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Research Article

Oscillation Theorems for Second-Order Half-Linear Advanced Dynamic Equations on Time Scales

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This paper is concerned with the oscillatory behavior of the second-order half-linear advanced dynamic equation $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(g(t)) = 0$ on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$, where $g(t) \geq t$ and $\int_{t_0}^{\infty}(\Delta s/({}_{r}1/\gamma_{(s)})) < \infty$. Some sufficient conditions for oscillation of the studied equation are established. Our results not only improve and complement those results in the literature but also unify the oscillation of the second-order half-linear advanced differential equation and the second-order half-linear advanced difference equation. Three examples are included to illustrate the main results.

1. Introduction

The study of dynamic equations on time scales, which has recently received a lot of attention, was introduced by Hilger [1] in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [2] and the references cited therein. For an excellent introduction to the calculus on time scales; see Bohner and Peterson [3]. Further information on working with dynamic equations on time scales can be found in [4].

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, that is, when $\mathbb{T} = \mathbb{R}$; $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$, where $q > 1$. Many other interesting time scales exist, and they give rise to many applications; see [3]. Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in, say, winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population; see [3].

The theory of oscillations is an important branch of the applied theory of dynamic equations related to the study of oscillatory phenomena in technology and natural and social sciences. In recent years, there has been much research activity concerning the oscillation of solutions of various dynamic equations on time scales, we refer the reader to the papers [5–18] and the references therein.

We are concerned with the oscillation of the second-order half-linear advanced dynamic equation

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p(t)x^{\gamma}(g(t))=0, \tag{1.1}$$

on a time scale $\mathbb{T}$ unbounded above, where $\gamma>0$ is the quotient of odd positive integers, r and p are real-valued rd-continuous positive functions defined on $\mathbb{T}$, $g\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$, $g(t)\geq t$.

Since we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above. We define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$.

By a solution of (1.1), we mean a nontrivial real-valued function x which has the properties $x(t)\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $r(t)(x^{\Delta}(t))^{\gamma}\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfying (1.1) for all $t\in[t_0,\infty)_{\mathbb{T}}$. We consider only those solutions x of (1.1) which satisfy $\sup\{|x(t)|:t\in[T,\infty)_{\mathbb{T}}\}>0$ for all $T\in[t_0,\infty)_{\mathbb{T}}$. We assume that (1.1) possesses such a solution. As usual, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)_{\mathbb{T}}$; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

We note that if $\mathbb{T}=\mathbb{R}$, (1.1) becomes the second-order advanced differential equation

$$\left(r(t)\left(x'(t)\right)^{\gamma}\right)'+p(t)x^{\gamma}(g(t))=0. \tag{1.2}$$

A special case of (1.2) is

$$\left(r(t)x'(t)\right)'+p(t)x(t)=0. \tag{1.3}$$

For the oscillation of (1.3); see [19–21]. Willett [19] gave a new version of Leighton's criterion and obtained the following oscillation criteria: if

$$\int_{t_0}^{\infty}\frac{1}{r(t)}\mathrm{d}t<\infty,\qquad \int_{t_0}^{\infty}p(t)\left(\int_t^{\infty}\frac{\mathrm{d}s}{r(s)}\right)^2\mathrm{d}t=\infty, \tag{1.4}$$

then every solution of (1.3) is oscillatory. Later, Li [20] obtained that

$$\left(t^2x'(t)\right)'+\lambda x(t)=0 \tag{1.5}$$

is oscillatory when $\lambda>1/4$.

The discrete analog of (1.3) is the second-order advanced difference equation

$$\Delta(r(n)\Delta x(n)) + q(n)x(n) = 0. \tag{1.6}$$

Budinčević [22] proved that if

$$\sum_{n=k_0}^{\infty} \frac{1}{r(n)} < \infty, \qquad \sum_{n=k_0}^{\infty} \frac{1}{r(n)} \sum_{k=k_0}^{n} q(k) = \infty, \tag{1.7}$$

then every solution x of (1.6) is either oscillatory or else $x(n) \to 0$ as $n \to \infty$.

Regarding the oscillation of (1.1), Agarwal et al. [5], Grace et al. [7], Saker [8], and Hassan [9] studied (1.1) when $\tau(t) = t$, that is,

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(t) = 0, \tag{1.8}$$

and established some oscillation criteria for the case when

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} = \infty. \tag{1.9}$$

Furthermore, the authors obtained some sufficient conditions which guarantee that every solution x of (1.8) oscillates or $\lim_{t\to\infty} x(t) = 0$ under the case when

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} < \infty. \tag{1.10}$$

Saker [14] studied the oscillation of the dynamic equation

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p(t)x(\sigma(t)) = 0 \tag{1.11}$$

for the cases when (1.9) and (1.10) hold with $\gamma = 1$. Very recently, Hassan [15] has investigated the oscillation of (1.1) under the conditions

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} = \infty \quad \text{or} \quad \int_{t_0}^{\infty} \left[\frac{1}{r(t)} \int_{t_0}^{t} p(s) \left(\int_{g(s)}^{\infty} \frac{\Delta u}{r^{1/\gamma}(u)}\right)^{\gamma} \Delta s\right]^{1/\gamma} \Delta t = \infty. \tag{1.12}$$

Now a problem is how to determine the oscillatory behavior of (1.1) when

$$\int_{t_0}^{\infty} \left[\frac{1}{r(t)} \int_{t_0}^{\infty} p(s) \left(\int_{g(s)}^{\infty} \frac{\Delta u}{r^{1/\gamma}(u)}\right)^{\gamma} \Delta s\right]^{1/\gamma} \Delta t < \infty. \tag{1.13}$$

In this paper, we will establish some new oscillation criteria for (1.1) under the case when (1.10) holds. Our results can be applied when (1.13) holds. The paper is organized as follows. In Section 2, we present some basic definitions and useful results from the theory of calculus on time scales. In Section 3, we shall establish several new oscillation criteria for (1.1).

Remark 1.1. All functional inequalities considered in this note are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. Preliminary Results

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, that is, it is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$. On any time scale, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} \mid s < t\}. \tag{2.1}$$

A point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) = t$, right dense if $\sigma(t) = t$, left scattered if $\rho(t) < t$, and right scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may actually be replaced with any Banach space), the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \tag{2.2}$$

if f is continuous at t and t is right scattered. If t is not right scattered, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s}, \tag{2.3}$$

provided that this limit exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

The derivative and the shift operator σ are related by the formula

$$f^{\sigma}(t) = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.4}$$

Let f be a real-valued function defined on an interval $[a, b]$. We say that f is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$ if $t_1, t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \geq f(t_1)$, and $f(t_2) \leq f(t_1)$, respectively. Let f be a differentiable function on $[a,b]$. Then f is increasing, decreasing, nondecreasing, and nonincreasing on $[a,b]$ if $f^{\Delta}(t)>0$, $f^{\Delta}(t)<0$, $f^{\Delta}(t)\geq 0$, and $f^{\Delta}(t)\leq 0$ for all $t\in[a,b)$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $g(t)g(\sigma(t))\neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g(\sigma(t)),$$
$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}. \tag{2.5}$$

For a, $b\in\mathbb{T}$, and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t=f(b)-f(a). \tag{2.6}$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t=f(b)g(b)-f(a)g(a)-\int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t, \tag{2.7}$$

and infinite integrals are defined as

$$\int_a^{\infty} f(s)\Delta s=\lim_{t\to\infty}\int_a^t f(s)\Delta s. \tag{2.8}$$

3. Main Results

In this section, by employing the Riccati transformation technique, we establish several oscillation criteria for (1.1). To prove the main theorems, we will use the following formula:

$$(x^{\gamma}(t))^{\Delta}=\gamma\int_0^1 [hx^{\sigma}(t)+(1-h)x(t)]^{\gamma-1}\mathrm{d}hx^{\Delta}(t), \tag{3.1}$$

which is a simple consequence of Keller's chain rule [3, Theorem 1.90].

Set

$$d_+(t):=\max\{0,d(t)\},\qquad R(t):=\int_t^{\infty}\frac{\Delta s}{r^{1/\gamma}(s)}, \tag{3.2}$$

and we assume that there exists a positive real-valued Δ-differentiable function m such that

$$\frac{m(t)}{r^{1/\gamma}(t)R(t)}+m^{\Delta}(t)\leq 0. \tag{3.3}$$

In order to prove the main results conveniently, we give the following known result.

Theorem A (see [12, Theorem 2.1]). *Assume that* (1.9) *holds. Further, assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that*

$$\limsup_{t\to\infty}\int_T^t\left[\delta(s)p(s)-\frac{r(s)\left(\left(\delta^{\Delta}(s)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s=\infty \tag{3.4}$$

holds for all sufficiently large T*. Then every solution of* (1.1) *is oscillatory.*

Remark 3.1. From the proof of [12, Theorem 2.1], if we let x be an eventually positive solution of (1.1), then $x^{\Delta}(t)>0$ due to condition (1.9). Hence, we can get a contradiction to (3.4) when $x^{\Delta}(t)>0$ occurs.

Theorem 3.2. *Assume that* (1.10) *holds,* $g(t)\geq\sigma(t)$ *and* $\gamma\leq 1$*. Furthermore, assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that* (3.4) *holds for all sufficiently large* T*. If*

$$\limsup_{t\to\infty}\int_T^t\left[p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma}R^{\gamma\sigma}(s)-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{1}{R^{\sigma}(s)r^{1/\gamma}(s)}\right]\Delta s=\infty, \tag{3.5}$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x(t)>0$ and $x(g(t))>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$. In view of (1.1), we obtain

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}=-p(t)x^{\gamma}(g(t))<0,\quad t\in[t_0,\infty)_{\mathbb{T}}. \tag{3.6}$$

Hence, $r(t)(x^{\Delta}(t))^{\gamma}$ is an eventually strictly decreasing function, and there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x^{\Delta}(t)>0,\ t\in[t_1,\infty)_{\mathbb{T}}$ or $x^{\Delta}(t)<0,\ t\in[t_1,\infty)_{\mathbb{T}}$.

Case 1. Assume that $x^{\Delta}(t)>0,\ t\in[t_1,\infty)_{\mathbb{T}}$. From Theorem A, we can obtain a contradiction to (3.4).

Case 2. Assume that $x^{\Delta}(t)<0,\ t\in[t_1,\infty)_{\mathbb{T}}$. Define the function ω by

$$\omega(t)=\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)}. \tag{3.7}$$

Then, $\omega(t)<0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. By (3.6), we get

$$x^{\Delta}(s)\leq\frac{r^{1/\gamma}(t)}{r^{1/\gamma}(s)}x^{\Delta}(t),\quad s\in[t,\infty)_{\mathbb{T}}. \tag{3.8}$$

Integrating it from t to l, we have

$$x(l) \leq x(t) + r^{1/\gamma}(t)x^{\Delta}(t)\int_t^l \frac{\Delta s}{r^{1/\gamma}(s)}, \quad l \in [t,\infty)_{\mathbb{T}}. \tag{3.9}$$

Taking $l \to \infty$ in the last inequality, we get

$$x(t) + r^{1/\gamma}(t)x^{\Delta}(t)R(t) \geq 0, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.10}$$

Thus, we obtain

$$r^{1/\gamma}(t)R(t)\frac{x^{\Delta}(t)}{x(t)} \geq -1. \tag{3.11}$$

By (3.7) and (3.11), we have

$$-1 \leq R^{\gamma}(t)\omega(t) \leq 0. \tag{3.12}$$

On the other hand, it follows from (3.11) that

$$\frac{x^{\Delta}(t)}{x(t)} \geq -\frac{1}{r^{1/\gamma}(t)R(t)}. \tag{3.13}$$

Then, we have

$$\left(\frac{x(t)}{m(t)}\right)^{\Delta} = \frac{x^{\Delta}(t)m(t) - x(t)m^{\Delta}(t)}{m(t)m^{\sigma}(t)} \geq -\frac{x(t)}{m(t)m^{\sigma}(t)}\left[\frac{m(t)}{r^{1/\gamma}(t)R(t)} + m^{\Delta}(t)\right] \geq 0. \tag{3.14}$$

Thus, $x(t)/m(t)$ is nondecreasing. Hence we obtain

$$\frac{x(g(t))}{x(\sigma(t))} \geq \frac{m(g(t))}{m(\sigma(t))}, \quad \text{since } g(t) \geq \sigma(t). \tag{3.15}$$

Δ-differentiating (3.7) and using (3.6), we obtain

$$\omega^{\Delta}(t) \leq -p(t)\left(\frac{m(g(t))}{m(\sigma(t))}\right)^{\gamma} - \frac{r(t)(x^{\Delta}(t))^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}. \tag{3.16}$$

In view of Keller's chain rule [3, Theorem 1.90], we see that

$$(x^{\gamma}(t))^{\Delta} \leq \gamma x^{\gamma-1}(t)x^{\Delta}(t), \quad \text{since} \quad \gamma \leq 1. \tag{3.17}$$

Thus, (3.16) yields

$$\omega^{\Delta}(t) \leq -p(t)\left(\frac{m(g(t))}{m(\sigma(t))}\right)^{\gamma} - \gamma\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma+1}}{x(t)x^{\gamma}(\sigma(t))}. \tag{3.18}$$

On the other hand, from $x^{\Delta}(t) < 0$, we have $x(t) \geq x^{\sigma}(t)$ and

$$-\gamma\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma+1}}{x(t)x^{\gamma}(\sigma(t))} \leq -\gamma\left(\frac{1}{r(t)}\right)^{1/\gamma}\omega^{(\gamma+1)/\gamma}(t). \tag{3.19}$$

Hence by (3.18), we have

$$\omega^{\Delta}(t) + p(t)\left(\frac{m(g(t))}{m(\sigma(t))}\right)^{\gamma} + \gamma r^{-1/\gamma}(t)\omega^{(\gamma+1)/\gamma}(t) \leq 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.20}$$

Multiplying (3.20) by $R^{\gamma\sigma}(t)$, we obtain

$$R^{\gamma\sigma}(t)\omega^{\Delta}(t) + p(t)\left(\frac{m(g(t))}{m(\sigma(t))}\right)^{\gamma}R^{\gamma\sigma}(t) + \gamma R^{\gamma\sigma}(t)r^{-1/\gamma}(t)\omega^{(\gamma+1)/\gamma}(t) \leq 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.21}$$

Integrating it from t_1 to t, we get

$$\int_{t_1}^{t} R^{\gamma\sigma}(s)\omega^{\Delta}(s)\Delta s + \int_{t_1}^{t} p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma}R^{\gamma\sigma}(s)\Delta s + \gamma\int_{t_1}^{t} R^{\gamma\sigma}(s)r^{-1/\gamma}(s)\omega^{(\gamma+1)/\gamma}(s)\Delta s \leq 0. \tag{3.22}$$

Integrating by parts, we have

$$\int_{t_1}^{t} R^{\gamma\sigma}(s)\omega^{\Delta}(s)\Delta s = R^{\gamma}(t)\omega(t) - R^{\gamma}(t_1)\omega(t_1) - \int_{t_1}^{t}\left(R^{\gamma}(s)\right)^{\Delta}\omega(s)\Delta s. \tag{3.23}$$

From Keller's chain rule [3, Theorem 1.90], we obtain

$$\left(R^{\gamma}(t)\right)^{\Delta} = \gamma\int_{0}^{1}\left[hR^{\sigma}(t) + (1-h)R(t)\right]^{\gamma-1}\mathrm{d}hR^{\Delta}(t). \tag{3.24}$$

Note that $R^{\Delta}(t) = -(1/r(t))^{1/\gamma} < 0$, we get

$$-\int_{t_1}^{t}\left(R^{\gamma}(s)\right)^{\Delta}\omega(s)\Delta s \geq \gamma\int_{t_1}^{t}\left(\frac{1}{r(s)}\right)^{1/\gamma}\left(R^{\sigma}(s)\right)^{\gamma-1}\omega(s)\Delta s \tag{3.25}$$

due to (3.24) and $\gamma \leq 1$. By (3.22), (3.23), and (3.25), we see that

$$R^{\gamma}(t)\omega(t) - R^{\gamma}(t_1)\omega(t_1) + \int_{t_1}^{t} p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma} R^{\gamma\sigma}(s)\Delta s + \gamma\int_{t_1}^{t}\left(\frac{1}{r(s)}\right)^{1/\gamma}(R^{\sigma}(s))^{\gamma-1}\omega(s)\Delta s$$
$$+\gamma\int_{t_1}^{t} R^{\gamma\sigma}(s)r^{-1/\gamma}(s)\omega^{(\gamma+1)/\gamma}(s)\Delta s \leq 0. \tag{3.26}$$

Set $p := (\gamma+1)/\gamma, q := \gamma+1$,

$$A := -(\gamma+1)^{\gamma/(\gamma+1)}\left(\frac{R^{\gamma\sigma}(t)}{r^{1/\gamma}(t)}\right)^{\gamma/(\gamma+1)}\omega(t),$$
$$B := \frac{\gamma}{\gamma+1}(\gamma+1)^{1/(\gamma+1)}\left(\frac{1}{r^{1/\gamma}(t)}\right)^{1/(\gamma+1)}\frac{1}{(R^{\sigma}(t))^{1/(\gamma+1)}}. \tag{3.27}$$

Then, using the inequality

$$\frac{A^p}{p} + \frac{B^q}{q} \geq AB, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{3.28}$$

we have

$$\gamma R^{\gamma\sigma}(t)r^{-1/\gamma}(t)\omega^{(\gamma+1)/\gamma}(t) + \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{1}{R^{\sigma}(t)r^{1/\gamma}(t)} \geq -\gamma\left(\frac{1}{r(t)}\right)^{1/\gamma}(R^{\sigma}(t))^{\gamma-1}\omega(t). \tag{3.29}$$

Thus, by (3.26) and (3.29), we get

$$R^{\gamma}(t)\omega(t) - R^{\gamma}(t_1)\omega(t_1) + \int_{t_1}^{t}\left[p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma}R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{1}{R^{\sigma}(s)r^{1/\gamma}(s)}\right]\Delta s \leq 0. \tag{3.30}$$

which contradicts (3.5) due to (3.12). The proof is complete. □

If $g(t) = t$, similar as in the proof of Theorem 3.2, we obtain the following result.

Theorem 3.3. *Assume that* (1.10) *holds,* $g(t) = t$ *and* $\gamma \leq 1$. *Furthermore, assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that* (3.4) *holds for all sufficiently large* T. *If*

$$\limsup_{t\to\infty}\int_{T}^{t}\left[p(s)R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{1}{R^{\sigma}(s)r^{1/\gamma}(s)}\right]\Delta s = \infty, \tag{3.31}$$

then (1.1) *is oscillatory.*

Theorem 3.4. *Assume that* (1.10) *holds,* $g(t) \geq \sigma(t)$ *and* $\gamma \geq 1$*. Suppose further that there exists a positive real-valued* Δ*-differentiable function* δ *such that* (3.4) *holds for all sufficiently large* T*. If*

$$\limsup_{t\to\infty} \int_T^t \left[p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma} R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2} r^{1/\gamma}(s)} \right] \Delta s = \infty, \tag{3.32}$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. In view of (1.1), we obtain (3.6). Therefore, $r(t)(x^{\Delta}(t))^{\gamma}$ is an eventually strictly decreasing function, and so there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x^{\Delta}(t) > 0,\ t \in [t_1, \infty)_{\mathbb{T}}$ or $x^{\Delta}(t) < 0,\ t \in [t_1, \infty)_{\mathbb{T}}$.

Case 1. Assume that $x^{\Delta}(t) > 0,\ t \in [t_1, \infty)_{\mathbb{T}}$. From Theorem A, we can obtain a contradiction to (3.4).

Case 2. Assume that $x^{\Delta}(t) < 0,\ t \in [t_1, \infty)_{\mathbb{T}}$. Define ω as in (3.7). We have (3.12). Δ-differentiating (3.7) and using (3.6), we obtain (3.16). In view of Keller's chain rule [3, Theorem 1.90], we see that

$$(x^{\gamma}(t))^{\Delta} \leq \gamma (x^{\sigma}(t))^{\gamma-1} x^{\Delta}(t), \quad \text{since } \gamma \geq 1. \tag{3.33}$$

Thus, we get

$$\omega^{\Delta}(t) \leq -p(t)\left(\frac{m(g(t))}{m(\sigma(t))}\right)^{\gamma} - \gamma \frac{r(t)(x^{\Delta}(t))^{\gamma+1}}{x^{\gamma}(t) x(\sigma(t))}. \tag{3.34}$$

On the other hand, from $x^{\Delta}(t) < 0$, we have $x(t) \geq x^{\sigma}(t)$ and

$$-\gamma \frac{r(t)(x^{\Delta}(t))^{\gamma+1}}{x^{\gamma}(t) x(\sigma(t))} \leq -\gamma \left(\frac{1}{r(t)}\right)^{1/\gamma} \omega^{(\gamma+1)/\gamma}(t). \tag{3.35}$$

Hence by (3.34), we get (3.20). Then we obtain that (3.22) and (3.23) hold. By Keller's chain rule [3, Theorem 1.90], we have (3.24). From (3.24), $\gamma \geq 1$ and $R^{\Delta}(t) = -(1/r(t))^{1/\gamma} < 0$, we see that

$$-\int_{t_1}^t (R^{\gamma}(s))^{\Delta} \omega(s) \Delta s \geq \gamma \int_{t_1}^t \left(\frac{1}{r(s)}\right)^{1/\gamma} R^{\gamma-1}(s) \omega(s) \Delta s. \tag{3.36}$$

It follows from (3.22), (3.23), and (3.36) that

$$R^{\gamma}(t)\omega(t) - R^{\gamma}(t_1)\omega(t_1) + \int_{t_1}^{t} p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma} R^{\gamma\sigma}(s)\Delta s + \gamma\int_{t_1}^{t}\left(\frac{1}{r(s)}\right)^{1/\gamma} R^{\gamma-1}(s)\omega(s)\Delta s$$
$$+\gamma\int_{t_1}^{t} R^{\gamma\sigma}(s)r^{-1/\gamma}(s)\omega^{(\gamma+1)/\gamma}(s)\Delta s \le 0. \tag{3.37}$$

Set $p := (\gamma+1)/\gamma, q := \gamma+1$,

$$A := -(\gamma+1)^{\gamma/(\gamma+1)}\left(\frac{R^{\gamma\sigma}(t)}{r^{1/\gamma}(t)}\right)^{\gamma/(\gamma+1)}\omega(t),$$
$$B := \frac{\gamma}{\gamma+1}(\gamma+1)^{1/(\gamma+1)}\left(\frac{1}{r^{1/\gamma}(t)}\right)^{1/(\gamma+1)}\frac{R^{\gamma-1}(t)}{(R^{\sigma}(t))^{\gamma^2/(\gamma+1)}}. \tag{3.38}$$

By the inequality (3.28), we have

$$\gamma R^{\gamma\sigma}(t)r^{-1/\gamma}(t)\omega^{(\gamma+1)/\gamma}(t) + \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{R^{\gamma^2-1}(t)}{(R^{\sigma}(t))^{\gamma^2}r^{1/\gamma}(t)} \ge -\gamma\left(\frac{1}{r(t)}\right)^{1/\gamma}R^{\gamma-1}(t)\omega(t). \tag{3.39}$$

Thus, (3.37) and (3.39) implies

$$R^{\gamma}(t)\omega(t)-R^{\gamma}(t_1)\omega(t_1)+\int_{t_1}^{t}\left[p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma}R^{\gamma\sigma}(s)-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2}r^{1/\gamma}(s)}\right]\Delta s \le 0. \tag{3.40}$$

which contradicts (3.32) when using (3.12). This completes the proof. □

If $g(t) = t$, similar as in the proof of Theorem 3.4, we establish the following result.

Theorem 3.5. *Assume that* (1.10) *holds,* $g(t) = t$ *and* $\gamma \ge 1$. *Moreover, assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that* (3.4) *holds for all sufficiently large* T. *If*

$$\limsup_{t\to\infty}\int_{T}^{t}\left[p(s)R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2}r^{1/\gamma}(s)}\right]\Delta s = \infty, \tag{3.41}$$

then (1.1) *is oscillatory.*

Theorem 3.6. *Assume that* (1.10) *holds,* $g(t) \geq \sigma(t)$ *and* $\gamma > 0$. *Assume further that there exists a positive real-valued* Δ*-differentiable function* δ *such that* (3.4) *holds for all sufficiently large* T. *If*

$$\int_T^t p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma} R^{\gamma+1}(\sigma(s))\Delta s = \infty, \tag{3.42}$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Similar as in the proof of Theorem 3.2 or Theorem 3.4, we consider two cases.

Case 1. Assume $x^{\Delta}(t) > 0$, $t \in [t_1, \infty)_{\mathbb{T}}$. By (3.4), this case is not true.

Case 2. Suppose $x^{\Delta}(t) < 0$, $t \in [t_1, \infty)_{\mathbb{T}}$. If $\gamma \leq 1$, proceeding as in the proof of Case 2 of Theorem 3.2, we obtain (3.12) and (3.20). Multiplying (3.20) by $R^{\gamma+1}(\sigma(t))$, and integrating it from t_1 to t, we get

$$\begin{aligned} &\int_{t_1}^t R^{\gamma+1}(\sigma(s))\omega^{\Delta}(s)\Delta s + \int_{t_1}^t p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma} R^{\gamma+1}(\sigma(s))\Delta s \\ &\quad + \gamma\int_{t_1}^t R^{\gamma+1}(\sigma(s))r^{-1/\gamma}(s)\omega^{(\gamma+1)/\gamma}(s)\Delta s \leq 0. \end{aligned} \tag{3.43}$$

Integrating by parts, we see that

$$\int_{t_1}^t R^{\gamma+1}(\sigma(s))\omega^{\Delta}(s)\Delta s = R^{\gamma+1}(t)\omega(t) - R^{\gamma+1}(t_1)\omega(t_1) - \int_{t_1}^t \left(R^{\gamma+1}(s)\right)^{\Delta}\omega(s)\Delta s. \tag{3.44}$$

By Keller's chain rule [3, Theorem 1.90], we obtain

$$\left(R^{\gamma+1}(t)\right)^{\Delta} = (\gamma+1)\int_0^1 \left[hR^{\sigma}(t) + (1-h)R(t)\right]^{\gamma} \mathrm{d}hR^{\Delta}(t). \tag{3.45}$$

Note that $R^{\Delta}(t) = -(1/r(t))^{1/\gamma} < 0$, we have

$$-\int_{t_1}^t \left(R^{\gamma+1}(s)\right)^{\Delta}\omega(s)\Delta s \geq (\gamma+1)\int_{t_1}^t \left(\frac{1}{r(s)}\right)^{1/\gamma} R^{\gamma}(s)\omega(s)\Delta s. \tag{3.46}$$

Thus, from (3.44), we get

$$R^{\gamma+1}(t)\omega(t) - R^{\gamma+1}(t_1)\omega(t_1) + \int_{t_1}^{t} p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma} R^{\gamma+1}(\sigma(s))\Delta s$$
$$+ (\gamma+1)\int_{t_1}^{t}\left(\frac{1}{r(s)}\right)^{1/\gamma} R^{\gamma}(s)\omega(s)\Delta s + \gamma\int_{t_1}^{t} R^{\gamma+1}(\sigma(s))r^{-1/\gamma}(s)\omega^{(\gamma+1)/\gamma}(s)\Delta s \le 0. \tag{3.47}$$

It follows from (3.12) that

$$-R^{\gamma+1}(t)\omega(t) \le R(t) < \infty, \quad t \longrightarrow \infty,$$
$$-\int_{t_1}^{\infty}\left(\frac{1}{r(s)}\right)^{1/\gamma} R^{\gamma}(s)\omega(s)\Delta s \le \int_{t_1}^{\infty}\left(\frac{1}{r(s)}\right)^{1/\gamma}\Delta s < \infty. \tag{3.48}$$

Noting that $R^{\sigma}(t)/R(t) \le 1$, we obtain

$$\int_{t_1}^{\infty} R^{\gamma+1}(\sigma(s))r^{-1/\gamma}(s)\omega^{(\gamma+1)/\gamma}(s)\Delta s = \int_{t_1}^{\infty} r^{-1/\gamma}(s)\left(\frac{R^{\sigma}(s)}{R(s)}\right)^{\gamma+1}(R^{\gamma}(s)\omega(s))^{(\gamma+1)/\gamma}\Delta s$$
$$\le \int_{t_1}^{\infty} r^{-1/\gamma}(s)\Delta s < \infty. \tag{3.49}$$

Thus, from (3.47), we get

$$\int_{t_1}^{\infty} p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma} R^{\gamma+1}(\sigma(s))\Delta s < \infty, \tag{3.50}$$

which contradicts (3.42).

When $\gamma \ge 1$, the proof is similar to that of the case where $\gamma \le 1$, so we omit the details. The proof is complete. □

If $g(t) = t$, similar as in the proof of Theorem 3.6, we have the following result.

Theorem 3.7. *Assume that* (1.10) *holds,* $g(t) = t$ *and* $\gamma > 0$*. Furthermore, assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that* (3.4) *holds for all sufficiently large* T*. If*

$$\int_{T}^{\infty} p(s)R^{\gamma+1}(\sigma(s))\Delta s = \infty, \tag{3.51}$$

then (1.1) *is oscillatory.*

4. Examples

For some applications, we give the following examples.

Example 4.1. Consider the second-order linear dynamic equation

$$\left(t\sigma(t)x^{\Delta}(t)\right)^{\Delta} + \lambda\frac{\sigma(t)}{t}x(\sigma(t)) = 0, \quad t \in [1,\infty)_{\mathbb{T}}, \tag{4.1}$$

where $\lambda > 0$ is a constant.

Let

$$r(t) = t\sigma(t), \qquad p(t) = \lambda\frac{\sigma(t)}{t}, \qquad g(t) = \sigma(t). \tag{4.2}$$

Then $R(t) = 1/t$. Set $m(t) = R(t) = 1/t$ and $\delta(t) = 1$. Using Theorem 3.2, it is easy to see that every solution of (4.1) is oscillatory if $\lambda > 1/4$. This result extends that of [20]. But recent results on the oscillation of second-order dynamic equations on time scales cannot be applied in (4.1).

Example 4.2. Consider the second-order advanced differential equation

$$\left(t^2x'(t)\right)' + p_0x(2t) = 0, \qquad t \geq 1, \tag{4.3}$$

where $p_0 > 0$ is a constant.

Let $m(t) = t^{-1}$. Applying Theorem 3.2, we see that (4.3) is oscillatory if $p_0 > 1/2$. However, results of [15] cannot be applied to (4.3), since

$$\int_1^{\infty}\left[\frac{1}{r(t)}\int_1^t p(s)\left(\int_{g(s)}^{\infty}\frac{\mathrm{d}u}{r^{1/\gamma}(u)}\right)^{\gamma}\mathrm{d}s\right]^{1/\gamma}\mathrm{d}t < \infty. \tag{4.4}$$

Example 4.3. Consider the second-order superlinear advanced dynamic equation

$$\left((t\sigma(t))^{\gamma}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \lambda\frac{g^{\gamma}(t)}{t}x^{\gamma}(g(t)) = 0, \quad t \in [1,\infty)_{\mathbb{T}}, \tag{4.5}$$

where $\gamma \geq 1$ is the ratio of odd positive integers, $\lambda > 0$ is a constant, $k_1t \leq \sigma(t) \leq k_2t$, and $g(t) \geq \sigma(t)$.

Let

$$r(t) = (t\sigma(t))^{\gamma}, \qquad p(t) = \lambda\frac{g^{\gamma}(t)}{t}. \tag{4.6}$$

Then $R(t) = 1/t$. Set $\delta(t) = 1$. Clearly, (3.4) holds. Put $m(t) = R(t) = 1/t$. Then

$$\begin{aligned}
&\limsup_{t\to\infty}\int_T^t\left[p(s)\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma}R^{\gamma\sigma}(s)-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2}r^{1/\gamma}(s)}\right]\Delta s\\
&\quad=\limsup_{t\to\infty}\int_T^t\left[\lambda\frac{g^{\gamma}(s)}{s}\frac{\sigma^{\gamma}(s)}{g^{\gamma}(s)}\frac{1}{\sigma^{\gamma}(s)}-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\left(\frac{\sigma(s)}{s}\right)^{\gamma^2}\frac{1}{\sigma(s)}\right]\Delta s\\
&\quad\geq\left(\lambda-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{k_2{}^{\gamma^2}}{k_1}\right)\limsup_{t\to\infty}\int_T^t\frac{1}{s}\Delta s\\
&\quad=\infty
\end{aligned}\tag{4.7}$$

if

$$\lambda>\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{k_2{}^{\gamma^2}}{k_1}.\tag{4.8}$$

Therefore, by Theorem 3.4, every solution of (4.5) is oscillatory when the above inequality holds.

5. Summary

This paper is concerned with the oscillatory behavior of advanced equation (1.1). By using the generalized Riccati substitution, we establish some new oscillation criteria for (1.1). On one hand, these criteria can be extended to the equation of the form

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p(t)x^{\beta}(g(t))=0.\tag{5.1}$$

On the other hand, the main results supplement and improve those results of [9] and extend those results of [20]. The established results are easily applicable and are illustrated on three suitable examples.

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