{aligned}
\tag{2.8}
$$

where $J_k[\cdot]$ is the first-kind Bessel function of the order k, $\tau^-(y)=u(0^-,y)$, $\nu^-(y)=u_x(0^-,y)$.

We calculate $u(-y/2,y/2)$ in order to use condition (2.5):

$$
\begin{aligned}
&u(-y/2,y/2)\\
&=\frac{1}{2}\left\{\tau^-(0)+\tau^-(y)-\int_0^y\nu^-(t)J_0\left[\sqrt{\lambda t(t-y)}\right]dt+\lambda\frac{y}{2}\int_0^y\tau^-(t)\frac{J_1\left[\sqrt{\lambda t(t-y)}\right]}{\sqrt{\lambda t(t-y)}}dt\right\}.
\end{aligned}
\tag{2.9}
$$

Considering the condition (2.5) and the following integral operator [23]

$$
B_{mx}^{n,\sqrt{\lambda}}[f(x)]=f(x)+\int_m^x f(t)\left(\frac{x-m}{t-m}\right)^{1-n}\frac{\partial}{\partial x}J_0\left[\sqrt{\lambda(t-m)(t-x)}\right]dt,\quad m,n=0,1,\tag{2.10}
$$

equality (2.9) can be written as follows

$$
\varphi_1(y)=\frac{1}{2}\left\{\varphi_1(0)+B_{0y}^{0,\sqrt{\lambda}}\left[\tau^-(y)\right]-\int_0^y B_{0t}^{1,\sqrt{\lambda}}\left[\nu^-(t)\right]dt\right\}.\tag{2.11}
$$

Now we use an integral operator

$$
A_{mx}^{n,\sqrt{\lambda}}[f(x)]=f(x)-\int_m^x f(t)\left(\frac{t-m}{x-m}\right)^{n}\frac{\partial}{\partial t}J_0\left[\sqrt{\lambda(x-m)(x-t)}\right]dt,\quad m,n=0,1,\tag{2.12}
$$

which is mutually inverse with the operator (2.10). Applying the operator (2.12) to both sides of (2.11), we obtain

$$A_{0y}^{0,\sqrt{\lambda}}[\psi_1(y)] = \frac{1}{2}\left\{\psi_1(0) + A_{0y}^{0,\sqrt{\lambda}}\left\{B_{0y}^{0,\sqrt{\lambda}}[\tau^-(y)]\right\} - A_{0y}^{0,\sqrt{\lambda}}\left\{\int_0^y B_{0t}^{1,\sqrt{\lambda}}[\nu^-(t)]\,dt\right\}\right\}. \tag{2.13}$$

Considering the following properties of operators (2.10) and (2.12)

$$A_{0y}^{0,\sqrt{\lambda}}\left\{B_{0y}^{0,\sqrt{\lambda}}[f(y)]\right\} = f(y), \qquad A_{0y}^{0,\sqrt{\lambda}}\left\{\int_0^y B_{0t}^{1,\sqrt{\lambda}}[f(t)]\,dt\right\} = \int_0^y f(t)J_0\left[\sqrt{\lambda}(y-t)\right]dt, \tag{2.14}$$

we derive

$$2A_{0y}^{0,\sqrt{\lambda}}[\psi_1(y)] = \psi_1(0) + \tau^-(y) - \int_0^y \nu^-(t)J_0\left[\sqrt{\lambda}(y-t)\right]dt. \tag{2.15}$$

Taking gluing conditions (2.7) into account, we have

$$D_{0y}^{\alpha-1}\nu^+(y) = D_{0y}^{\alpha-1}\tau^+(y) - 2A_{0y}^{0,\sqrt{\lambda}}[\psi_1(y)] + \psi_1(0). \tag{2.16}$$

Applying operator $D_{0y}^{1-\alpha}$ to both sides of (2.16) and considering the following composition rule [11]:

$$D_{at}^{\alpha}D_{at}^{\beta}f(t) = D_{at}^{\alpha+\beta}f(t), \quad \beta \le 0, \tag{2.17}$$

we get

$$\tau^+(y) = \nu^+(y) + \psi_1^*(y), \quad 0 < y < 1, \tag{2.18}$$

where $\psi_1^*(y) = D_{0y}^{1-\alpha}\{2A_{0y}^{0,\sqrt{\lambda}}[\psi_1(y)] - \psi_1(0)\}$.

Let us consider the following auxiliary problem:

$$\begin{gathered} u_{xx} - D_{0y}^{\alpha}u - \lambda u = 0, \\ u_x(0,y) = \nu^+(y), \qquad u(1,y) = \psi_2(y), \qquad \lim_{y\to 0} y^{1-\alpha}u(x,y) = \omega(x). \end{gathered} \tag{2.19}$$

Solution of this problem can be defined as [24]

$$\begin{aligned} u(x,y) = {}& \int_0^1 \omega(\xi)G(x,y,\xi,0)\,d\xi - \int_0^y \nu^+(\eta)G(x,y,0,\eta)\,d\eta \\ & + \int_0^y \psi_2(\eta)G_\xi(x,y,1,\eta)\,d\eta - \lambda\int_0^1\int_0^y u(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta, \end{aligned} \tag{2.20}$$

where

$$G(x,y,\xi,\eta) = \frac{(y-\eta)^{\beta-1}}{2} \sum_{n=-\infty}^{\infty} \left[e_{1,\beta}^{1,\beta}\left(-\frac{|x-\xi+2n|}{(y-\eta)^{\beta}} \right) + e_{1,\beta}^{1,\beta}\left(-\frac{|x+\xi+2n|}{(y-\eta)^{\beta}} \right) \right] \tag{2.21}$$

is the Green function of the problem (2.19),

$$e_{1,\beta}^{1,\beta}(z) = \Phi(-\beta,\beta,z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(-\beta n+\beta)} \tag{2.22}$$

is the function of Wright [25], $\beta = \alpha/2$.

Considering (2.20) as an integral equation regarding the function $u(x,y)$, we write solution via resolvent of the kernel $\lambda G(x,y,\xi,\eta)$:

$$u(x,y) = P(x,y) - \int_0^y \nu^+(\eta) K_1(x,y,\eta)\,d\eta, \tag{2.23}$$

where

$$\begin{aligned} P(x,y) = {} & \int_0^1 \omega(\xi) G(x,y,\xi,0)\,d\xi + \int_0^y \int_0^1 \int_0^1 \omega(\xi) G(s,t,\xi,0) R(x,y,\xi,0)\,d\xi\,ds\,dt \\ & + \int_0^y \varphi_2(\eta) \left[G(x,y,1,\eta) + \int_\eta^y \int_0^1 G(s,t,1,\eta) R(x,y,1,\eta)\,ds\,dt \right] d\eta, \end{aligned} \tag{2.24}$$

$$K_1(x,y,\eta) = G(x,y,0,\eta) + \int_\eta^y \int_0^1 G(s,t,0,\eta) R(x,y,0,\eta)\,ds\,dt,$$

$R(x,y,\xi,\eta)$ is a resolvent of the kernel $\lambda G(x,y,\xi,\eta)$.

From (2.23), tending x to 0^+, we obtain

$$u(0^+,y) = \tau^+(y) = P(0^+,y) - \int_0^y \nu^-(\eta) K_1(0^+,y,\eta)\,d\eta. \tag{2.25}$$

Considering functional relation (2.18), from (2.25) we get

$$\nu^+(y) + \int_0^y \nu^+(\eta) K_1(y,\eta)\,d\eta = \varphi_1^*(y) - P(0,y). \tag{2.26}$$

Equality (2.26) is the second-kind Volterra-type integral equation regarding the function $\nu^+(y)$. Since kernel $K_1(y,\eta)$ has weak singularity and functions on the right-hand side are continuous, we can conclude that (2.26) is uniquely solvable [26], and solution can be represented as

$$\nu^+(y) = \Psi(y) + \int_0^y \Psi(\eta) K_2(y,\eta)\,d\eta, \tag{2.27}$$

where $\Psi(y) = \varphi_1^*(y) - P(0,y)$, $K_2(y,\eta)$ is the resolvent of the kernel $K_1(y,\eta)$.

Once we have obtained $\nu^+(y)$, considering (2.18) or (2.25) we find function $\tau^+(y)$. Then using gluing conditions (2.7) we find functions $\tau^-(\mathrm{y})$, $\nu^-(\mathrm{y})$. Finally, we can define solution of the considered problem by the formula (2.23) in the domain Ω_1, by formula (2.8) in the domain Ω_2.

Hence, we prove the following theorem.

Theorem 2.1. *If*

$$\omega(x) \in C^2[0,1], \qquad \varphi_i(y) \in C^1[0,1] \cap C^2(0,1) \quad (i = 1,2), \tag{2.28}$$

then there exists unique solution of the Problem AT and is defined by formulas (2.23) *and* (2.8) *in the domains* Ω_1, Ω_2, *respectively.*

3. Analog of the Gellerstedt Problem

We would like to note some related works. Regarding the consideration of Gellerstedt problem for parabolic-hyperbolic equations with constant coefficients we refer the readers to [3] and for loaded parabolic-hyperbolic equations work by Khubiev [27], and also for Lavrent'ev-Bitsadze equation [28].

Consider an equation

$$0 = \begin{cases} u_{xx} - D_{0y}^{\alpha} u - \lambda u, & \Phi_0 \\ u_{xx} - u_{yy} + \lambda u, & \Phi_i, \ (i = 1,2) \end{cases} \tag{3.1}$$

in the domain $\Phi = (\bigcup_{k=0}^{2} \Phi_k) \cup I_0$, where Φ_0 is a domain, bounded by segments AA_0, BB_0, A_0B_0 of straight lines $x = 0$, $x = 1$, $y = 1$, respectively; Φ_1 is a domain, bounded by the segment AE of the axe x and by characteristics of (3.1) $AC_1 : \ x + y = 0$, $EC_1 : \ x - y = r$; Φ_2 is a domain, bounded by the segment EB of the axe x and by characteristics of (3.1) $EC_2 : \ x - y = r$, $BC_2 : \ x - y = 1$; I_0 is an interval $0 < x < 1$, I_1 is an interval $0 < x < r$, and I_2 is an interval $r < x < 1$.

Problem AG

To find a solution of (3.1) from the class of functions

$$W_2 = \left\{ u : D_{0y}^{\alpha-1} u \in C\left(\overline{\Phi_0}\right),\ u_{xx}, D_{0y}^{\alpha} u \in C(\Phi_0),\ u_y(x, 0^{\pm}) \in H(I_0),\ u \in C\left(\overline{\Phi_i}\right) \cap C^2(\Phi_i) \ (i = 1,2) \right\}, \tag{3.2}$$

satisfying boundary conditions

$$u(0,y) = \varphi_1(y), \quad u(1,y) = \varphi_2(y), \quad 0 \le y \le 1, \tag{3.3}$$

$$u\,|_{AC_1} = u\left(\frac{x}{2}, -\frac{x}{2}\right) = \varphi_3(x), \quad 0 \le x \le r, \tag{3.4}$$

$$u\,|_{EC_2} = u\left(\frac{(x+r)}{2}, \frac{(r-x)}{2}\right) = \varphi_4(x), \quad r \le x \le 1, \tag{3.5}$$

together with gluing conditions

$$\lim_{y\to 0^+} y^{1-\alpha} u(x,y) = \lim_{y\to 0^+} u(x,y), \quad x \in \overline{I_0}, \tag{3.6}$$

$$\lim_{y\to 0^+}\left[y^{1-\alpha}\left(y^{1-\alpha}u(x,y)\right)_y\right] = \int_0^x \lim_{y\to 0^+} u_y(t,y) J_0\left[\sqrt{\alpha}(x-t)\right]dt, \quad x \in I_0 \setminus \{\mathrm{r}\}. \tag{3.7}$$

Here $\varphi_j(\cdot)$ $(j=\overline{1,4})$ are given functions such as $\lim_{y\to 0^+} y^{1-\alpha}\varphi_1(y) = \varphi_3(0)$, $\lim_{\mathrm{y}\to 0^+} y^{1-\alpha}u(r,y) = \varphi_4(r)$.

Theorem 3.1. *If the following conditions*

$$\lambda \ge 0, \quad \varphi_i(y) \in C^1[0,1] \cap C^2(0,1), \quad \varphi_j(x) \in C^1\left(\overline{I_i}\right) \cap C^2(I_i) \quad (i=1,2;\ j=3,4) \tag{3.8}$$

are fulfilled, then the Problem AG has a unique solution.

Proof. Introduce the following designations:

$$\begin{gathered}
\lim_{y\to 0^+} y^{1-\alpha}u(x,y) = \tau^+(x), \quad \lim_{y\to 0^-} u(x,y) = \tau^-(x), \quad x \in \overline{I_0}, \\
\lim_{y\to 0^+}\left[y^{1-\alpha}\left(y^{1-\alpha}u(x,y)\right)_y\right] = \nu^+(x), \quad \lim_{y\to 0^-} u_y(x,y) = \nu^-(x), \quad x \in I_0.
\end{gathered} \tag{3.9}$$

Solution of the Cauchy problem for (3.1) in the domain Φ_i $(i=1,2)$ in case, when $\lambda \ge 0$ has a form

$$\begin{aligned}
u(x,y) = \frac{1}{2}\Bigg\{ &\tau^-(x+y) + \tau^-(x-y) + \int_{x-y}^{x+y} \nu^-(t) J_0\left[\sqrt{\lambda\left[(x-t)^2 - y^2\right]}\right]dt \\
&+\lambda y \int_{x-y}^{x+y} \tau^-(t) \frac{J_1\left[\sqrt{\lambda\left[(x-t)^2-y^2\right]}\right]}{\sqrt{\lambda\left[(x-t)^2-y^2\right]}}dt \Bigg\}.
\end{aligned} \tag{3.10}$$

Using boundary conditions (3.4), (3.5), and gluing conditions (3.6), (3.7), from (3.10) we obtain

$$\nu^{+}(x) = \tau^{+}(x) + \varphi_3^{*}(x), \quad x \in I_1, \tag{3.11}$$

$$\nu^{+}(x) = \tau^{+}(x) + \varphi_4^{*}(x), \quad x \in I_2, \tag{3.12}$$

where

$$\varphi_3^{*}(x) = \varphi_3(0) - A_{0x}^{0,\sqrt{\lambda}}\left[2\varphi_3(x)\right], \qquad \varphi_4^{*}(x) = \varphi_4(r) - A_{rx}^{0,\sqrt{\lambda}}\left[2\varphi_4(x)\right]. \tag{3.13}$$

According to [10], tending y to +0, from (3.1) we get

$$\nu^{+}(y) = \frac{1}{\Gamma(1+\alpha)}\left[\tau^{+''}(x) - \lambda\tau^{+}(x)\right]. \tag{3.14}$$

In order to prove the uniqueness of the solution for the Problem AG, we need estimate the following integral:

$$\mathbb{I} = \int_0^1 \tau^{+}(x)\nu^{+}(x)dx. \tag{3.15}$$

Considering homogeneous case of the condition (3.3) and taking designation (3.9) into account, after some evaluations we derive

$$\mathbb{I} = -\int_0^1 \left\{\left[\tau^{+'}(x)\right]^2 + \lambda\left[\tau^{+}(x)\right]^2\right\}dx. \tag{3.16}$$

If $\lambda \geq 0$, then $\mathbb{I} \leq 0$. On the other hand, if we consider homogeneous cases of (3.11) and (3.12), one can easily be sure that $\mathbb{I} \geq 0$. Hence, we get that $\mathbb{I} \equiv 0$. Based on (3.16) we can conclude that $\tau^{+}(x) = 0$ for all $x \in \overline{I_0}$. Due to the solution of the first boundary problem [24] we can conclude that $u(x,y) \equiv 0$ in $\overline{\Phi_0}$. Further, according to the gluing conditions and the solution of Cauchy problem, we have $u(x,y) \equiv 0$ in $\overline{\Phi}$.

Considering functional relations (3.11)–(3.14) and conditions (3.3)–(3.5), we get the following problems:

$$\begin{gathered}\tau^{+''}(x) - (\lambda + \Gamma(1+\alpha))\tau^{+}(x) = \varphi_3^{*}(x)\Gamma(1+\alpha), \\ \tau^{+}(0) = \varphi_3(0), \qquad \tau^{+}(r) = \varphi_4(r), \quad x \in \overline{I_1},\end{gathered} \tag{3.17}$$

$$\begin{gathered}\tau^{+''}(x) - (\lambda + \Gamma(1+\alpha))\tau^{+}(x) = \varphi_4^{*}(x)\Gamma(1+\alpha), \\ \tau^{+}(r) = \varphi_4(r), \qquad \tau^{+}(1) = \lim_{y\to+0} y^{1-\alpha}\varphi_2(y), \quad x \in \overline{I_1}.\end{gathered} \tag{3.18}$$

The problems (3.17) and (3.18) are model problems and can be solved directly. After the finding function $\tau^+(x)$ for all $x \in \overline{I_0}$, functions $\nu^+(x)$ and $\tau^-(x)$, $\nu^-(x)$ can be defined by formulas (3.14) and (3.6), (3.7), respectively. Finally, solution of the Problem AG can be recovered by formulas (3.10) and (2.23) in the domains Φ_i $(i = 1, 2)$ and Φ_0, respectively, but only with some changes in (2.23), precisely, Green function $G(x, y, \xi, \eta)$ should be replaced by

$$G^*(x, y, \xi, \eta) = \frac{(y-\eta)^{\beta-1}}{2} \sum_{n=-\infty}^{\infty} \left[e_{1,\beta}^{1,\beta}\left(-\frac{|x-\xi+2n|}{(y-\eta)^{\beta}} \right) - e_{1,\beta}^{1,\beta}\left(-\frac{|x+\xi+2n|}{(y-\eta)^{\beta}} \right) \right], \tag{3.19}$$

which is the Green function of the first boundary problem for the (3.1) in Φ_0 [24].

Theorem 3.1 is proved. □

Acknowledgment

This paper is partially supported by the foundation ERASMUSMUNDUS (Project number 155778-EM-1-2009-1-BEERASMUNMUNDUS-ECW-LO9).

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