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Research Article

Multiple Positive Solutions for First-Order Impulsive Integrodifferential Equations on the Half Line in Banach Spaces

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The author discusses the multiple positive solutions for an infinite boundary value problem of first-order impulsive superlinear integrodifferential equations on the half line in a Banach space by means of the fixed point theorem of cone expansion and compression with norm type.

1. Introduction

Let E be a real Banach space and P a cone in E which defines a partial ordering in E by $x \le y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$, where θ denotes the zero element of E and the smallest N is called the normal constant of P. If $x \le y$ and $x \ne y$, we write $x < y$. For details on cone theory, see [1].

In paper [2], we considered the infinite boundary value problem (IBVP) for first-order impulsive nonlinear integrodifferential equation of mixed type on the half line in E:

$$\begin{aligned} u'(t) &= f(t, u(t), (Tu)(t), (Su)(t)), \quad \forall t \in J', \\ \Delta u|_{t=t_k} &= I_k(u(t_k)) \quad (k = 1, 2, 3, \ldots), \\ u(\infty) &= \beta u(0), \end{aligned} \tag{1.1}$$

where $J = [0, \infty)$, $0 < t_1 < \cdots < t_k < \ldots, t_k \to \infty$, $J' = J \setminus \{t_1, \ldots, t_k, \ldots\}$, $f \in C[J \times P \times P \times P, P]$, $I_k \in C[P, P]$ $(k = 1, 2, 3, \ldots)$, $\beta > 1$, $u(\infty) = \lim_{t\to\infty} u(t)$, and

$$(Tu)(t) = \int_0^t K(t, s)u(s)ds, \qquad (Su)(t) = \int_0^\infty H(t, s)u(s)ds, \tag{1.2}$$

$K \in C[D, R_+]$, $D = \{(t,s) \in J \times J : t \geq s\}$, $H \in C[J \times J, R_+]$, R_+ denotes the set of all nonnegative numbers. $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, that is,

$$\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-), \tag{1.3}$$

where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. By using the fixed point index theory, we discussed the multiple positive solutions of IBVP(1.1). But the discussion dealt with sublinear equations, that is, we assume that there exists $c \in C[J, R_+] \cap L[J, R_+]$ such that

$$\frac{\|f(t,u,v,w)\|}{c(t)(\|u\| + \|v\| + \|w\|)} \longrightarrow 0 \quad \text{as } u,v,w \in P,\ \|u\| + \|v\| + \|w\| \longrightarrow \infty \tag{1.4}$$

uniformly for $t \in J$ (see condition (H_5) in [2]).

Now, in this paper, we discuss the multiple positive solutions of an infinite three-point boundary value problem (which includes IBVP(1.1) as a special case) for superlinear case by means of different method, that is, by using the fixed point theorem of cone expansion and compression with norm type, which was established by the author in [3] (see also [1]), and the key point is to introduce a new cone Q.

Consider the infinite three-point boundary value problem for first-order impulsive nonlinear integrodifferential equation of mixed type on the half line in E:

$$\begin{gathered} u'(t) = f(t, u(t), (Tu)(t), (Su)(t)), \quad \forall t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k)) \quad (k = 1,2,3,\ldots), \\ u(\infty) = \gamma u(\eta) + \beta u(0), \end{gathered} \tag{1.5}$$

where $0 \leq \gamma < 1$, $\beta + \gamma > 1$, and $t_{m-1} < \eta \leq t_m$ (for some m). It is clear that IBVP(1.5) includes IBVP(1.1) as a special case when $\gamma = 0$.

Let $\mathrm{PC}[J, E] = \{u : u$ is a map from J into E such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1,2,3,\ldots\}$ and $\mathrm{BPC}[J, E] = \{u \in \mathrm{PC}[J, E] : \sup_{t\in J}\|u(t)\| < \infty\}$. It is clear that $\mathrm{BPC}[J, E]$ is a Banach space with norm

$$\|u\|_B = \sup_{t\in J}\|u(t)\|. \tag{1.6}$$

Let $\mathrm{BPC}[J, P] = \{u \in \mathrm{BPC}[J, E] : u(t) \geq \theta,\ \forall t \in J\}$ and $Q = \{u \in \mathrm{BPC}[J, P] : u(t) \geq \beta^{-1}(1-\gamma)u(s),\ \forall t, s \in J\}$. Obviously, $\mathrm{BPC}[J, P]$ and Q are two cones in space $\mathrm{BPC}[J, E]$ and $Q \subset \mathrm{BPC}[J, P]$. $u \in \mathrm{BPC}[J, P] \cap C^1[J', E]$ is called a positive solution of IBVP(1.5) if $u(t) > \theta$ for $t \in J$ and $u(t)$ satisfies (1.5).

2. Several Lemmas

Let us list some conditions.

(H_1) $\sup_{t\in J}\int_0^t K(t,s)ds < \infty$, $\sup_{t\in J}\int_0^\infty H(t,s)ds < \infty$, and

$$\lim_{t'\to t}\int_0^\infty \left|H(t', s) - H(t, s)\right| ds = 0, \quad \forall t \in J. \tag{2.1}$$

In this case, let

$$k^* = \sup_{t\in J}\int_0^t K(t,s)ds, \qquad h^* = \sup_{t\in J}\int_0^\infty H(t,s)ds. \tag{2.2}$$

(H_2) There exist $a \in C[J,R_+]$ and $g \in C[R_+ \times R_+ \times R_+, R_+]$ such that

$$\begin{gathered}\|f(t,u,v,w)\| \leq a(t)g(\|u\|,\|v\|,\|w\|), \quad \forall t\in J,\ u,v,w\in P,\\ a^* = \int_0^\infty a(t)dt < \infty.\end{gathered} \tag{2.3}$$

(H_3) There exist $\gamma_k \geq 0$ $(k = 1,2,3,\ldots)$ and $F \in C[R_+, R_+]$ such that

$$\begin{gathered}\|I_k(u)\| \leq \gamma_k F(\|u\|), \quad \forall u \in P\ (k=1,2,3,\ldots),\\ \gamma^* = \sum_{k=1}^\infty \gamma_k < \infty.\end{gathered} \tag{2.4}$$

(H_4) For any $t \in J$ and $r > 0$, $f(t,P_r,P_r,P_r) = \{f(t,u,v,w) : u,v,w \in P_r\}$ and $I_k(P_r) = \{I_k(u) : u \in P_r\}$ $(k = 1,2,3,\ldots)$ are relatively compact in E, where $P_r = \{u \in P : \|u\| \leq r\}$.

Remark 2.1. Obviously, condition (H_4) is satisfied automatically when E is finite dimensional.

Remark 2.2. It is clear that if condition (H_1) is satisfied, then the operators T and S defined by (1.2) are bounded linear operators from $\mathrm{BPC}[J,E]$ into $\mathrm{BPC}[J,E]$ and $\|T\| \leq k^*$, $\|S\| \leq h^*$; moreover, we have $T(\mathrm{BPC}[J,P]) \subset \mathrm{BPC}[J,P]$ and $S(\mathrm{BPC}[J,P]) \subset \mathrm{BPC}[J,P]$.

We shall reduce IBVP(1.5) to an impulsive integral equation. To this end, we consider the operator A defined by

$$\begin{aligned}(Au)(t) = {} & \frac{1}{\beta+\gamma-1}\Bigg\{\int_\eta^\infty f(s,u(s),(Tu)(s),(Su)(s))ds + (1-\gamma)\\ & \times \int_0^\eta f(s,u(s),(Tu)(s),(Su)(s))ds + \sum_{k=m}^\infty I_k(u(t_k)) + (1-\gamma)\sum_{k=1}^{m-1} I_k(u(t_k))\Bigg\}\\ & + \int_0^t f(s,u(s),(Tu)(s),(Su)(s))ds + \sum_{0<t_k<t} I_k(u(t_k)), \quad \forall t \in J.\end{aligned} \tag{2.5}$$

In what follows, we write $J_1 = [0,t_1]$, $J_k = (t_{k-1}, t_k]$ $(k = 2,3,4,\ldots)$.

Lemma 2.3. *If conditions (H_1)–(H_4) are satisfied, then operator A defined by* (2.5) *is a completely continuous (i.e., continuous and compact) operator from $BPC[J,P]$ into Q.*

Proof. Let $r > 0$ be given. Let

$$M_r = \max\{g(x,y,z) : 0 \le x \le r,\ 0 \le y \le k^* r,\ 0 \le z \le h^* r\}, \tag{2.6}$$

$$N_r = \max\{F(x) : 0 \le x \le r\}. \tag{2.7}$$

For $u \in \mathrm{BPC}[J,P]$, $\|u\|_B \le r$, we see that by virtue of condition (H_2) and (2.6),

$$\|f(t,u(t),(Tu)(t),(Su)(t))\| \le M_r a(t), \quad \forall t \in J, \tag{2.8}$$

which implies the convergence of the infinite integral

$$\int_0^\infty f(t,u(t),(Tu)(t),(Su)(t))dt, \tag{2.9}$$

$$\left\|\int_0^\infty f(t,u(t),(Tu)(t),(Su)(t))dt\right\| \le \int_0^\infty \|f(t,u(t),(Tu)(t),(Su)(t))\|dt \le M_r a^*. \tag{2.10}$$

On the other hand, condition (H_3) and (2.7) imply the convergence of the infinite series

$$\sum_{k=1}^{\infty} I_k(u(t_k)), \tag{2.11}$$

$$\left\|\sum_{k=1}^{\infty} I_k(u(t_k))\right\| \le \sum_{k=1}^{\infty} \|I_k(u(t_k))\| \le N_r \gamma^*. \tag{2.12}$$

It follows from (2.5), (2.10), and (2.12) that

$$\begin{aligned}\|(Au)(t)\| \le \frac{1}{\beta+\gamma-1}\Bigg\{&\int_\eta^\infty \|f(s,u(s),(Tu)(s),(Su)(s))\|ds + (1-\gamma)\\ &\times \int_0^\eta \|f(s,u(s),(Tu)(s),(Su)(s))\|ds\\ &+\sum_{k=m}^{\infty}\|I_k(u(t_k))\| + (1-\gamma)\sum_{k=1}^{m-1}\|I_k(u(t_k))\|\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \|f(s,u(s),(Tu)(s),(Su)(s))\|ds+\sum_{0<t_k<t}\|I_k(u(t_k))\| \\
&\leq \frac{1}{\beta+\gamma-1}\left\{\int_0^\infty \|f(s,u(s),(Tu)(s),(Su)(s))\|ds+\sum_{k=1}^{\infty}\|I_k(u(t_k))\|\right\} \\
&+\int_0^\infty \|f(s,u(s),(Tu)(s),(Su)(s))\|ds+\sum_{k=1}^{\infty}\|I_k(u(t_k))\| \\
&=\frac{\beta+\gamma}{\beta+\gamma-1}\left\{\int_0^\infty \|f(s,u(s),(Tu)(s),(Su)(s))\|ds+\sum_{k=1}^{\infty}\|I_k(u(t_k))\|\right\} \\
&\leq \frac{\beta+\gamma}{\beta+\gamma-1}(M_r a^*+N_r\gamma^*), \quad \forall t\in J,
\end{aligned}
\tag{2.13}
$$

which implies that $Au\in \mathrm{BPC}[J,P]$ and

$$
\|Au\|_B \leq \frac{\beta+\gamma}{\beta+\gamma-1}(M_r a^*+N_r\gamma^*). \tag{2.14}
$$

Moreover, by (2.5), we have

$$
\begin{aligned}
(Au)(t) \geq \frac{1}{\beta+\gamma-1}\Bigg\{&\int_\eta^\infty f(s,u(s),(Tu)(s),(Su)(s))ds+(1-\gamma) \\
&\times\int_0^\eta f(s,u(s),(Tu)(s),(Su)(s))ds+\sum_{k=m}^{\infty}I_k(u(t_k))+(1-\gamma)\sum_{k=1}^{m-1}I_k(u(t_k))\Bigg\}, \\
&\forall t\in J,
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
(Au)(t) \leq \frac{1}{\beta+\gamma-1}\Bigg\{&\int_\eta^\infty f(s,u(s),(Tu)(s),(Su)(s))ds+(1-\gamma) \\
&\times\int_0^\eta f(s,u(s),(Tu)(s),(Su)(s))ds \\
&+\sum_{k=m}^{\infty}I_k(u(t_k))+(1-\gamma)\sum_{k=1}^{m-1}I_k(u(t_k))\Bigg\} \\
&+\int_0^\infty f(s,u(s),(Tu)(s),(Su)(s))ds+\sum_{k=1}^{\infty}I_k(u(t_k)), \quad \forall t\in J.
\end{aligned}
\tag{2.16}
$$

It is clear that

$$\int_0^\infty f(s,u(s),(Tu)(s),(Su)(s))ds + \sum_{k=1}^{\infty} I_k(u(t_k))$$
$$\leq \frac{1}{1-\gamma}\left\{\int_\eta^\infty f(s,u(s),(Tu)(s),(Su)(s))ds + (1-\gamma)\int_0^\eta f(s,u(s),(Tu)(s),(Su)(s))ds \right.$$
$$\left. +\sum_{k=m}^{\infty} I_k(u(t_k)) + (1-\gamma)\sum_{k=1}^{m-1} I_k(u(t_k))\right\}, \tag{2.17}$$

so, (2.16) and (2.17) imply

$$(Au)(t) \leq \left\{\frac{1}{\beta+\gamma-1} + \frac{1}{1-\gamma}\right\}$$
$$\times\left\{\int_\eta^\infty f(s,u(s),(Tu)(s),(Su)(s))ds + (1-\gamma)\int_0^\eta f(s,u(s),(Tu)(s),(Su)(s))ds \right.$$
$$\left. +\sum_{k=m}^{\infty} I_k(u(t_k)) + (1-\gamma)\sum_{k=1}^{m-1} I_k(u(t_k))\right\}, \quad \forall t \in J. \tag{2.18}$$

It follows from (2.15) and (2.18) that

$$(Au)(t) \geq \frac{1}{\beta+\gamma-1}\left(\frac{1}{\beta+\gamma-1} + \frac{1}{1-\gamma}\right)^{-1}(Au)(s) = \beta^{-1}(1-\gamma)(Au)(s), \quad \forall t,s \in J. \tag{2.19}$$

Hence, $Au \in Q$. That is, A maps $\mathrm{BPC}[J,P]$ into Q.

Now, we are going to show that A is continuous. Let $u_n, \overline{u} \in \mathrm{BPC}[J,P]$, $\|u_n - \overline{u}\|_B \to 0$ $(n \to \infty)$. Then $r = \sup_n \|u_n\|_B < \infty$ and $\|\overline{u}\|_B \leq r$. Similar to (2.14), it is easy to get

$$\|Au_n - A\overline{u}\|_B$$
$$\leq \frac{\beta+\gamma}{\beta+\gamma-1}\left\{\int_0^\infty \left\| f(s,u_n(s),(Tu_n)(s),(Su_n)(s)) - f(s,\overline{u}(s),(T\overline{u})(s),(S\overline{u})(s))ds\right\| \right.$$
$$\left. +\sum_{k=1}^{\infty}\|I_k(u_n(t_k)) - I_k(\overline{u}(t_k))\|\right\} \quad (n = 1,2,3,\ldots). \tag{2.20}$$

It is clear that

$$f(t,u_n(t),(Tu_n)(t),(Su_n)(t)) \longrightarrow f(t,\overline{u}(t),(T\overline{u})(t),(S\overline{u})(t)) \quad \text{as } n \longrightarrow \infty,\ \forall t \in J. \tag{2.21}$$

Moreover, we see from (2.8) that

$$\begin{aligned}&\left\| f(t,u_n(t),(Tu_n)(t),(Su_n)(t)) - f(t,\overline{u}(t),(T\overline{u})(t),(S\overline{u})(t)) \right\| \\ &\quad \le 2M_r a(t) = \sigma(t), \quad \forall t \in J \ (n = 1,2,3,\ldots);\ \sigma \in L[J,R_+].\end{aligned} \tag{2.22}$$

It follows from (2.21), (2.22) and the dominated convergence theorem that

$$\lim_{n\to\infty}\int_0^{\infty} \left\| f(t,u_n(t),(Tu_n)(t),(Su_n)(t)) - f(t,\overline{u}(t),(T\overline{u})(t),(S\overline{u})(t)) \right\| dt = 0. \tag{2.23}$$

On the other hand, for any $\epsilon > 0$, we can choose a positive integer j such that

$$N_r \sum_{k=j+1}^{\infty} \gamma_k < \epsilon. \tag{2.24}$$

And then, choose a positive integer n_0 such that

$$\sum_{k=1}^{j} \|I_k(u_n(t_k)) - I_k(\overline{u}(t_k))\| < \epsilon, \quad \forall n > n_0. \tag{2.25}$$

From (2.24) and (2.25), we get

$$\sum_{k=1}^{\infty} \|I_k(u_n(t_k)) - I_k(\overline{u}(t_k))\| < \epsilon + 2N_r \sum_{k=j+1}^{\infty} \gamma_k < 3\epsilon, \quad \forall n > n_0, \tag{2.26}$$

hence

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \|I_k(u_n(t_k)) - I_k(\overline{u}(t_k))\| = 0. \tag{2.27}$$

It follows from (2.20), (2.23), and (2.51) that $\|Au_n - A\overline{u}\|_B \to 0$ as $n \to \infty$, and the continuity of A is proved.

Finally, we prove that A is compact. Let $V = \{u_n\} \subset \mathrm{BPC}[J,P]$ be bounded and $\|u_n\|_B \le r$ $(n = 1,2,3,\ldots)$. Consider $J_i = (t_{i-1}, t_i]$ for any fixed i. By (2.5) and (2.8), we have

$$\begin{aligned}\left\| (Au_n)(t') - (Au_n)(t) \right\| &\le \int_t^{t'} \left\| f(s,u_n(s),(Tu_n)(s),(Su_n)(s)) \right\| ds \\ &\le M_r \int_t^{t'} a(s)ds, \quad \forall t,\ t' \in J_i,\ t' > t \ (n = 1,2,3,\ldots),\end{aligned} \tag{2.28}$$

which implies that the functions $\{w_n(t)\}$ $(n = 1,2,3,\ldots)$ defined by

$$w_n(t) = \begin{cases} (Au_n)(t), & \forall t \in J_i = (t_{i-1}, t_i], \\ (Au_n)(t_{i-1}^+), & \forall t = t_{i-1} \end{cases} \quad (n = 1,2,3,\ldots) \tag{2.29}$$

($(Au_n)(t_{i-1}^+)$ denotes the right limit of $(Au_n)(t)$ at $t = t_{i-1}$) are equicontinuous on $\overline{J}_i = [t_{i-1}, t_i]$. On the other hand, for any $\epsilon > 0$, choose a sufficiently large $\tau > \eta$ and a sufficiently large positive integer $j > m$ such that

$$M_r \int_\tau^\infty a(s)ds < \epsilon, \qquad N_r \sum_{k=j+1}^\infty \gamma_k < \epsilon. \tag{2.30}$$

We have, by (2.29), (2.5), (2.8), (2.30), and condition (H_3),

$$\begin{aligned} w_n(t) &= \frac{1}{\beta+\gamma-1}\Bigg\{ \int_\eta^\tau f(s,u_n(s),(Tu_n)(s),(Su_n)(s))ds \\ &\qquad + \int_\tau^\infty f(s,u_n(s),(Tu_n)(s),(Su_n)(s)ds) + (1-\gamma) \\ &\qquad \times \int_0^\eta f(s,u_n(s),(Tu_n)(s),(Su_n)(s))ds + \sum_{k=m}^{j} I_k(u_n(t_k)) + \sum_{k=j+1}^\infty I_k(u_n(t_k)) \\ &\qquad + (1-\gamma)\sum_{k=1}^{m-1} I_k(u_n(t_k)) \Bigg\} \\ &\quad + \int_0^t f(s,u_n(s),(Tu_n)(s),(Su_n)(s))ds \\ &\quad + \sum_{k=1}^{i-1} I_k(u_n(t_k)), \quad \forall t \in \overline{J}_i \ (n = 1,2,3,\ldots), \end{aligned} \tag{2.31}$$

$$\left\| \int_\tau^\infty f(s,u_n(s),(Tu_n)(s),(Su_n)(s))ds \right\| < \epsilon \quad (n = 1,2,3,\ldots), \tag{2.32}$$

$$\left\| \sum_{k=j+1}^\infty I_k(u_n(t_k)) \right\| < \epsilon \quad (n = 1,2,3,\ldots). \tag{2.33}$$

It follows from (2.31), (2.32), (2.33), (2.8), and [4, Theorem 1.2.3] that

$$\begin{aligned}\alpha(W(t)) \leq \frac{1}{\beta+\gamma-1}\Bigg\{ & 2\int_{\eta}^{\tau} \alpha\big(f(s,V(s),(TV)(s),(SV)(s))\big)ds + 2\epsilon \\ & + 2(1-\gamma)\int_{0}^{\eta} \alpha\big(f(s,V(s),(TV)(s),(SV)(s))\big)ds \\ & + \sum_{k=m}^{j}\alpha(I_k(V(t_k))) + 2\epsilon + (1-\gamma)\sum_{k=1}^{m-1}\alpha(I_k(V(t_k)))\Bigg\} \\ & + 2\int_{0}^{t}\alpha\big(f(s,V(s),(TV)(s),(SV)(s))\big)ds + \sum_{k=1}^{i-1}\alpha(I_k(V(t_k))), \quad \forall t \in \overline{J}_i,\end{aligned} \tag{2.34}$$

where $W(t) = \{w_n(t) : n = 1,2,3,\ldots\}$, $V(s) = \{u_n(s) : n = 1,2,3,\ldots\}$, $(TV)(s) = \{(Tu_n)(s) : n=1,2,3,\ldots\}$, $(SV)(s) = \{(Su_n)(s) : n = 1,2,3,\ldots\}$ and $\alpha(U)$ denotes the Kuratowski measure of noncompactness of bounded set $U \subset E$ (see [4, Section 1.2]). Since $V(s), (TV)(s), (SV)(s) \subset P_{r^*}$ for $s \in J$, where $r^* = \max\{r, k^*r, h^*r\}$, we see that, by condition (H_4),

$$\alpha\big(f(s,V(s),(TV)(s),(SV)(s))\big) = 0, \quad \forall t \in J, \tag{2.35}$$

$$\alpha(I_k(V(t_k))) = 0 \quad (k = 1,2,3,\ldots). \tag{2.36}$$

It follows from (2.34)–(2.36) that

$$\alpha(W(t)) \leq \frac{4\epsilon}{\beta+\gamma-1}, \quad \forall t \in \overline{J}_i, \tag{2.37}$$

which implies by virtue of the arbitrariness of ϵ that $\alpha(W(t)) = 0$ for $t \in \overline{J}_i$.

By Ascoli-Arzela theorem (see [4, Theorem 1.2.5]), we conclude that $W = \{w_n : n = 1,2,3,\ldots\}$ is relatively compact in $C[\overline{J}_i, E]$; hence, $\{w_n(t)\}$ has a subsequence which is convergent uniformly on $\overline{J}_i$, so, $\{(Au_n(t)\}$ has a subsequence which is convergent uniformly on J_i. Since i may be any positive integer, so, by diagonal method, we can choose a subsequence $\{(Au_{n_i})(t)\}$ of $\{(Au_n)(t)\}$ such that $\{(Au_{n_i})(t)\}$ is convergent uniformly on each J_k $(k = 1,2,3,\ldots)$. Let

$$\lim_{i\to\infty}(Au_{n_i})(t) = v(t), \quad \forall t \in J. \tag{2.38}$$

It is clear that $v \in \mathrm{PC}[J,P]$. By (2.14), we have

$$\|Au_{n_i}\|_B \leq \frac{\beta+\gamma}{\beta+\gamma-1}(M_r a^* + N_r \gamma^*), \quad (i = 1,2,3,\ldots), \tag{2.39}$$

which implies that $v \in \mathrm{BPC}[J, P]$ and

$$\|v\|_B \leq \frac{\beta + \gamma}{\beta + \gamma - 1}(M_r a^* + N_r \gamma^*). \tag{2.40}$$

Let $\epsilon > 0$ be arbitrarily given and choose a sufficiently large positive number τ such that

$$M_r \int_\tau^\infty a(s)ds + N_r \sum_{t_k \geq \tau} \gamma_k < \epsilon. \tag{2.41}$$

For any $\tau < t < \infty$, we have, by (2.5),

$$\begin{aligned}(Au_{n_i})(t) - (Au_{n_i})(\tau) = & \int_\tau^t f(s, u_{n_i}(s), (Tu_{n_i})(s), (Su_{n_i})(s))ds \\ & + \sum_{\tau \leq t_k < t} I_k(u_{n_i}(t)), \quad (i = 1, 2, 3, \ldots),\end{aligned} \tag{2.42}$$

which implies by virtue of (2.8), condition (H_3) and (2.41) that

$$\|(Au_{n_i})(t) - (Au_{n_i})(\tau)\| \leq M_r \int_\tau^t a(s)ds + N_r \sum_{\tau \leq t_k < t} \gamma_k < \epsilon, \quad (i = 1, 2, 3, \ldots). \tag{2.43}$$

Letting $i \to \infty$ in (2.43), we get

$$\|v(t) - v(\tau)\| \leq \epsilon, \quad \forall t > \tau. \tag{2.44}$$

On the other hand, since $\{(Au_{n_i})(t)\}$ converges uniformly to $v(t)$ on $[0, \tau]$ as $i \to \infty$, there exists a positive integer i_0 such that

$$\|(Au_{n_i})(t) - v(t)\| < \epsilon, \quad \forall t \in [0, \tau], \ i > i_0. \tag{2.45}$$

It follows from (2.43)–(2.45) that

$$\begin{aligned}\|(Au_{n_i})(t) - v(t)\| & \leq \|(Au_{n_i})(t) - (Au_{n_i})(\tau)\| + \|(Au_{n_i})(\tau) - v(\tau)\| + \|v(\tau) - v(t)\| \\ & < 3\epsilon, \quad \forall t > \tau, \ i > i_0.\end{aligned} \tag{2.46}$$

By (2.45) and (2.46), we have

$$\|Au_{n_i} - v\|_B \leq 3\epsilon, \quad \forall i > i_0, \tag{2.47}$$

hence $\|Au_{n_i} - v\|_B \to 0$ as $i \to \infty$, and the compactness of A is proved. □

Lemma 2.4. *Let conditions (H_1)–(H_4) be satisfied. Then $u \in BPC[J, P] \cap C^1[J', E]$ is a solution of* IBVP(1.5) *if and only if $u \in Q$ is a solution of the following impulsive integral equation:*

$$u(t) = \frac{1}{\beta+\gamma-1}\left\{\int_\eta^\infty f(s,u(s),(Tu)(s),(Su)(s))ds + (1-\gamma)\right.$$
$$\left.\times\int_0^\eta f(s,u(s),(Tu)(s),(Su)(s))ds + \sum_{k=m}^{\infty} I_k(u(t_k)) + (1-\gamma)\sum_{k=1}^{m-1} I_k(u(t_k))\right\}$$
$$+\int_0^t f(s,u(s),(Tu)(s),(Su)(s))ds + \sum_{0<t_k<t} I_k(u(t_k)), \quad \forall t \in J. \tag{2.48}$$

that is, u is a fixed point of operator A defined by (2.5) *in Q.*

Proof. For $u \in \mathrm{PC}[J, E] \cap C^1[J', E]$, it is easy to get the following formula:

$$u(t) = u(0) + \int_0^t u'(s)ds + \sum_{0<t_k<t}\left[u(t_k^+) - u(t_k)\right], \quad \forall t \in J. \tag{2.49}$$

Let $u \in \mathrm{BPC}[J, P] \cap C^1[J', E]$ be a solution of IBVP(1.5). By (1.5) and (2.49), we have

$$u(t) = u(0) + \int_0^t f(s,u(s),(Tu)(s),(Su)(s))ds + \sum_{0<t_k<t} I_k(u(t_k)), \quad \forall t \in J. \tag{2.50}$$

We have shown in the proof of Lemma 2.3 that the infinite integral (2.9) and the infinite series (2.11) are convergent, so, by taking limits as $t \to \infty$ in both sides of (2.50), we get

$$u(\infty) = u(0) + \int_0^\infty f(s,u(s),(Tu)(s),(Su)(s))ds + \sum_{k=1}^{\infty} I_k(u(t_k)). \tag{2.51}$$

On the other hand, by (1.5) and (2.50), we have

$$u(\infty) = \gamma u(\eta) + \beta u(0), \tag{2.52}$$

$$u(\eta) = u(0) + \int_0^\eta f(s,u(s),(Tu)(s),(Su)(s))ds + \sum_{k=1}^{m-1} I_k(u(t_k)). \tag{2.53}$$

It follows from (2.51)–(2.53) that

$$u(0)=\frac{1}{\beta+\gamma-1}\left\{\int_{\eta}^{\infty}f(s,u(s),(Tu)(s),(Su)(s))ds+(1-\gamma)\int_{0}^{\eta}f(s,u(s),(Tu)(s),(Su)(s))ds\right.$$
$$\left.+\sum_{k=m}^{\infty}I_k(u(t_k))+(1-\gamma)\sum_{k=1}^{m-1}I_k(u(t_k))\right\},\tag{2.54}$$

and, substituting it into (2.50), we see that $u(t)$ satisfies (2.48), that is, $u=Au$. Since $Au\in Q$ by virtue of Lemma 2.3, we conclude that $u\in Q$.

Conversely, assume that $u\in Q$ is a solution of (2.48). We have, by (2.48),

$$u(0)=\frac{1}{\beta+\gamma-1}\left\{\int_{\eta}^{\infty}f(s,u(s),(Tu)(s),(Su)(s))ds+(1-\gamma)\int_{0}^{\eta}f(s,u(s),(Tu)(s),(Su)(s))ds\right.$$
$$\left.+\sum_{k=m}^{\infty}I_k(u(t_k))+(1-\gamma)\sum_{k=1}^{m-1}I_k(u(t_k))\right\},\tag{2.55}$$

$$u(\eta)=\frac{1}{\beta+\gamma-1}\left\{\int_{\eta}^{\infty}f(s,u(s),(Tu)(s),(Su)(s))ds+(1-\gamma)\right.$$
$$\left.\times\int_{0}^{\eta}f(s,u(s),(Tu)(s),(Su)(s))ds+\sum_{k=m}^{\infty}I_k(u(t_k))+(1-\gamma)\sum_{k=1}^{m-1}I_k(u(t_k))\right\}$$
$$+\int_{0}^{\eta}f(s,u(s),(Tu)(s),(Su)(s))ds+\sum_{k=1}^{m-1}I_k(u(t_k)).\tag{2.56}$$

Moreover, by taking limits as $t\to\infty$ in (2.33), we see that $u(\infty)$ exists and

$$u(\infty)=\frac{1}{\beta+\gamma-1}\left\{\int_{\eta}^{\infty}f(s,u(s),(Tu)(s),(Su)(s))ds+(1-\gamma)\int_{0}^{\eta}f(s,u(s),(Tu)(s),(Su)(s))ds\right.$$
$$\left.+\sum_{k=m}^{\infty}I_k(u(t_k))+(1-\gamma)\sum_{k=1}^{m-1}I_k(u(t_k))\right\}$$
$$+\int_{0}^{\infty}f(s,u(s),(Tu)(s),(Su)(s))ds+\sum_{k=1}^{\infty}I_k(u(t_k)).\tag{2.57}$$

It follows from (2.55)–(2.57) that

$$\gamma u(\eta) + \beta u(0) = u(\infty). \tag{2.58}$$

On the other hand, direct differentiation of (2.48) gives

$$u'(t) = f(t, u(t), (Tu)(t), (Su)(t)), \quad \forall t \in J', \tag{2.59}$$

and, it is clear, by (2.48),

$$\Delta u|_{t=t_k} = I_k(u(t_k)) \quad (k = 1, 2, 3, \ldots). \tag{2.60}$$

Hence, $u \in C^1[J', E]$ and $u(t)$ satisfies (1.5). □

Corollary 2.5. *Let cone P be normal. If u is a fixed point of operator A defined by* (1.5) *in Q and $\|u\|_B > 0$, then $u(t) > \theta$ for $t \in J$, so, u is a positive solution of* IBVP(1.5).

Proof. For $u \in Q$, we have

$$u(t) \geq \beta^{-1}(1 - \gamma)u(s) \geq \theta, \quad \forall t,\ s \in J, \tag{2.61}$$

so,

$$\|u(t)\| \geq N^{-1}\beta^{-1}(1 - \gamma)\|u\|_B, \quad \forall t \in J, \tag{2.62}$$

where N denotes the normal constant of P. Since $\|u\|_B > 0$, (2.61) and (2.62) imply that $u(t) > \theta$ for $t \in J$. □

Lemma 2.6 (Fixed point theorem of cone expansion and compression with norm type, see [3, Theorem 3] or [1, Theorem 2.3.4]). *Let P be a cone in real Banach space E and Ω_1, Ω_2 two bounded open sets in E such that $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, where θ denotes the zero element of E and $\overline{\Omega_2}$ denotes the closure of Ω_2. Let operator $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be completely continuous. Suppose that one of the following two conditions is satisfied:*

(a)

$$\|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_1; \qquad \|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_2, \tag{2.63}$$

where $\partial\Omega_i$ denotes the boundary of Ω_i ($i = 1, 2$).

(b)

$$\|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \qquad \|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_2. \tag{2.64}$$

Then A has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Main Theorems

Let us list more conditions.

(H_5) There exist $u_0 \in P \setminus \{\theta\}$, $b \in C[J, R_+]$, and $\tau \in C[P, R_+]$ such that

$$f(t,u,v,w) \geq b(t)\tau(u)u_0, \quad \forall t \in J,\ u,v,w \in P,$$
$$\frac{\tau(u)}{\|u\|} \longrightarrow \infty \quad \text{as } u \in P,\ \|u\| \longrightarrow \infty, \tag{3.1}$$
$$b^* = \int_0^\infty b(t)dt < \infty.$$

Remark 3.1. Condition (H_5) means that $f(t,u,v,w)$ is superlinear with respect to u.

(H_6) There exist $u_1 \in P \setminus \{\theta\}$, $c \in C[J, R_+]$, and $\sigma \in C[P, R_+]$ such that

$$f(t,u,v,w) \geq c(t)\sigma(u)u_1, \quad \forall t \in J,\ u,v,w \in P,$$
$$\frac{\sigma(u)}{\|u\|} \longrightarrow \infty \quad \text{as } u \in P,\ \|u\| \longrightarrow 0, \tag{3.2}$$
$$c^* = \int_0^\infty c(t)dt < \infty.$$

Theorem 3.2. *Let cone P be normal and conditions (H_1)–(H_6) satisfied. Assume that there exists a $\xi > 0$ such that*

$$\frac{N(\beta+\gamma)}{\beta+\gamma-1}\left(M_\xi a^* + N_\xi \gamma^*\right) < \xi, \tag{3.3}$$

where

$$M_\xi = \max\{g(x,y,z) : 0 \leq x \leq \xi,\ 0 \leq y \leq k^*\xi,\ 0 \leq z \leq h^*\xi\},$$
$$N_\xi = \max\{F(x) : 0 \leq x \leq \xi\}. \tag{3.4}$$

(for $g(x,y,z), F(x), a^*$ and γ^*, see conditions (H_2) and (H_3)). Then IBVP(1.5) *has at least two positive solutions $u^*, u^{**} \in Q \cap C^1[J', E]$ such that $0 < \|u^*\|_B < \xi < \|u^{**}\|_B$.*

Proof. By Lemmas 2.3, 2.4, and Corollary 2.5, operator A defined by (2.5) is completely continuous from Q into Q, and we need to prove that A has two fixed points u^* and u^{**} in Q such that $0 < \|u^*\|_B < \xi < \|u^{**}\|_B$.

By condition (H_5), there exists an $r_1 > 0$ such that

$$\tau(u) \geq \frac{\beta(\beta+\gamma-1)N^2}{(1-\gamma)^2 b^* \|u_0\|}\|u\|, \quad \forall u \in P,\ \|u\| \geq r_1, \tag{3.5}$$

where N denotes the normal constant of P, so,

$$f(t,u,v,w) \geq \frac{\beta(\beta+\gamma-1)N^2\|u\|}{(1-\gamma)^2 b^* \|u_0\|} b(t)u_0, \quad \forall t \in J,\ u,v,w \in P,\ \|u\| \geq r_1. \tag{3.6}$$

Choose

$$r_2 > \max\left\{N\beta(1-\gamma)^{-1}r_1, \xi\right\}. \tag{3.7}$$

For $u \in Q$, $\|u\|_B = r_2$; we have by (2.62) and (3.7),

$$\|u(t)\| \geq N^{-1}\beta^{-1}(1-\gamma)\|u\|_B = N^{-1}\beta^{-1}(1-\gamma)r_2 > r_1, \quad \forall t \in J, \tag{3.8}$$

so, (2.5), (3.8), (3.6), and (2.62) imply

$$\begin{aligned}
(Au)(t) &\geq \frac{1-\gamma}{\beta+\gamma-1}\left(\int_0^\infty f(s,u(s),(Tu)(s),(Su)(s))ds\right) \\
&\geq \frac{\beta N^2}{(1-\gamma)b^*\|u_0\|}\left(\int_0^\infty \|u(s)\|b(s)ds\right)u_0 \\
&\geq \frac{N\|u\|_B}{b^*\|u_0\|}\left(\int_0^\infty b(s)ds\right)u_0 = \frac{N\|u\|_B}{\|u_0\|}u_0, \quad \forall t \in J,
\end{aligned} \tag{3.9}$$

and consequently,

$$\|Au\|_B \geq \|u\|_B, \quad \forall u \in Q,\ \|u\|_B = r_2. \tag{3.10}$$

Similarly, by condition (H_6), there exists $r_3 > 0$ such that

$$\sigma(u) \geq \frac{\beta(\beta+\gamma-1)N^2}{(1-\gamma)^2 c^* \|u_1\|}\|u\|, \quad \forall u \in P,\ 0 < \|u\| < r_3, \tag{3.11}$$

so,

$$f(t,u,v,w) \geq \frac{\beta(\beta+\gamma-1)N^2\|u\|}{(1-\gamma)^2 c^* \|u_1\|} c(t)u_1, \quad \forall t \in J,\ u,v,w \in P,\ 0 < \|u\| < r_3. \tag{3.12}$$

Choose

$$0 < r_4 < \min\{r_3, \xi\}. \tag{3.13}$$

For $u \in Q$, $\|u\|_B = r_4$, we have by (3.13) and (2.62),

$$r_3 > \|u(t)\| \geq N^{-1}\beta^{-1}(1-\gamma)\|u\|_B = N^{-1}\beta^{-1}(1-\gamma)r_4 > 0, \tag{3.14}$$

so, similar to (3.9), we get by (2.5), (3.12), and (3.14)

$$\begin{aligned}(Au)(t) &\geq \frac{1-\gamma}{\beta+\gamma-1}\left(\int_0^{\infty} f(s,u(s),(Tu)(s),(Su)(s))ds\right)\\ &\geq \frac{\beta N^2}{(1-\gamma)c^*\|u_1\|}\left(\int_0^{\infty}\|u(s)\|c(s)ds\right)u_1\\ &\geq \frac{N\|u\|_B}{c^*\|u_1\|}\left(\int_0^{\infty}c(s)ds\right)u_1 = \frac{N\|u\|_B}{\|u_1\|}u_1, \quad \forall t\in J;\end{aligned} \tag{3.15}$$

hence

$$\|Au\|_B \geq \|u\|_B, \quad \forall u\in Q,\ \|u\| = r_4. \tag{3.16}$$

On the other hand, for $u\in Q$, $\|u\|_B = \xi$, by condition (H_2), condition (H_3), (3.4), we have

$$\left\|f(t,u(t),(Tu)(t),(Su)(t))\right\| \leq M_{\xi}a(t), \quad \forall t\in J, \tag{3.17}$$

$$\|I_k(u(t_k))\| \leq N_{\xi}\gamma_k \quad (k=1,2,3,\ldots). \tag{3.18}$$

It is clear that

$$(Au)(t) \leq \frac{\beta+\gamma}{\beta+\gamma-1}\left(\int_0^{\infty} f(s,u(s),(Tu)(s),(Su)(s))ds + \sum_{k=1}^{\infty} I_k(u(t_k))\right) \quad \forall t\in J. \tag{3.19}$$

It follows from (3.17)–(3.19) that

$$\|Au\|_B \leq \frac{N(\beta+\gamma)}{\beta+\gamma-1}\left(M_{\xi}a^* + N_{\xi}\gamma^*\right). \tag{3.20}$$

Thus, (3.20) and (3.3) imply

$$\|Au\|_B < \|u\|_B, \quad \forall u\in Q,\ \|u\|_B = \xi. \tag{3.21}$$

From (3.7) and (3.13), we know $0 < r_4 < \xi < r_2$; hence, (3.10), (3.16), (3.21), and Lemma 2.6 imply that A has two fixed points $u^*, u^{**} \in Q$ such that $r_4 < \|u^*\|_B < \xi < \|u^{**}\|_B < r_2$. The proof is complete. □

Theorem 3.3. *Let cone P be normal and conditions (H_1)–(H_5) satisfied. Assume that*

$$\frac{g(x,y,z)}{x+y+z} \longrightarrow 0 \quad \text{as } x+y+z \longrightarrow 0^+,$$
$$\frac{F(x)}{x} \longrightarrow 0 \quad \text{as } x \longrightarrow 0^+. \tag{3.22}$$

(for $g(x,y,z)$ and $F(x)$, see conditions (H_2) and (H_3)). Then IBVP(1.5) *has at least one positive solution $u^* \in Q \cap C^1[J', E]$.*

Proof. As in the proof of Theorem 3.2, we can choose $r_2 > 0$ such that (3.10) holds (in this case, we only choose $r_2 > N\beta(1-\gamma)^{-1}r_1$ instead of (3.7)). On the other hand, by (3.22), there exists $r_5 > 0$ such that

$$g(x,y,z) \leq \epsilon_0(x+y+z), \quad \forall 0 < x+y+z < r_5,$$
$$F(x) \leq \epsilon_0 x, \quad \forall 0 < x < r_5, \tag{3.23}$$

where

$$\epsilon_0 = \frac{\beta+\gamma-1}{N(\beta+\gamma)\left[(1+k^*+h^*)a^*+\gamma^*\right]}. \tag{3.24}$$

Choose

$$0 < r_6 < \min\left\{\frac{r_5}{1+k^*+h^*},\ r_2\right\}. \tag{3.25}$$

For $u \in Q$, $\|u\|_B = r_6$, we have by (2.62) and (3.25),

$$0 < N^{-1}\beta^{-1}(1-\gamma)r_6 \leq \|u(t)\| \leq r_6 < r_5, \quad \forall t \in J,$$
$$0 < N^{-1}\beta^{-1}(1-\gamma)r_6 \leq \|u(t)\| + \|(Tu)(t)\| + \|(Su)(t)\| \leq (1+k^*+h^*)r_6 < r_5, \quad \forall t \in J, \tag{3.26}$$

so, (3.23) imply

$$\begin{aligned} g(\|u(t)\|, \|(Tu)(t)\|, \|(Su)(t)\|) &\leq \epsilon_0(\|u(t)\| + \|(Tu)(t)\| + \|(Su)(t)\|) \\ &\leq \epsilon_0(1+k^*+h^*)r_6, \quad \forall t \in J, \end{aligned} \tag{3.27}$$
$$F(\|u(t_k)\|) \leq \epsilon_0\|u(t_k)\| \leq \epsilon_0 r_6, \quad (k = 1,2,3,\ldots).$$

It follows from (3.19), condition (H_2), condition (H_3), (3.27), and (3.24) that

$$\begin{aligned}\|(Au)(t)\| &\leq \frac{N(\beta+\gamma)}{\beta+\gamma-1}\left\{\epsilon_0(1+k^*+h^*)r_6\int_0^\infty a(s)ds+\epsilon_0 r_6\sum_{k=1}^\infty \gamma_k\right\}\\ &= \frac{N(\beta+\gamma)\epsilon_0 r_6}{\beta+\gamma-1}\{(1+k^*+h^*)a^*+\gamma^*\}=r_6, \quad \forall t\in J,\end{aligned} \tag{3.28}$$

and consequently,

$$\|Au\|_B \leq \|u\|_B, \quad \forall u\in Q,\ \|u\|_B = r_6. \tag{3.29}$$

Since $0 < r_6 < r_2$ by virtue of (3.25), we conclude from (3.10), (3.29), and Lemma 2.6 that A has a fixed point $u^* \in Q$ such that $r_6 \leq \|u^*\|_B \leq r_2$. The theorem is proved. □

Example 3.4. Consider the infinite system of scalar first-order impulsive integrodifferential equations of mixed type on the half line:

$$\begin{aligned}u_n'(t) &= \frac{1}{8n^2}e^{-5t}\left(\left[u_{n+1}(t)+\sum_{m=1}^\infty u_m(t)\right]^2+\sqrt{3u_{2n}(t)+\sum_{m=1}^\infty u_m(t)}\right)\\ &\quad+\frac{1}{9n^3}e^{-6t}\left\{\left(\int_0^t e^{-(t+1)s}u_n(s)ds\right)^2+\left(\int_0^\infty \frac{u_{n+2}(s)ds}{(1+t+s)^2}\right)^3\right\},\\ &\qquad \forall 0\leq t<\infty,\ t\neq k\ (k=1,2,3,\ldots;\ n=1,2,3,\ldots),\\ \Delta u_n|_{t=k} &= \frac{1}{6n^2}3^{-k}\left([u_n(k)]^2+[u_{n+2}(k)]^2\right), \quad (k=1,2,3,\ldots;\ n=1,2,3,\ldots),\\ u(\infty) &= \frac{1}{2}u_n\left(\frac{9}{2}\right)+6u_n(0), \quad (n=1,2,3,\ldots).\end{aligned} \tag{3.30}$$

Evidently, $u_n(t)\equiv 0 \ \ (n=1,2,3,\ldots)$ is the trivial solution of infinite system (3.30).

Conclusion. Infinite system (3.30) has at least two positive solutions $\{u_n^*(t)\}$ $(n=1,2,3,\ldots)$ and $\{u_n^{**}(t)\}$ $(n=1,2,3,\ldots)$ such that

$$0<\inf_{0\leq t<\infty}\sum_{n=1}^\infty u_n^*(t)\leq \sup_{0\leq t<\infty}\sum_{n=1}^\infty u_n^*(t)<1<\sup_{0\leq t<\infty}\sum_{n=1}^\infty u_n^{**}(t), \qquad \inf_{0\leq t<\infty}\sum_{n=1}^\infty u_n^{**}(t)>0. \tag{3.31}$$

Proof. Let $E = l^1 = \{u = (u_1,\ldots,u_n,\ldots) : \sum_{n=1}\infty|u_n| < \infty\}$ with norm $\|u\| = \sum_{n=1}^\infty |u_n|$ and $P = (u_1,\ldots,u_n,\ldots) : u_n \geq 0, \ \ n=1,2,3,\ldots\}$. Then P is a normal cone in E with normal constant $N=1$, and infinite system (3.30) can be regarded as an infinite three-point boundary value problem of form (1.5). In this situation, $u = (u_1,\ldots,u_n,\ldots)$, $v = (v_1,\ldots,v_n,\ldots)$,

$w = (w_1, \ldots, w_n, \ldots)$, $t_k = k$ $(k = 1, 2, 3, \ldots)$, $K(t,s) = e^{-(t+1)s}$, $H(t,s) = (1+t+s)^{-2}$, $\eta = 9/2$, $\gamma = 1/2$, $\beta = 6$, $f = (f_1, \ldots, f_n, \ldots)$, and $I_k = (I_{k1}, \ldots, I_{kn} \ldots)$, in which

$$f_n(t,u,v,w) = \frac{1}{8n^2} e^{-5t} \left(\left[u_{n+1} + \sum_{m=1}^{\infty} u_m \right]^2 + \sqrt{3u_{2n} + \sum_{m=1}^{\infty} u_m} \right) + \frac{1}{9n^3} e^{-6t} \left(v_n^2 + w_{n+2}^3 \right), \quad \forall t \in J = [0, \infty),\ u, v, w \in P\ (n = 1, 2, 3, \ldots), \tag{3.32}$$

$$I_{kn}(u) = \frac{1}{6n^2} 3^{-k} \left(u_n^2 + u_{2n+1}^2 \right), \quad \forall u \in P\ (k = 1, 2, 3, \ldots;\ n = 1, 2, 3, \ldots). \tag{3.33}$$

It is easy to see that $f \in C[J \times P \times P \times P, P]$, $I_k \in C[P, P]$ $(k = 1, 2, 3, \ldots)$, and condition (H_1) is satisfied and $k^* \le 1$, $h^* \le 1$. We have, by (3.32),

$$\begin{aligned} 0 &\le f_n(t,u,v,w) \\ &\le \frac{1}{8n^2} e^{-5t} \left([2\|u\|]^2 + \sqrt{4\|u\|} \right) + \frac{1}{9n^3} e^{-6t} \left(\|v\|^2 + \|w\|^3 \right) \\ &\le \frac{1}{n^2} e^{-5t} \left(\frac{1}{2}\|u\|^2 + \frac{1}{4}\sqrt{\|u\|} + \frac{1}{9}\|v\|^2 + \frac{1}{9}\|w\|^3 \right), \quad \forall t \in J,\ u, v, w \in P \quad (n = 1, 2, 3, \ldots), \end{aligned} \tag{3.34}$$

so, observing the inequality $\sum_{n=1}^{\infty} (1/n^2) < 2$, we get

$$\|f(t,u,v,w)\| = \sum_{n=1}^{\infty} f_n(t,u,v,w) \le e^{-5t} \left(\|u\|^2 + \frac{1}{2}\sqrt{\|u\|} + \frac{2}{9}\|v\|^2 + \frac{2}{9}\|w\|^3 \right), \quad \forall t \in J,\ u, v, w \in P, \tag{3.35}$$

which implies that condition (H_2) is satisfied for $a(t) = e^{-5t}$ $(^* = 1/5)$ and

$$g(x,y,z) = x^2 + \frac{1}{2}\sqrt{x} + \frac{2}{9}y^2 + \frac{2}{9}z^3. \tag{3.36}$$

By (3.33), we have

$$0 \le I_{kn}(u) \le \frac{1}{6n^2} 3^{-k} \|u\|^2, \quad \forall u \in P\ (k = 1, 2, 3, \ldots;\ n = 1, 2, 3, \ldots), \tag{3.37}$$

so, condition (H_3) is satisfied for $\gamma_k = 3^{-k-1}$ $(\gamma^* = 1/6)$ and

$$F(x) = x^2. \tag{3.38}$$

On the other hand, (3.32) implies

$$f_n(t,u,v,w) \geq \frac{1}{8n^2}e^{-5t}\|u\|^2, \quad \forall t \in J,\ u,v,w \in P\ (n = 1,2,3,\ldots),$$
$$f_n(t,u,v,w) \geq \frac{1}{8n^2}e^{-5t}\sqrt{\|u\|}, \quad \forall t \in J,\ u,v,w \in P\ (n = 1,2,3,\ldots), \tag{3.39}$$

so, we see that condition (H_5) is satisfied for $b(t) = (1/8)e^{-5t}(b^* = 1/40)$, $\tau(u) = \|u\|^2$ and $u_0 = (1,\ldots,1/n^2,\ldots)$, and condition (H_6) is satisfied for $c(t) = (1/8)e^{-5t}(c^* = 1/40)$, $\sigma(u) = \sqrt{\|u\|}$, and let $u_1 = (1,\ldots,1/n^2,\ldots)$. Now, we check that condition (H_4) is satisfied. Let $t \in J$ and $r > 0$ be fixed, and $\{z^{(m)}\}$ be any sequence in $f(t,P_r,P_r,P_r)$, where $z^{(m)} = (z_1^{(m)},\ldots,z_n^{(m)},\ldots)$. Then, we have, by (3.34),

$$0 \leq z_n^{(m)} \leq \frac{1}{n^2}\left(\frac{11}{18}r^2 + \frac{1}{4}\sqrt{r} + \frac{1}{9}r^3\right), \quad (n,m = 1,2,3,\ldots). \tag{3.40}$$

So, $\{z_n^{(m)}\}$ is bounded, and, by diagonal method, we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$z^{(m_i)} \longrightarrow \overline{z}_n \quad \text{as } i \longrightarrow \infty \quad (n = 1,2,3,\ldots), \tag{3.41}$$

which implies by virtue of (3.40) that

$$0 \leq \overline{z}_n \leq \frac{1}{n^2}\left(\frac{11}{18}r^2 + \frac{1}{4}\sqrt{r} + \frac{1}{9}r^3\right), \quad (n = 1,2,3,\ldots). \tag{3.42}$$

Consequently, $\overline{z} = (\overline{z}_1,\ldots,\overline{z}_n,\ldots) \in l^1 = E$. Let $\epsilon > 0$ be given. Choose a positive integer n_0 such that

$$\left(\sum_{n=n_0+1}^{\infty}\frac{1}{n^2}\right)\left(\frac{11}{18}r^2 + \frac{1}{4}\sqrt{r} + \frac{1}{9}r^3\right) < \frac{\epsilon}{3}. \tag{3.43}$$

By (3.41), we see that there exists a positive integer i_0 such that

$$\left|z_n^{(m_i)} - \overline{z}_n\right| < \frac{\epsilon}{3n_0}, \quad \forall i > i_0\ (n = 1,2,\ldots,n_0). \tag{3.44}$$

It follows from (3.40)–(3.44) that

$$\left\|z^{(m_i)} - \overline{z}\right\| = \sum_{n=1}^{\infty}\left|z_n^{(m_i)} - \overline{z}_n\right| \leq \sum_{n=1}^{n_0}\left|z_n^{(m_i)} - \overline{z}_n\right| + \sum_{n=n_0+1}^{\infty}\left|z_n^{(m_i)}\right|$$
$$+ \sum_{n=n_0+1}^{\infty}|\overline{z}_n| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \quad \forall i > i_0. \tag{3.45}$$

Thus, we have proved that $f(t, P_r, P_r, P_r)$ is relatively compact in E. Similarly, by using (3.37), we can prove that $I_k(P_r)$ is relatively compact in E. Hence, condition (H_4) is satisfied. Finally, it is easy to check that inequality (3.3) is satisfied for $\xi = 1$ (in this case, $M_\xi \le 17/36$ and $N_\xi = 1$). Hence, our conclusion follows from Theorem 3.2. □

Example 3.5. Consider the infinite system of scalar first-order impulsive integrodifferential equations of mixed type on the half line:

$$u_n'(t) = \frac{1}{n^3(1+t)^3}\left(u_n(t) + 2u_{n+1}(t) + \sum_{m=1}^{\infty} u_m(t)\right)^3 + \frac{1}{n^4(1+t)^4}\left(\int_0^t \frac{u_{2n}(s)ds}{1+ts+s^2}\right)^4$$
$$+ \frac{1}{n^5(1+t)^5}\left(\int_0^\infty e^{-s}\sin^2(t-s)u_{3n}(s)ds\right)^5,$$
$$\forall 0 \le t < \infty,\ t \ne 2k\ (k = 1,2,3,\ldots;\ n = 1,2,3,\ldots),$$
$$\Delta u_n\,|_{t=2k} = \frac{1}{n^2}e^{-k}[u_n(2k)]^3 + \frac{1}{n^3}2^{-k}[u_{2n+1}(2k)]^4, \quad (k = 1,2,3,\ldots;\ n = 1,2,3,\ldots),$$
$$4u_n(\infty) = 3u_n(7) + 2u_n(0), \quad (n = 1,2,3,\ldots). \tag{3.46}$$

Evidently, $u_n(t) \equiv 0 \ \ (n = 1,2,3,\ldots)$ is the trivial solution of infinite system (3.46).

Conclusion. Infinite system (3.46) has at least one positive solution $\{u_n^*(t)\}$ $(n = 1,2,3,\ldots)$ such that

$$\inf_{0\le t<\infty} \sum_{n=1}^{\infty} u_n^*(t) > 0. \tag{3.47}$$

Proof. Let $E = l^1 = (\{u = (u_1,\ldots,u_n,\ldots) : \sum_{n=1}^{\infty}|u_n| < \infty\}$ with norm $\|u\| = \sum_{n=1}^{\infty}|u_n|$ and $P = \{u = (u_1,\ldots,u_n,\ldots) \in l^1 : u_n \ge 0,\ n = 1,2,3,\ldots\}$. Then P is a normal cone in E with normal constant $N = 1$, and infinite system (3.46) can be regarded as an infinite three-point boundary value problem of form (1.5) in E. In this situation, $u = (u_1,\ldots,u_n,\ldots)$, $v = (v_1,\ldots,v_n,\ldots)$, $w = (w_1,\ldots,w_n,\ldots)$, $t_k = 2k$ $(k = 1,2,3,\ldots)$, $K(t,s) = (1+ts+s^2)^{-1}$, $H(t,s) = e^{-s}\sin^2(t-s)$, $\eta = 7$, $\gamma = 3/4$, $\beta = 1/2$, $f = (f_1,\ldots,f_n,\ldots)$, and $I_k = (I_{k1},\ldots,I_{kn},\ldots)$, in which

$$f_n(t,u,v,w) = \frac{1}{n^3}(1+t)^{-3}\left(u_n + 2u_{n+1} + \sum_{m=1}^{\infty} u_m\right)^3 + \frac{1}{n^4}(1+t)^{-4}v_{2n}^4 + \frac{1}{n^5}(1+t)^{-5}w_{3n}^5,$$
$$\forall t \in J = [0,\infty),\ u,v,w \in P\ (n = 1,2,3,\ldots), \tag{3.48}$$
$$I_{kn}(u) = \frac{1}{n^2}e^{-k}u_n^3 + \frac{1}{n^3}2^{-k}u_{2n+1}^4, \quad (k = 1,2,3,\ldots;\ n = 1,2,3,\ldots).$$

It is clear that $f \in C[J \times P \times P \times P, P]$, $I_k \in C[P, P]$ $(k = 1, 2, 3, \ldots)$, and condition (H_1) is satisfied and $k^* \le \pi/2$, $h^* \le 1$. We have

$$0 \le f_n(t, u, v, w) \le \frac{1}{n^3}(1+t)^{-3}\left((3\|u\|)^3 + \|v\|^4 + \|w\|^5\right), \quad \forall t \in J,\ u, v, w \in P\ (n = 1, 2, 3, \ldots),$$
$$0 \le I_{kn}(u) \le \frac{1}{n^2} 2^{-k}\left(\|u\|^3 + \|u\|^4\right), \quad \forall u \in P\ (k = 1, 2, 3, \ldots,\ n = 1, 2, 3, \ldots), \tag{3.49}$$

so, condition (H_2) is satisfied for $a(t) = (1+t)^{-3}(a^* = (1/2))$ and

$$g(x, y, z) = 54x^3 + 2y^4 + 2z^5, \tag{3.50}$$

and (H_3) is satisfied for $\gamma_k = 2^{-k}(\gamma^* = 1)$ and

$$F(x) = 2x^3 + 2x^4. \tag{3.51}$$

From

$$f_n(t, u, v, w) \ge \frac{1}{n^3}(1+t)^{-3}\|u\|^3, \quad \forall t \in J,\ u, v, w \in P\ (n = 1, 2, 3, \ldots), \tag{3.52}$$

we see that condition (H_5) is satisfied for $b(t) = (1+t)^{-3}$ $(b^* = 1/2)$, $\tau(u) = \|u\|^3$, and $u_0 = (1, \ldots, 1/n^3, \ldots)$. Moreover, it is clear that (3.22) are satisfied. Similar to the discussion in Example 3.4, we can prove that $f(t, P_r, P_r, P_r)$ and $I_k(P_r)$ (for fixed $t \in J$ and $r > 0$; $k = 1, 2, 3, \ldots$) are relatively compact in $E = l^1$; so, condition (H_4) is satisfied. Hence, our conclusion follows from Theorem 3.3. □

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