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Research Article

On Spectrum of the Laplacian in a Circle Perforated along the Boundary: Application to a Friedrichs-Type Inequality

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In this paper, we construct and verify the asymptotic expansion for the spectrum of a boundary-value problem in a unit circle periodically perforated along the boundary. It is assumed that the size of perforation and the distance to the boundary of the circle are of the same smallness. As an application of the obtained results, the asymptotic behavior of the best constant in a Friedrichs-type inequality is investigated.

1. Introduction

We study a two-dimensional eigenvalue problem for the Laplace operator in a unit circle periodically perforated along the boundary. It is assumed that the size of perforation and the distance to the boundary of the circle are of the same smallness. The asymptotic behavior of the spectrum of the considered boundary-value problem is investigated in this paper. We construct and verify the asymptotic expansion for the eigenvalues with respect to the small parameter describing the microinhomogeneous structure of the domain. A similar problem was considered in [1] for the case of perforation located along the plane part of the boundary. The case studied in this paper is much more complicated since the eigenvalues of multiplicity more than one can appear. The technique for asymptotic analysis of such kind of problem can be found, for example, in [2, 3].

The obtained results are used for asymptotic expansion of the best constant in a Friedrichs-type inequality for functions from the space H^1, vanishing on the boundary of the perforation and satisfying homogeneous Neuman condition on the boundary of the circle. Analogous questions concerning the asymptotic behavior of the best constant in Friedrichs-type inequality in domains having microinhomogeneous structure in a neighborhood of the boundary were studied in [1, 4–11]. In the remaining part of this introduction, we will give a short description of some of the most important results in these papers to put the results obtained in this paper into a more general frame.

In paper [4], the authors proved a Friedrichs-type inequality for functions, having zero trace on the small periodically alternating pieces of the boundary of a two-dimensional domain. The total measure of the set, where the function vanishes, tends to zero. It turns out that for this case the constant in the Friedrichs-type inequality is bounded. Moreover, the precise asymptotics of the constant in the derived Friedrichs-type inequality is described as the small parameter characterizing the microinhomogeneous structure of the boundary, tends to zero.

Paper [5] is devoted to the asymptotic analysis of functions depending on the small parameter, which characterizes the microinhomogeneous structure of the domain where the functions are defined. The authors considered a boundary-value problem in a two-dimensional domain perforated nonperiodically along the boundary in the case when the diameter of circles and the distance between them have the same order. In particular, it was proved that the Dirichlet problem is the limit for the original problem. Moreover, some numerical simulations were used to illustrate the results. As an application, a Friedrichs-type inequality was derived for functions vanishing on the boundary of the cavities. It was proved that the constant in the obtained inequality is close to the constant in the inequality for functions from $\overset{\circ}{H}{}^1$. The three-dimensional case of the same problem is considered in [8].

In paper [9], the author considered a three-dimensional domain, which is aperiodically perforated along the boundary in the case when the diameter of the holes and the distance between them have the same order. A Friedrichs-type inequality was derived for functions from the space H^1 vanishing on the boundaries of cavities. In particular, it was shown that the constant in the derived inequality tends to the constant of the classical inequality for functions from $\overset{\circ}{H}{}^1$ when the small parameter describing the size of perforation tends to zero.

Paper [1] (see also [7]) deals with the construction of the asymptotic expansion for the first eigenvalue of a boundary-value problem for the Laplacian in a perforated domain. This asymptotics gives an asymptotic expansion for the best constant in a corresponding Friedrichs-type inequality.

Paper [11], is devoted to the Friedrichs-type inequality, where the domain is periodically and rarely perforated along the boundary. It is assumed that the functions satisfy homogeneous Neumann boundary conditions on the outer boundary and that they vanish on the perforation. In particular, it is proved that the best constant in the inequality converges to the best constant in a Friedrichs-type inequality as the size of the perforation goes to zero much faster than the period of perforation. The limit Friedrichs-type inequality is valid for functions in the Sobolev space H^1.

Some generalizations of Friedrichs-type inequalities are Hardy-type inequalities. There exist several books devoted to this topic, see [12–16]. The first attempts to generalize the classical results concerning Hardy-type inequalities in fixed domains to domains with microinhomogeneous structure one can find in [6, 10].

Paper [6] deals with a three-dimensional weighted Hardy-type inequality in the case when the domain Ω is bounded and has nontrivial microstructure. It is assumed that the small holes are distributed periodically along the boundary. The main result is the validity of a weighted Hardy-type inequality for the class of functions from the Sobolev space H^1 having zero trace on the small holes under the assumption that a weight function decreases to zero in a neighborhood of the microinhomogenity on the boundary.

In paper [10], the author derived a new two-dimensional weighted Hardy-type inequality in a rectangle for the class of functions from the Sobolev space H^1 vanishing on small alternating pieces of the boundary. The dependence of the best constant in the derived inequality on the small parameter describing the size of microinhomogenity was established.

This paper is organized as follows: in Section 2 we give all necessary definitions and state the spectral problem. Section 3 is devoted to the construction of the leading terms of asymptotic expansion, while the complete expansions for the simple and multiple eigenvalues are constructed in Sections 4 and 5, respectively. The verification of the constructed asymptotics is given in Section 6. Finally, in Section 7, the obtained results are applied to describe the asymptotic behavior for the best constant in a Friederichs-type inequality considered in a perforated domain.

2. Preliminaries

Consider a unit circle Ω centered at the origin. We introduce the polar system of coordinates (θ, r) in Ω. Introduce a small parameter $\varepsilon = 2/N,\ N \gg 1$, and consider the open set B_ε which is the union of small sets periodically distributed along the boundary. Each of these small sets can be obtained from the neighboring one by rotation about the origin through the angle $\varepsilon\pi$. Finally, we define $\Omega_\varepsilon = \Omega \setminus \overline{B}_\varepsilon$ and $\partial B_\varepsilon = \Gamma_\varepsilon$, see Figure 1. Let us describe the geometry of B_ε in details. Consider the semi-strip:

$$\Pi = \left\{\xi : -\frac{\pi}{2} < \xi_1 < \frac{\pi}{2}, \xi_2 > 0\right\}, \qquad \Gamma := \left\{\xi : -\frac{\pi}{2} < \xi_1 < \frac{\pi}{2}, \xi_2 = 0\right\}. \tag{2.1}$$

Let B be an arbitrary two-dimensional open domain with a smooth boundary that is symmetric the vertical axis and lies in a disk of a fixed radius $a < 1$ centered at the point $(0, 1)$, see Figure 2. Let B_a be the union of the π-integer translations of B along the axis ξ_1. Then we define B_ε as the image of B_a under the mapping $\theta = \varepsilon\xi_1, r = 1 - \varepsilon\xi_2$.

Consider the following spectral problem:

$$\begin{gathered} -\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial r} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.2}$$

The problem,

$$\begin{gathered} -\Delta u_0 = \lambda_0 u_0 \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.3}$$

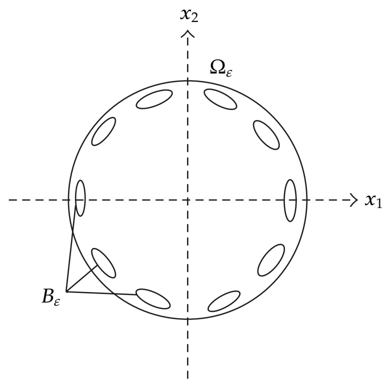

Figure 1: Perforated circle.

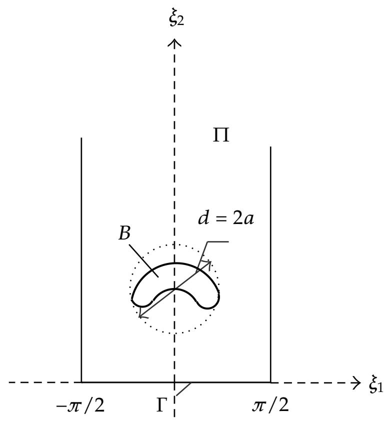

Figure 2: Cell of periodicity.

is the limit one for (2.2). This fact can be established analogously as in [17, 18], by using the same technique.

Remark 2.1. In particular, it can be proved that the number of eigenvalues (bearing in mind the multiplicities) of the original problem converging to the eigenvalue of the limit (homogenized) problem is equal to the multiplicity of the mentioned eigenvalue of the limit problem (for the method of proof see, e.g., [19]).

Remark 2.2. The limit spectral problem (2.3) is studied very well. In particular, if the eigenvalue λ_0 is simple, then the corresponding eigenfrequency $k_0 = \sqrt{\lambda_0}$ of (2.3) is the zero-point of the Bessel-function $\mathcal{J}_0$, and the corresponding eigenfunction has the form $\mathcal{J}_0(k_0 r)$. One can find the definition of Bessel-functions, for example, in [20, Section 4.7].

The goal of this paper is to construct and verify the asymptotic expansion for the eigenvalues of (2.2). The obtained asymptotics is used for studying the behavior of the best constant in a Friedrichs-type inequality for functions belonging to the Sobolev class $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$

(see the definition of $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ in Section 7). One of the main results of this paper is the following asymptotics for λ_ε converging to λ_0:

$$\lambda_\varepsilon = \lambda_0 + \sum_{i=1}^{\infty} \varepsilon^i \lambda_i, \tag{2.4}$$

where λ_i are some fixed constants which can be calculated according to (4.23) and (4.15) in the case of simple λ_ε and according to (5.10) and (4.15) when λ_ε is of multiplicity two. In particular, $\lambda_1 < 0$ which implies that $\lambda_\varepsilon < \lambda_0$.

3. Construction of the Leading Terms of the Asymptotic Expansion

Suppose that λ_0 is the simple eigenvalue for (2.3) and the corresponding eigenfunction u_0 is normalized in $L_2(\Omega)$. Our aim is to construct the leading terms of the asymptotic expansions for λ_ε converging to λ_0 as well as u_ε converging to u_0. We use the method of boundary-layer functions (see [21]) for this purpose. We are looking for eigenvalues and eigenfunctions in the following form:

$$\begin{gathered} \lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \cdots, \\ u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon\alpha_0(\theta) v(\xi) + \cdots, \end{gathered} \tag{3.1}$$

where $\xi = (\xi_1, \xi_2)$, $\xi_1 = \theta/\varepsilon$, $\xi_2 = (1-r)/\varepsilon$, and

$$\begin{gathered} u_0(x) = \alpha_0(\theta)(1-r) + O\Big((1-r)^2\Big) \quad \text{as } r \longrightarrow 1,\ \alpha_0(\theta) = -\left.\frac{\partial u_0}{\partial r}\right|_{r=1}, \\ u_1(x) = u_1|_{r=1} + \alpha_1(\theta)(1-r) + O\Big((1-r)^2\Big) \quad \text{as } r \longrightarrow 1,\ \alpha_1(\theta) = -\left.\frac{\partial u_1}{\partial r}\right|_{r=1}. \end{gathered} \tag{3.2}$$

Substituting the first expansion from (3.1) and the sum $u_0+\varepsilon u_1$ from the second expansion in (2.2) and equating terms at the same power of ε, we get the equation for u_1:

$$-\Delta_x u_1 = \lambda_0 u_1 + \lambda_1 u_0 \quad \text{in } \Omega. \tag{3.3}$$

The existence of the solution for (3.3) is given in the following proposition.

Proposition 3.1. *For any λ_1, there exists the smooth solution of* (3.3) *satisfying the boundary condition*

$$u_1 = -\lambda_1 \alpha_0(\theta)\left(\int_0^{2\pi} \alpha_0^2\, d\theta\right)^{-1} \quad \text{on } \partial\Omega. \tag{3.4}$$

Proof. The existence of the smooth solution follows from the classical results on regular solutions of elliptic equations (see e.g., [22]). In order to get u_1 as the unique solution, one can add the condition of mutual orthogonality:

$$\int_{\Omega} u_0 u_1 dx = 0. \tag{3.5}$$

By multiplying (3.3) by u_0, integrating (3.3) over Ω, and twice integrating by parts the obtained equation, we find that

$$-\int_{\Omega} u_1 \Delta u_0 dx - \int_{\partial\Omega} u_1 \frac{\partial u_0}{\partial r} d\theta + \int_{\partial\Omega} u_0 \frac{\partial u_1}{\partial r} d\theta = \lambda_1 \int_{\Omega} u_0^2 dx + \lambda_0 \int_{\Omega} u_1 u_0 dx. \tag{3.6}$$

Taking into account the fact that u_0 is the normalized (in $L_2(\Omega)$) solution of (2.3) and since u_1 satisfies (3.5), we can deduce that

$$\lambda_1 = -\int_{\partial\Omega} \frac{\partial u_0}{\partial r} u_1 d\theta = -\int_{\partial\Omega} \alpha_0(\theta) u_1 d\theta. \tag{3.7}$$

Then (3.7) leads to (3.4) and the proof is complete. □

However, the approximation $u_0 + \varepsilon u_1$ does not satisfy the condition on Γ_ε. This forces us to introduce an additional term $\alpha_0 v$ in second expansion of (3.1) to satisfy the appropriate boundary condition. We assume that the function v has exponential decay as $\xi_2 \to \infty$ and is π-periodical with respect to ξ_1. Under this assumption, $\alpha_0 v$ "almost" does not destroy (2.2) in the sense that the norm of additional contribution is small. The rigorous explanation is given in Section 6. Proceeding, we have that

$$-\Delta_x (u_0 + \varepsilon u_1 + \varepsilon \alpha_0 v + \cdots) = (\lambda_0 + \varepsilon \lambda_1 + \cdots)(u_0 + \varepsilon u_1 + \varepsilon \alpha_0 v + \cdots). \tag{3.8}$$

Taking into account (2.3) and (3.3), we see that v has to satisfy the equation

$$-\Delta_x (\alpha_0 v) = \lambda_0 \alpha_0 v. \tag{3.9}$$

Rewrite Δ_x in polar coordinates and pass to the ξ-variables in the argument of v:

$$\begin{aligned}
\Delta_x (\alpha_0 v) &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} (\alpha_0 v) \right) + \frac{1}{r^2} \frac{\partial^2 (\alpha_0 v)}{\partial \theta^2} \\
&= \alpha_0 \frac{\partial^2 v}{\partial r^2} + \frac{\alpha_0}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \left(v \frac{\partial^2 \alpha_0}{\partial \theta^2} + 2 \frac{\partial \alpha_0}{\partial \theta} \frac{\partial v}{\partial \theta} + \alpha_0 \frac{\partial^2 v}{\partial \theta^2} \right) \\
&= \frac{\alpha_0}{\varepsilon^2} \frac{\partial^2 v}{\partial \xi_2^2} - \frac{\alpha_0}{(\varepsilon - \varepsilon^2 \xi_2)} \frac{\partial v}{\partial \xi_2} + \frac{1}{(1 - \varepsilon \xi_2)^2} \left[v \frac{\partial^2 \alpha_0}{\partial \theta^2} + \frac{2}{\varepsilon} \frac{\partial \alpha_0}{\partial \theta} \frac{\partial v}{\partial \xi_1} + \frac{\alpha_0}{\varepsilon^2} \frac{\partial^2 v}{\partial \xi_1^2} \right].
\end{aligned} \tag{3.10}$$

Finally, replacing formulas $1/(\varepsilon - \varepsilon^2\xi_2)$ and $1/(1 - \varepsilon\xi_2)^2$ with Taylor series with respect to ε, substituting the obtained formula for $\Delta_x(\alpha_0 v)$ in (3.9), and equating terms at ε^{-2}, we deduce that

$$\Delta_\xi v = 0. \tag{3.11}$$

Now we derive the boundary conditions for function v. Substituting the second series from (3.1) in boundary conditions from (2.2) and using (3.2), we have

$$\begin{aligned} 0 &= u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon\alpha_0 v + \cdots = \varepsilon\left(\alpha_0\xi_2 + u_1|_{r=1} + \alpha_0 v\right) + O\left(\varepsilon^2\right), \\ 0 &= \frac{\partial u_\varepsilon}{\partial r} = \frac{\partial u_0}{\partial r} + \varepsilon\frac{\partial u_1}{\partial r} + \varepsilon\alpha_0\frac{\partial v}{\partial r} + \cdots = -\alpha_0 - \varepsilon\alpha_1 - \alpha_0\frac{\partial v}{\partial \xi_2} + \cdots, \end{aligned} \tag{3.12}$$

which implies that

$$\begin{gathered} \alpha_0\xi_2 + u_1|_{r=1} + \alpha_0 v = 0, \\ -\alpha_0 - \alpha_0\frac{\partial v}{\partial \xi_2} = 0. \end{gathered} \tag{3.13}$$

Taking into account (3.4), we derive the boundary conditions for v on ∂B and on Γ:

$$\begin{gathered} v = -\xi_2 + \lambda_1\left(\int_0^{2\pi} \alpha_0^2\, d\theta\right)^{-1} \quad \text{on } \partial B, \\ \frac{\partial v}{\partial \xi_2} = -1 \quad \text{on } \Gamma. \end{gathered} \tag{3.14}$$

Summing up (3.11) and (3.14), we get the following boundary-value problem for v:

$$\begin{gathered} \Delta_\xi v = 0 \quad \Pi \setminus \overline{B}, \\ v = -\xi_2 + \lambda_1\left(\int_0^{2\pi} \alpha_0^2\, d\theta\right)^{-1} \quad \text{on } \partial B, \\ \frac{\partial v}{\partial \xi_2} = -1 \quad \text{on } \Gamma. \end{gathered} \tag{3.15}$$

Define the function Y as the solution of the following boundary-value problem in the cell of periodicity:

$$\begin{gathered}
\Delta Y = 0 \quad \text{in } \Pi \setminus \overline{B}, \\
Y = 0 \quad \text{on } \partial B, \\
\frac{\partial Y}{\partial \xi_1} = 0 \quad \text{on } \partial \Pi \setminus \Gamma, \\
\frac{\partial Y}{\partial \xi_2} = 0 \quad \text{on } \Gamma, \\
\frac{\partial Y}{\partial \xi_2} = 1 \quad \text{as } \xi_2 \longrightarrow \infty.
\end{gathered} \tag{3.16}$$

It was proved in [7] that there exists the solution of (3.16), which is even with respect to ξ_1 and has the asymptotics:

$$Y(\xi) = \xi_2 + C(B) + O\left(e^{-\alpha \xi_2}\right) \quad \text{as } \xi_2 \longrightarrow \infty, \tag{3.17}$$

where

$$C(B) = \int_{\Pi \setminus \overline{B}} |\nabla (Y - \xi_2)|^2 d\xi + |B| > 0, \tag{3.18}$$

and $|B|$ is the area of the domain B.

The following lemma gives the conditions to obtain v as an exponentially decaying function as $\xi_2 \to \infty$.

Lemma 3.2. *Assume that F is π-periodic with respect to ξ_1 function with exponential decay as $\xi_2 \to \infty$, and let v be a π-periodic solution of the boundary-value problem:*

$$\Delta v = F, \quad \xi_2 > 0; \qquad v = A_1, \quad \xi \in \partial B; \qquad \frac{\partial v}{\partial \xi_2} = A_2, \quad \xi \in \Gamma; \tag{3.19}$$

with finite Dirichlet integral in Π. Then there exists the unique weak solution, which has asymptotics $v = C + O(e^{-\alpha \xi_2}), \alpha > 0$. To obtain v as a function with exponential decay as $\xi_2 \to \infty$, it is necessary and sufficient to have

$$\int_{\Pi \setminus B} YF \, d\xi + \int_{\partial B} A_1 \frac{\partial Y}{\partial \nu_B} dS_B + \int_{\Gamma} A_2 Y \, d\xi_1 = 0. \tag{3.20}$$

Proof. The existence of the solution with asymptotics $v = C + O(e^{-\alpha \xi_2})$ follows from the classical results on elliptic boundary-value problems in cylindric domains (see, e.g., [23] and [24, Chapters 2, 5]). Let us verify (3.20). Define $\Pi_R = \Pi \cap \{0 < \xi_2 < R\}$ and $\Gamma_R = \{\xi :$

$-\pi/2 < \xi_1 < \pi/2, \xi_2 = R\}$. By multiplying the equation from (3.15) by Y, integrating it over $\Pi_R \setminus \overline{B}$, and using the property of Y, we get that

$$\begin{aligned}\int_{\Pi_R\setminus\overline{B}} FY\,d\xi &= -\int_{\Pi_R\setminus\overline{B}} \nabla v \nabla Y\,d\xi + \int_{\Gamma_R} \frac{\partial v}{\partial \xi_2} Y\,d\xi_1 - \int_{\Gamma} \frac{\partial v}{\partial \xi_2} Y\,d\xi_1 \\ &= \int_{\Pi_R\setminus\overline{B}} v\Delta Y\,d\xi - \int_{\Gamma_R} v\frac{\partial Y}{\partial \xi_2}\,d\xi_1 + \int_{\Gamma} v\frac{\partial Y}{\partial \xi_2}\,d\xi_1 - \int_{\partial B} v\frac{\partial Y}{\partial \nu}\,dS_B \\ &\quad + \int_{\Gamma_R} \frac{\partial v}{\partial \xi_2} Y\,d\xi_1 - \int_{\Gamma} \frac{\partial v}{\partial \xi_2} Y\,d\xi_1 = -\int_{\Gamma_R} v\frac{\partial Y}{\partial \xi_2}\,d\xi_1 - \int_{\partial B} A_1\frac{\partial Y}{\partial \nu}\,dS_B \\ &\quad + \int_{\Gamma_R} \frac{\partial v}{\partial \xi_2} Y\,d\xi_1 - \int_{\Gamma} A_2 Y\,d\xi_1.\end{aligned} \tag{3.21}$$

Passing to the limit as $R \to \infty$, we obtain that

$$\int_{\Pi\setminus\overline{B}} FY\,d\xi = -\pi C - \int_{\partial B} A_1\frac{\partial Y}{\partial \nu}\,dS_B - \int_{\Gamma} A_2 Y\,d\xi_1. \tag{3.22}$$

This can be rewritten as

$$C = \frac{1}{\pi}\left(-\int_{\Gamma} A_2 Y\,d\xi_1 - \int_{\partial B} A_1\frac{\partial Y}{\partial \nu}\,dS_B - \int_{\Pi\setminus\overline{B}} FY\,d\xi\right). \tag{3.23}$$

Then v has exponential decay as $\xi_2 \to \infty$ if and only if $C = 0$ which is equivalent to (3.20). The proof is complete. □

In order to obtain v as function with exponential decay as $\xi_2 \to \infty$, one must have

$$0 = -\int_{\partial B} (-\xi_2 + K)\frac{\partial Y}{\partial \nu_B}\,dS_B + \int_{\Gamma} Y\,d\xi_1, \tag{3.24}$$

where we denote $K = \lambda_1 \left(\int_0^{2\pi} \alpha_0^2\,d\theta\right)^{-1}$. However, (3.24) implies that

$$K = \left(\int_{\partial B} \xi_2\frac{\partial Y}{\partial \nu_B}\,dS_B + \int_{\Gamma} Y\,d\xi_1\right)\left(\int_{\partial B} \frac{\partial Y}{\partial \nu_B}\,dS_B\right)^{-1}. \tag{3.25}$$

Integrate the identities $0 = \int_{\Pi_R \setminus \overline{B}} \Delta Y \, d\xi, 0 = \int_{\Pi_R \setminus \overline{B}} \xi_2 \Delta Y \, d\xi$:

$$\begin{aligned}
0 &= \int_{\Pi_R \setminus \overline{B}} \Delta Y \, d\xi = \int_{\partial(\Pi_R \setminus \overline{B})} \frac{\partial Y}{\partial n} dS = \int_{\partial B} \frac{\partial Y}{\partial \nu_B} dS_B + \int_{\Gamma_R} \frac{\partial Y}{\partial \xi_2} d\xi_1, \\
0 &= \int_{\Pi_R \setminus \overline{B}} (\xi_2 \Delta Y - Y \Delta \xi_2) d\xi = \int_{\partial(\Pi_R \setminus \overline{B})} \left(\xi_2 \frac{\partial Y}{\partial n} - Y \frac{\partial \xi_2}{\partial n} \right) dS \\
&= \int_{\partial B} \xi_2 \frac{\partial Y}{\partial \nu_B} dS_B + \int_{\Gamma} Y \, d\xi_1 + \int_{\Gamma_R} \left(\xi_2 \frac{\partial Y}{\partial \xi_2} - Y \right) d\xi_1.
\end{aligned} \tag{3.26}$$

Passing to the limit as $R \to \infty$, we find that

$$\begin{aligned}
0 &= \int_{\partial B} \frac{\partial Y}{\partial \nu_B} dS_B + \pi, \\
0 &= \int_{\partial B} \xi_2 \frac{\partial Y}{\partial \nu_B} dS_B + \int_{\Gamma} Y \, d\xi_1 - \pi C(B).
\end{aligned} \tag{3.27}$$

Then (3.25) and (3.27) together with Remark 2.2 imply that

$$\lambda_1 = -C(B) \int_0^{2\pi} \alpha_0^2 \, d\theta = -2\pi C(B) k_0^2 (\mathcal{J}_0')^2 (k_0) < 0. \tag{3.28}$$

4. Complete Expansion in the Case of the Simple Eigenvalue λ_0

Assume that λ_0 is the simple eigenvalue of the limit problem. Now we construct the complete expansion in the following form:

$$u_\varepsilon(x) = u_\varepsilon^{\text{ex}}(x) + \chi(1-r) u_\varepsilon^{\text{in}} \left(\frac{1-r}{\varepsilon}, \frac{\theta}{\varepsilon} \right), \tag{4.1}$$

where χ is a smooth cutoff function, which equals to one when $1/2 < r < 1$ and zero when $r < 1/4$:

$$u_\varepsilon^{\text{ex}}(x) = \mathcal{J}_0(k(\varepsilon) r), \tag{4.2}$$

$$u_\varepsilon^{\text{in}}(\xi) = \sum_{i=1}^{\infty} \varepsilon^i v_i(\xi). \tag{4.3}$$

Here $k(\varepsilon) = \sqrt{\lambda_\varepsilon}, v_i(\xi)$ are π-periodic in ξ_1 functions with exponential decay as $\xi_2 \to \infty$. One can easily show that (4.2) solves the equation:

$$-\Delta_x u_\varepsilon^{\text{ex}}(x) = \lambda_\varepsilon u_\varepsilon^{\text{ex}}(x) \tag{4.4}$$

if and only if $k(\varepsilon) = \sqrt{\lambda_\varepsilon}$.

We are looking for $u_\varepsilon^{\text{in}}(\xi)$, which solves the equation:

$$-\Delta_x u_\varepsilon^{\text{in}}(\xi) = \lambda_\varepsilon u_\varepsilon^{\text{in}}(\xi). \tag{4.5}$$

If (4.4) and (4.5) are satisfied, then u_ε from (4.1) is the solution of

$$-\Delta_x u_\varepsilon = \lambda_\varepsilon u_\varepsilon + F, \tag{4.6}$$

where $F = -u_\varepsilon^{\text{in}} \Delta_x \chi - 2\nabla_x u_\varepsilon^{\text{in}} \nabla_x \chi$. Our aim is to construct $u_\varepsilon^{\text{in}}$ so that F will be of small order as $\varepsilon \to 0$. This is the reason why we need to have v_i as exponentially decaying functions.

Now we derive the formula for the Laplacian in ξ-variables:

$$\begin{aligned} \Delta_x &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \\ &= \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \xi_2^2} + \frac{1}{\varepsilon(\varepsilon\xi_2 - 1)}\frac{\partial}{\partial \xi_2} + \frac{1}{\varepsilon^2(\varepsilon\xi_2 - 1)^2}\frac{\partial^2}{\partial \xi_1^2} \\ &= \frac{1}{\varepsilon^2}\Delta_\xi + \frac{1}{\varepsilon(\varepsilon\xi_2 - 1)}\frac{\partial}{\partial \xi_2} + \frac{1}{\varepsilon^2}\left(\frac{1}{(\varepsilon\xi_2 - 1)^2} - 1\right)\frac{\partial^2}{\partial \xi_1^2}. \end{aligned} \tag{4.7}$$

By substituting the Taylor series for the functions

$$\frac{1}{\varepsilon(\varepsilon\xi_2 - 1)}, \qquad \frac{1}{\varepsilon^2}\left(\frac{1}{(\varepsilon\xi_2 - 1)^2} - 1\right) \tag{4.8}$$

in (4.7), we get the final formula for Δ_x:

$$\Delta_x = \frac{1}{\varepsilon^2}\Delta_\xi + \sum_{n=0}^{\infty}(n+1)\varepsilon^{n-2}\xi_2^n \frac{\partial^2}{\partial \xi_1^2} - \sum_{n=0}^{\infty}\varepsilon^{n-1}\xi_2^n \frac{\partial}{\partial \xi_2}. \tag{4.9}$$

Substituting (2.4) and (4.3) in (4.5) and taking into account (4.9), we deduce the following formula:

$$\begin{aligned} \sum_{i=1}^{\infty}\varepsilon^i \Delta_\xi v_i &= \left(\varepsilon + \varepsilon^2\xi_2 + \cdots + \varepsilon^{n+1}\xi_2^n + \cdots\right)\sum_{i=1}^{\infty}\varepsilon^i \frac{\partial v_i}{\partial \xi_2} \\ &\quad - \left(2\varepsilon\xi_2 + 3\varepsilon^2\xi_2^2 + \cdots + (n+1)\varepsilon^n\xi_2^n + \cdots\right)\sum_{i=1}^{\infty}\varepsilon^i \frac{\partial^2 v_i}{\partial \xi_1^2} \\ &\quad - \left(\varepsilon^2\lambda_0 + \varepsilon^3\lambda_1 + \cdots + \varepsilon^{n+2}\lambda_n\right)\sum_{i=1}^{\infty}\varepsilon^i v_i. \end{aligned} \tag{4.10}$$

By equating terms of the same power of ε, we obtain that

$$\begin{aligned}
&\varepsilon^1: \quad \Delta_\xi v_1 = 0, \\
&\varepsilon^2: \quad \Delta_\xi v_2 = \frac{\partial v_1}{\partial \xi_2} - 2\xi_2 \frac{\partial^2 v_1}{\partial \xi_1^2}, \\
&\vdots, \\
&\varepsilon^k: \quad \Delta_\xi v_k = \sum_{j=1}^{k-1} \left(\xi_2^{j-1} \frac{\partial v_{k-j}}{\partial \xi_2} - (j+1)\xi_2^j \frac{\partial^2 v_{k-j}}{\partial \xi_1^2} \right) - \sum_{j=0}^{k-3} \lambda_j v_{k-j-2}, \\
&\vdots.
\end{aligned} \tag{4.11}$$

Consider now the boundary conditions from (2.2). According to the property of χ,

$$u_\varepsilon(x) = u_\varepsilon^{\mathrm{ex}}(x) + u_\varepsilon^{\mathrm{in}}\left(\frac{1-r}{\varepsilon}, \frac{\theta}{\varepsilon}\right) = \mathcal{J}_0(k(\varepsilon)r) + \sum_{i=1}^{\infty} \varepsilon^i v_i(\xi), \tag{4.12}$$

in a small neighborhood of $\partial\Omega$. Moreover, on $\partial\Omega$, it yields that

$$0 = \frac{\partial u_\varepsilon}{\partial r} = k(\varepsilon)\mathcal{J}'_0(k(\varepsilon)) - \sum_{i=1}^{\infty} \varepsilon^{i-1} \left.\frac{\partial v_i}{\partial \xi_2}\right|_{\xi_2=0}. \tag{4.13}$$

We assume that the function $k(\varepsilon)$ has asymptotics:

$$k(\varepsilon) = k_0 + \varepsilon k_1 + \cdots + \varepsilon^n k_n + \cdots, \tag{4.14}$$

and since $\lambda_\varepsilon = k^2(\varepsilon)$, we can derive the following formulas for λ_i:

$$\lambda_0 = k_0^2, \qquad \lambda_1 = 2k_0k_1, \ldots, \qquad \lambda_i = \sum_{j=0}^{i} k_j k_{i-j}. \tag{4.15}$$

Rewriting $\mathcal{J}'_0(k(\varepsilon))$ as a Taylor series with respect to ε, we have

$$\mathcal{J}'_0(k(\varepsilon)) = \mathcal{J}'_0(k_0) + \frac{\mathcal{J}''_0(k_0)k_1\varepsilon}{1!} + \frac{\left(\mathcal{J}'''_0(k_0)k_1^2 + \mathcal{J}''_0(k_0)k_2\right)\varepsilon^2}{2!} + \cdots. \tag{4.16}$$

Substituting (4.16) in (4.13), using (4.14), and equating the terms with the same powers of ε, we get the following boundary condition for $v_i, i = 1, 2, \ldots$:

$$\frac{\partial v_i}{\partial \xi_2} = g_i(k_1, \ldots, k_{i-1}) \quad \text{on } \Gamma, \tag{4.17}$$

where

$$g_1 = k_0 \mathcal{J}_0'(k_0), \qquad g_2 = k_1 \mathcal{J}_0'(k_0) + k_0 k_1 \mathcal{J}_0''(k_0) \equiv 0. \tag{4.18}$$

Consider now the boundary conditions on small holes. Analogously,

$$u_\varepsilon(x) = \mathcal{J}_0(k(\varepsilon)r) + \sum_{i=1}^{\infty} \varepsilon^i v_i(\xi) = \mathcal{J}_0(k(\varepsilon)(1-\varepsilon\xi_2)) + \sum_{i=1}^{\infty} \varepsilon^i v_i(\xi). \tag{4.19}$$

Substituting the Taylor series for $\mathcal{J}_0(k(\varepsilon)(1-\varepsilon\xi_2))$ with respect to ε in the last formula, using (4.14), and equating the terms with the same powers of ε in equation $u_\varepsilon = 0$ on Γ_ε, we get the following boundary condition for v_i, $i = 1, 2, \ldots$, on ∂B:

$$v_i = -k_i \mathcal{J}_0'(k_0) + f_i(\xi_2; k_0, k_1, \ldots, k_{i-1}) \quad \text{on } \partial B, \tag{4.20}$$

where f_i are polynomials of power i with respect to ξ_2 with coefficients which depend on $(k_0, k_1, \ldots, k_{i-1})$. The precise formula for f_i can be derived for each fixed i. For example, we have that

$$f_1 = k_0 \mathcal{J}_0'(k_0)\xi_2, \qquad f_2 = k_1 \mathcal{J}_0'(k_0)\xi_2 - \frac{1}{2}\mathcal{J}_0''(k_0)(k_1 - k_0\xi_2)^2. \tag{4.21}$$

The following Lemma is useful for our analysis. For the proof see for example,[3].

Lemma 4.1. *Suppose that F and v satisfy the conditions of Lemma 3.2.* (a) *If F is even with respect to ξ_1, then v is even;* (b) *if F is odd with respect to ξ_1 and $A_1 = A_2 = 0$, then v is odd with respect to ξ_1 and decays exponentially as $\xi_2 \to \infty$.*

Theorem 4.2. *There exist numbers k_i and π-periodic in ξ_1 functions v_i with finite Dirichlet integral in Π and exponential decay as $\xi_2 \to \infty$, such that these functions are solutions of the following boundary-value problems:*

$$\begin{gathered}\Delta v_i = F_i \equiv \sum_{j=1}^{i-1}\left(\xi_2^{j-1}\frac{\partial v_{i-j}}{\partial \xi_2} - (j+1)\xi_2^{j}\frac{\partial^2 v_{i-j}}{\partial \xi_1^2}\right) - \sum_{j=0}^{i-3}\lambda_j v_{i-j-2} \quad \textit{in } \Pi \setminus \overline{B}, \\ v_i = -k_i \mathcal{J}_0'(k_0) + f_i(\xi_2; k_0, k_1, \ldots, k_{i-1}) \quad \textit{on } \partial B, \\ \frac{\partial v_i}{\partial \xi_2} = g_i(k_1, \ldots, k_{i-1}) \quad \textit{on } \Gamma.\end{gathered} \tag{4.22}$$

Moreover, the constants are defined by the formula:

$$k_i = -\frac{1}{\pi \mathcal{J}_0'(k_0)}\left(\int_{\Pi\setminus B} Y F_i d\xi + \int_{\partial B} f_i(\xi_2; k_0, k_1, \ldots, k_{i-1})\frac{\partial Y}{\partial \nu_B} dS_B + g_i(k_1, \ldots, k_{i-1})\int_\Gamma Y\, d\xi_1\right) \tag{4.23}$$

In particular,

$$k_1 = -\pi C(B) k_0 (\mathcal{J}')_0^2(k_0), \tag{4.24}$$

$$k_2 = \frac{k_1^2}{2k_0}. \tag{4.25}$$

Proof. Let v be the solution of boundary-value problem (3.15). It can be easily verified that

$$v_1 = -k_0 \mathcal{J}_0'(k_0) v \tag{4.26}$$

is a solution of (4.22), (4.20), (4.17) for f_1, g_1, and k_1 defined by (4.21), (4.18), and (4.24). For any k_2 boundary-value problem (4.22), (4.20), (4.17) for v_2 has a π-periodic solution with finite Dirichlet integral. By Lemma 3.2 and (3.27), v_2 has exponential decay as $\xi_2 \to \infty$ if and only if k_2 is given by (4.23) for $i = 2$. Let us verify formula (4.25) without applying the general (4.23). It is obvious that

$$\frac{\partial^2 v_1}{\partial \xi_1^2} = -\frac{\partial^2 v_1}{\partial \xi_2^2}. \tag{4.27}$$

By using that fact one can write the boundary-value problem for v_2 as

$$\begin{gathered}
\Delta v_2 = \left(\frac{\partial v_1}{\partial \xi_2} - 2\xi_2 \frac{\partial^2 v_1}{\partial \xi_1^2} \right) = \left(\frac{\partial v_1}{\partial \xi_2} + 2\xi_2 \frac{\partial^2 v_1}{\partial \xi_2^2} \right) \quad \text{in } \Pi \setminus \overline{B}, \\
v_2 = -k_2 \mathcal{J}_0'(k_0) + k_1 \mathcal{J}_0'(k_0)\xi_2 - \frac{1}{2}\mathcal{J}_0''(k_0)(k_1 - k_0\xi_2)^2 \quad \text{on } \partial B, \\
\frac{\partial v_2}{\partial \xi_2} = 0 \quad \text{on } \Gamma.
\end{gathered} \tag{4.28}$$

It can be verified that the function

$$v_2 = \frac{1}{2}\xi_2^2 \frac{\partial v_1}{\partial \xi_2} \tag{4.29}$$

is π-periodic with finite Dirichlet integral in Π, has exponential decay as $\xi_2 \to \infty$, and satisfies problem (4.28) for k_2 defined by (4.25). We can use the induction process to finalize the proof. □

Since k_i are defined by (4.23), we can calculate λ_i by using (4.15). Denote

$$u_{\varepsilon,N} = \mathcal{J}_0\left(\sqrt{\lambda_{\varepsilon,N}}\, r\right) + \chi(1-r) v_{\varepsilon,N}, \tag{4.30}$$

where $\lambda_{\varepsilon,N}$ and $v_{\varepsilon,N}$ are the partial sums of (2.4) and (4.3), respectively.

Theorem 4.2 implies the validity of the following useful result.

Theorem 4.3. *For any integer $N > 0$, the function $u_{\varepsilon,N}$ is the solution of the boundary-value problem*

$$\begin{gathered} -\Delta u_{\varepsilon,N} = \lambda_{\varepsilon,N} u_{\varepsilon,N} + F_{\varepsilon,N} \quad \text{in } \Omega_\varepsilon, \\ u_{\varepsilon,N} = a_{\varepsilon,N}(\theta) \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial u_{\varepsilon,N}}{\partial r} = b_{\varepsilon,N}(\theta) \quad \text{on } \partial\Omega, \end{gathered} \tag{4.31}$$

where $\|a_{\varepsilon,N}\|_{L_2(\Gamma_\varepsilon)} = O(\varepsilon^{N_1})$, $\|b_{\varepsilon,N}\|_{L_2(\partial\Omega)} = O(\varepsilon^{N_1})$, $\|F_{\varepsilon,N}\|_{L_2(\Omega_\varepsilon)} = O(\varepsilon^{N_1})$, and $N_1 \to \infty$ as $N \to \infty$.

Proof. According to the definition of $u_{\varepsilon,N}$, we have that

$$\begin{aligned} -\Delta_x u_{\varepsilon,N} &= -\Delta_x \left(\mathcal{J}_0 \left(\sqrt{\lambda_{\varepsilon,N}} r \right) + \chi(1-r) v_{\varepsilon,N} \right) \\ &= -\Delta_x \mathcal{J}_0 \left(\sqrt{\lambda_{\varepsilon,N}} r \right) - \Delta_x \chi v_{\varepsilon,N} - 2\nabla_x \chi \nabla_x v_{\varepsilon,N} - \chi \Delta_x v_{\varepsilon,N} \\ &= \lambda_{\varepsilon,N} \mathcal{J}_0 \left(\sqrt{\lambda_{\varepsilon,N}} r \right) + \lambda_{\varepsilon,N} \chi(1-r) v_{\varepsilon,N} - \lambda_{\varepsilon,N} \chi(1-r) v_{\varepsilon,N} - \Delta_x \chi v_{\varepsilon,N} - 2\nabla_x \chi \nabla_x v_{\varepsilon,N} \\ &\quad - \chi \Delta_x v_{\varepsilon,N} \\ &= \lambda_{\varepsilon,N} u_{\varepsilon,N} + F_{\varepsilon,N}, \end{aligned} \tag{4.32}$$

where

$$F_{\varepsilon,N} = -v_{\varepsilon,N} \Delta_x \chi - \chi(\lambda_{\varepsilon,N} v_{\varepsilon,N} + \Delta_x v_{\varepsilon,N}) - 2\nabla_x \chi \nabla_x v_{\varepsilon,N} =: I_1 + I_2 + I_3. \tag{4.33}$$

Passing from (x_1, x_2) variables to polar coordinates (r, θ), we get that

$$\frac{\partial}{\partial x_1} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial x_2} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}, \tag{4.34}$$

$$\Delta_x = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \tag{4.35}$$

By using the fact that $\lim_{x\to\infty} x e^{-\alpha x} = 0$ and due to the result of Theorem 4.2, we have that, for any $1 \le i \le N$,

$$\varepsilon^i v_i = \varepsilon^i O\left(e^{-\alpha\xi_2}\right) = \varepsilon^N O\left(\varepsilon^{i-N} e^{-\alpha(1-r)/\varepsilon}\right) = \varepsilon^N O(\varepsilon^m) = O\left(\varepsilon^{N+m}\right), \tag{4.36}$$

where m is fixed. Hence, $v_{\varepsilon,N} = O(\varepsilon^{N+m})$. Similarly, taking into account (4.34) and (4.35), we can deduce that

$$\begin{gathered}\nabla_x v_{\varepsilon,N} = O\left(\varepsilon^{N+m}\right)\left(\frac{\alpha\cos\theta}{\varepsilon} - \frac{\sin\theta}{r}, \frac{\alpha\sin\theta}{\varepsilon} + \frac{\cos\theta}{r}\right),\\ \nabla_x\chi = \left(-\cos\theta\chi', -\sin\theta\chi'\right).\end{gathered} \tag{4.37}$$

Consequently,

$$\nabla_x v_{\varepsilon,N}\nabla_x\chi = O\left(\varepsilon^{N+m}\right)O\left(\frac{1}{\varepsilon r}\right). \tag{4.38}$$

Furthermore,

$$\begin{gathered}\Delta_x v_{\varepsilon,N} = -\frac{\alpha}{\varepsilon r}O\left(\varepsilon^{N+m}\right) + \frac{\alpha^2}{\varepsilon^2}O\left(\varepsilon^{N+m}\right) + \frac{1}{r^2}O\left(\varepsilon^{N+m}\right) = O\left(\frac{1}{\varepsilon^2 r^2}\right)O\left(\varepsilon^{N+m}\right),\\ \Delta_x\chi = -\frac{1}{r}\chi' + \chi'' = O\left(\frac{1}{r}\right).\end{gathered} \tag{4.39}$$

According to the definition of χ, the support of $\nabla_x\chi$ and $\Delta_x\chi$ is the set $\{1/4 \le r \le 1/2\}$. Summarizing, we have that

$$I_1 = O\left(\varepsilon^{N+m}\right)O\left(\frac{1}{r}\right), \qquad I_2 = O\left(\varepsilon^{N+m}\right)O\left(\frac{1}{\varepsilon^2 r^2}\right), \qquad I_3 = O\left(\varepsilon^{N+m}\right)O\left(\frac{1}{\varepsilon r}\right), \tag{4.40}$$

and we can derive that

$$\begin{aligned}\|F_{\varepsilon,N}\|^2_{L_2(\Omega_\varepsilon)} &= \int_\Omega F^2_{\varepsilon,N} r\,dr\,d\theta\\ &= \int_{\Omega\cap\{1/4\le r\le 1\}} I_2^2 r\,dr\,d\theta + \int_{\Omega\cap\{1/4\le r\le 1/2\}}\left[(I_1+I_3)^2 r + 2I_2(I_1+I_3)r\right]dr\,d\theta\\ &= O\left(\varepsilon^{2N+2m}\right)O\left(\frac{1}{\varepsilon^3}\right) + O\left(\varepsilon^{2N+2m}\right)O\left(\frac{1}{\varepsilon^4}\right) = O\left(\varepsilon^{2N+2m}\right)O\left(\frac{1}{\varepsilon^4}\right).\end{aligned} \tag{4.41}$$

Therefore,

$$\|F_{\varepsilon,N}\|_{L_2(\Omega_\varepsilon)} = O\left(\varepsilon^{N+m-2}\right) = O\left(\varepsilon^{N_1}\right), \quad N_1 \longrightarrow \infty \text{ as } N \longrightarrow \infty. \tag{4.42}$$

Consider now $u_{\varepsilon,N}$ on Γ_ε:

$$u_{\varepsilon,N} = \mathcal{J}_0\left(\sqrt{\lambda_{\varepsilon,N}}r\right) + v_{\varepsilon,N} = \varepsilon^{N+1}\beta_{N+1} + \varepsilon^{N+2}\beta_{N+2} + \cdots, \tag{4.43}$$

where β_j are the coefficients of the Taylor series of the function $\mathcal{J}_0(\sqrt{\lambda_{\varepsilon,N}}r)$. Hence, $a_{\varepsilon,N} = O(\varepsilon^{N+1})$ and

$$\|a_{\varepsilon,N}\|^2_{L_2(\Gamma_\varepsilon)} = \frac{2}{\varepsilon}\int_{\partial B_\varepsilon} a^2_{\varepsilon,N}\, d\theta \sim \frac{2}{\varepsilon}2\pi a\varepsilon O\left(\varepsilon^{2N+2}\right) = O\left(\varepsilon^{2N+2}\right), \tag{4.44}$$

which yields that $\|a_{\varepsilon,N}\|_{L_2(\Gamma_\varepsilon)} = O(\varepsilon^{N+1}) = O(\varepsilon^{N_1}), N_1 \to \infty$ as $N \to \infty$. Analogously, one can verify that $\|b_{\varepsilon,N}\|_{L_2(\partial\Omega)} = O(\varepsilon^{N_1}), N_1 \to \infty$ as $N \to \infty$. The proof is complete. □

5. Complete Expansion in the Case of Multiple Eigenvalue λ_0

In this section we consider the case when λ_0 is of multiplicity two. The asymptotics of the eigenvalue were constructed in the form (2.4) and

$$u_\varepsilon(x) = u_\varepsilon^{\mathrm{ex}}(x) + \chi(1-r)u_\varepsilon^{\mathrm{in}}\left(\frac{1-r}{\varepsilon}, \frac{\theta}{\varepsilon}, \theta\right), \tag{5.1}$$

$$u_\varepsilon^{\mathrm{ex}}(x) = \cos(n\theta)\mathcal{J}_n(k(\varepsilon)r), \tag{5.2}$$

$$u_\varepsilon^{\mathrm{in}}(x) = \cos(n\theta)\sum_{i=1}^{\infty}\varepsilon^i v_i^{\mathrm{even}}(\xi) + \sin(n\theta)\sum_{i=2}^{\infty}\varepsilon^i v_i^{\mathrm{odd}}(\xi). \tag{5.3}$$

In this case,

$$v_i^{\mathrm{even}} = -k_i\mathcal{J}'_n(k_0) + f_i^{(n)}(\xi_2; k_0, k_1, \dots, k_{i-1}) \quad \text{on } \partial B, \tag{5.4}$$

where $f_i^{(n)}$ are polynomials of power i with respect to ξ_2 with coefficients which depend on $(k_0, k_1, \dots, k_{i-1})$. Moreover,

$$\frac{\partial v_i^{\mathrm{even}}}{\partial \xi_2} = g_i^{(n)}(k_1, \dots, k_{i-1}) \quad \text{on } \Gamma, \tag{5.5}$$

where

$$\begin{gathered} f_1^{(n)} = k_0\mathcal{J}'_n(k_0)\xi_2, \qquad f_2^{(n)} = k_1\mathcal{J}'_0(k_0)\xi_2 - \frac{1}{2}\mathcal{J}''_0(k_0)(k_1 - k_0\xi_2)^2, \\ g_1^{(n)} = k_0\mathcal{J}'_n(k_0), \qquad g_2^{(n)} = k_1\mathcal{J}'_n(k_0) + k_0k_1\mathcal{J}''_n(k_0) \equiv 0, \end{gathered} \tag{5.6}$$

$$v_i^{\mathrm{odd}} = 0, \quad \xi \in \partial B, \qquad \frac{\partial v_i^{\mathrm{odd}}}{\partial \xi_2} = 0, \quad \xi \in \Gamma. \tag{5.7}$$

Substituting (5.3) and (2.4) in (4.5), passing to the variables ξ and (θ,ρ), and collecting all the terms with equal order of ε, we get two systems of equations for v_i^{even} and v_i^{odd}:

$$\begin{aligned}\Delta v_i^{\text{even}} &= \sum_{j=1}^{i-1}\left(\xi_2^{j-1}\frac{\partial v_{i-j}^{\text{even}}}{\partial \xi_2} - (j+1)\xi_2^j\frac{\partial^2 v_{i-j}^{\text{even}}}{\partial \xi_1^2}\right) - n\sum_{j=0}^{i-3}(j+1)\xi_2^j\frac{\partial v_{i-j-1}^{\text{odd}}}{\partial \xi_1}\\ &\quad - n^2\sum_{j=0}^{i-3}(j+1)\xi_2^j v_{i-j-2}^{\text{even}} - \sum_{j=0}^{i-3}\lambda_j v_{i-j-2}^{\text{even}} \quad \text{in } \Pi\setminus\overline{B},\end{aligned} \tag{5.8}$$

$$\begin{aligned}\Delta v_i^{\text{odd}} &= \sum_{j=1}^{i-2}\left(\xi_2^{j-1}\frac{\partial v_{i-j}^{\text{even}}}{\partial \xi_2} - (j+1)\xi_2^j\frac{\partial^2 v_{i-j}^{\text{odd}}}{\partial \xi_1^2}\right) + n\sum_{j=0}^{i-2}(j+1)\xi_2^j\frac{\partial v_{i-j-1}^{\text{even}}}{\partial \xi_1}\\ &\quad + n^2\sum_{j=0}^{i-3}(j+1)\xi_2^j v_{i-j-2}^{\text{odd}} - \sum_{j=0}^{i-3}\lambda_j v_{i-j-2}^{\text{odd}} \quad \text{in } \Pi\setminus\overline{B}.\end{aligned} \tag{5.9}$$

Theorem 5.1. *There exist numbers k_i and π-periodic in ξ_1 even functions v_i^{even} and odd functions v_i^{odd} with finite Dirichlet integral in Π, which have exponential decay as $\xi_2 \to \infty$, such that these functions are solutions of the boundary-value problems* (5.8), (5.4), (5.5), *and* (5.9), (5.7), *respectively. Moreover, the constants k_i are defined by the formula:*

$$k_i = -\frac{1}{\pi \mathcal{J}_n'(k_0)}\left(\int_{\Pi\setminus B} YF_i\,d\xi + \int_{\partial B} f_i^{(n)}(\xi_2;k_0,k_1,\ldots,k_{i-1})\frac{\partial Y}{\partial \nu_B}dS_B + g_i^{(n)}(k_1,\ldots,k_{i-1})\int_\Gamma Y\,d\xi_1\right), \tag{5.10}$$

Proof. The problems (5.8), (5.5), (5.4) for functions $v_1^{\text{even}}, v_2^{\text{even}}$ coincide with problems (4.22), (4.17), and (4.20) (if one change $\mathcal{J}_0'(k_0)$ by $\mathcal{J}_n'(k_0)$ and f_i, g_i by the respective $f_i^{(n)}, g_i^{(n)}$). Therefore the construction of $v_1^{\text{even}}, v_2^{\text{even}}$ and k_1, k_2 is just the same as the construction from the proof of Theorem 4.2. Due to (5.9), (5.7), the problem for v_2^{odd} is as follows:

$$\begin{aligned}\Delta v_2^{\text{odd}} &= n\xi_2\frac{\partial v_1^{\text{even}}}{\partial \xi_1} \quad \text{in } \Pi\setminus\overline{B},\\ v_2^{\text{odd}} &= 0 \quad \text{on } \partial B,\\ \frac{\partial v_2^{\text{odd}}}{\partial \xi_2} &= 0 \quad \text{on } \Gamma.\end{aligned} \tag{5.11}$$

The function v_1^{even} is even (due to (4.26)) and, hence, the right-hand side is odd in (5.11) and is even in (5.8). By Lemma 3.2 and Theorem 4.2, we conclude that there exists the even solution v_2^{odd} of (5.11) with exponential decay. Then we can use the iteration process to complete the proof. □

Denote

$$u_{\varepsilon,N} = \cos(n\theta)\mathcal{J}_0\left(\sqrt{\lambda_{\varepsilon,N}}r\right) + \chi(1-r)v_{\varepsilon,N}, \tag{5.12}$$

where $\lambda_{\varepsilon,N}$ and $v_{\varepsilon,N}$ are the partial sums of (2.4) and (5.3), respectively.

Theorem 5.1 implies the validity of the following result.

Theorem 5.2. *For any integer $N > 0$, the function $u_{\varepsilon,N}$ is the solution of the boundary-value problem:*

$$\begin{aligned} -\Delta u_{\varepsilon,N} &= \lambda_{\varepsilon,N} u_{\varepsilon,N} + F_{\varepsilon,N} \quad \text{in } \Omega_\varepsilon, \\ u_{\varepsilon,N} &= a_{\varepsilon,N}(\theta)\cos(n\theta) \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial u_{\varepsilon,N}}{\partial r} &= b_{\varepsilon,N}(\theta)\cos(n\theta) \quad \text{on } \partial\Omega, \end{aligned} \tag{5.13}$$

where $\|a_{\varepsilon,N}\|_{L_2(\Gamma_\varepsilon)} = O(\varepsilon^{N_1})$, $\|b_{\varepsilon,N}\|_{L_2(\partial\Omega)} = O(\varepsilon^{N_1})$, $\|F_{\varepsilon,N}\|_{L_2(\Omega_\varepsilon)} = O(\varepsilon^{N_1})$, and $N_1 \to \infty$ as $N \to \infty$.

Proof. The proof is analogous to the proof of Theorem 4.3. Hence, we omit the details. □

6. Verification of the Asymptotics

Consider the boundary-value problem:

$$\begin{aligned} -\Delta U_\varepsilon &= \lambda U_\varepsilon + F \quad \text{in } \Omega_\varepsilon, \\ U_\varepsilon &= 0 \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial U_\varepsilon}{\partial r} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{6.1}$$

where $F \in L_2(\Omega)$ and $\lambda \neq \lambda_0$ is some fixed number.

Similarly to the techniques used in [3, 18], one can show that the boundary-value problem (6.1) has the solution $U_\varepsilon \in H^1(\Omega)$ and the following representation holds:

$$U_\varepsilon = \frac{u_\varepsilon}{\lambda_\varepsilon - \lambda} \int_\Omega u_\varepsilon F \, dx + \widetilde{U}_\varepsilon, \tag{6.2}$$

for λ close to the simple eigenvalue λ_0 of the problem (2.3) and

$$U_\varepsilon = \frac{1}{\lambda_\varepsilon - \lambda} \sum_{i=1}^{2} u_\varepsilon^i \int_\Omega u_\varepsilon^i F \, dx + \widetilde{U}_\varepsilon, \tag{6.3}$$

for λ close to multiple eigenvalue λ_0 of the problem (2.3). Here u_ε is normalized in $L_2(\Omega)$ eigenfunctions to (2.2) and u_ε^i is orthonormalized in $L_2(\Omega)$ eigenfunctions to (2.2). Moreover,

$$\left\| \widetilde{U}_\varepsilon \right\|_{H^1} \leq C \|F\|_{L_2}, \tag{6.4}$$

where the constant C is independent on ε and λ. It follows from (6.2) and (6.4) that

$$\|U_\varepsilon\|_{H^1} \le \frac{C}{\lambda_\varepsilon - \lambda}\|F\|_{L_2}. \tag{6.5}$$

Consider now the case of simple λ_0. Define the function:

$$U_\varepsilon^N(x) = \left(1+\frac{1}{\varepsilon}\right)u_{\varepsilon,N}(x) - \left(1+\frac{1}{\varepsilon}\right)a_{\varepsilon,N} + b_{\varepsilon,N}\chi(1-r)(v(\xi)+\xi_2+C(B)), \tag{6.6}$$

where $u_{\varepsilon,N}$ and v are the solutions of (4.31) and (3.15), respectively, and $C(B)$ is given by (3.18). Then, by Theorem 4.3, U_ε^N is the solution of (6.1) if

$$\lambda = \lambda_{\varepsilon,N}, \qquad \|F\|_{L_2} = O\left(\varepsilon^{N_2}\right), \quad N_2 \longrightarrow \infty \quad \text{as } N \longrightarrow \infty. \tag{6.7}$$

Taking into account (6.5), (6.7), and the fact that $\|U_\varepsilon\|_{H^1} < \infty$, we can conclude that for each fixed N,

$$\lambda_\varepsilon - \lambda_{\varepsilon,N} = O\left(\varepsilon^{N_2}\right) = o\left(\varepsilon^{N}\right) \quad \text{as } \varepsilon \longrightarrow 0. \tag{6.8}$$

Therefore the asymptotics constructed in Section 4 coincide with the expansion of λ_ε. For the case of multiple λ_0, one can use the same technique. The difference is follows: one should use (6.3) instead of (6.2) and Theorem 5.2 instead of Theorem 4.3. The asymptotics of λ_ε are completely verified.

7. Application to a Friedrichs-Type Inequality

Consider the sets $\Omega_\varepsilon, \Gamma_\varepsilon$, which were defined in Section 2.

Definition 7.1. The Sobolev class $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ is the class of functions from $H^1(\Omega_\varepsilon)$ having zero trace on Γ_ε.

Theorem 7.2. *Let $u \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$. Then a Friedrichs-type inequality*

$$\int_{\Omega_\varepsilon} u^2(x)\,dx \le K_\varepsilon \int_{\Omega_\varepsilon} |\nabla u(x)|^2\,dx \tag{7.1}$$

holds, where the best constant K_ε has the asymptotics

$$K_\varepsilon = \frac{1}{k_0^2} + \frac{4\pi C(B)\left(\mathcal{J}_0'\right)^2(k_0)}{k_0^2}\varepsilon + o(\varepsilon), \tag{7.2}$$

as $\varepsilon \to 0$. Here k_0 is the smallest root of the Bessel function $\mathcal{J}_0$ and the constant $C(B)$ is given by (3.18).

Proof. The geometric approach developed in [5, 9] allows us to state that there is a constant $K > 0$ such that

$$\int_{\Omega_\varepsilon} u^2(x)\,dx \le K \int_{\Omega_\varepsilon} |\nabla u(x)|^2\,dx. \tag{7.3}$$

The idea and method of proof are exactly similar to the ones which were used in the mentioned papers. We are interested in the behavior of the best possible constant as $\varepsilon \to 0$. Clearly, the best constant $K_\varepsilon = 1/\lambda_\varepsilon^1$, where λ_ε^1 is the smallest eigenvalue of the boundary-value problem (2.2) (due to the variational formulation of the smallest eigenvalue). Therefore, we can apply (2.4) and (3.28) to derive the asymptotic expansion for K_ε:

$$K_\varepsilon = \left(\lambda_\varepsilon^1\right)^{-1} = \left(\lambda_0^1 + \varepsilon\lambda_1^1 + o(\varepsilon)\right)^{-1} = \frac{1}{\lambda_0^1} - \frac{2\lambda_1^1}{\left(\lambda_0^1\right)^2}\varepsilon + o(\varepsilon). \tag{7.4}$$

Since we are interested in the smallest eigenvalue λ_0^1, we have to choose the smallest positive root of $\mathcal{J}_0(k_0) = 0$ as k_0, precisely, $k_0 = 2,405$. Then, we get, after some simple calculations and using (4.15) and (3.28),

$$K_\varepsilon = \frac{1}{k_0^2} + \frac{4\pi C(B)\left(\mathcal{J}'_0\right)^2(k_0)}{k_0^2}\varepsilon + o(\varepsilon). \tag{7.5}$$

The proof is complete. □

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