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Research Article

Solutions of the Force-Free Duffing-van der Pol Oscillator Equation

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A new approximate method for solving the nonlinear Duffing-van der pol oscillator equation is proposed. The proposed scheme depends only on the two components of homotopy series, the Laplace transformation and, the Padé approximants. The proposed method introduces an alternative framework designed to overcome the difficulty of capturing the behavior of the solution and give a good approximation to the solution for a large time. The Runge-Kutta algorithm was used to solve the governing equation via numerical solution. Finally, to demonstrate the validity of the proposed method, the response of the oscillator, which was obtained from approximate solution, has been shown graphically and compared with that of numerical solution.

1. Introduction

Considerable attention has been directed toward the solution of oscillator equations since they play crucial role in applied mathematics, physics, and engineering problems. In general, the analytical approximation to solution of a given oscillator problem is more difficult than the numerical solution approximation. Many powerful methods for solving nonlinear oscillator problems were appeared in open literature, such as variational iteration method [1–3], homotopy perturbation method [4–6], Hamiltonian method [7], Lindstedt-Poincare method [8], Variational method [9, 10], parameter-expansion method [11], max-min approach [12], iterative harmonic balance method [13] and differential transformation method [14].

Our main concern in this paper is to study the dynamics of approximate solution for the Duffing-van der pol oscillator equation of the form [15–17]

$$\ddot{u} + \left(\alpha + \beta u^2\right)\dot{u} - \gamma u + \lambda u^3 = 0. \tag{1.1}$$

With initial conditions

$$u(0) = a, \qquad \dot{u}(0) = b, \tag{1.2}$$

where the overdot denotes differentiation with respect to time, α, β, γ, and λ are arbitrary parameters. Equation (1.1) is an autonomous equation which describes the propagation of voltage pulses along a neural axon. Some progress was made on the integrability of the Duffing-van der pol equation (1.1) until Chandrasekar and coworkers [15] established the complete integrability of this equation and derived a general solution for a specific choice of arbitrary parameters α, β, γ, given by

$$\alpha = \frac{4}{\beta}, \qquad \gamma = \frac{-3}{\beta^2} \tag{1.3}$$

and taking $\lambda = 1$. Under the specific choice of parameters (1.3) and using a special transformation, Chandrasekar et al. were able to find the solution of this equation.

Mukherjee and colleagues [16] employed the differential transformation method to solve the Duffing-van der pol equation (1.1). Since there are some limitations in using the differential transformation method together with the fact that this method gives the solution in a very small region, developing the method for different applications is very difficult.

In the present study, we used a modified version of homotopy perturbation method which is based on two components of homotopy series. In order to improve the accuracy of the solution, we first apply the Laplace transformation, then convert the transformed series into a meromorphic function by forming the Padé approximants, and finally adopt an inverse Laplace transform to obtain an analytic solution.

Next, Runge-Kutta's RK algorithm has been introduced to solve the governing equation (1.1). Finally, numerical examples are given to demonstrate the validity of the proposed method, and the effect of parameters on the accuracy of the method is investigated. Here, we also point out that for $\alpha = -\varepsilon$, $\gamma = \varepsilon$, $\beta = \varepsilon$, and $\lambda = 0$, (1.1) reduces to a classical van der pol equation

$$\ddot{u} + \varepsilon\left(u^2 - 1\right)\dot{u} + u = 0. \tag{1.4}$$

2. Analysis of New Homotopy Perturbation Method

Let us consider the nonlinear differential equation:

$$\mathscr{A}(u) = f(z), \quad z \in \Omega, \tag{2.1}$$

where $\mathcal{A}$ is operator, f is a known function, and u is a sought function. Assume that operator $\mathcal{A}$ can be written as

$$\mathcal{A}(u) = \mathcal{L}(u) + \mathcal{N}(u), \tag{2.2}$$

where $\mathcal{L}$ is the linear operator and $\mathcal{N}$ is the nonlinear operator. Hence, (2.1) can be rewritten as follows:

$$\mathcal{L}(u) + \mathcal{N}(u) = f(z), \quad z \in \Omega. \tag{2.3}$$

We define an operator $\mathcal{H}$ as

$$\mathcal{H}(v;p) \equiv (1-p)(\mathcal{L}(v) - \mathcal{L}(u_0)) + p(\mathcal{A}(v) - f), \tag{2.4}$$

where $p \in [0,\ 1]$ is an embedding or homotopy parameter, $v(z;p) : \Omega \times [0,1] \rightarrow \Re$, and u_0 is an initial approximation of solution of the problem in (2.4) which can be written as

$$\mathcal{H}(v;p) \equiv \mathcal{L}(v) - \mathcal{L}(u_0) + p\mathcal{L}(u_0) + p(\mathcal{N}(v) - f(z)) = 0. \tag{2.5}$$

Clearly, the operator equations $\mathcal{H}(v,0) = 0$ and $\mathcal{H}(v,1) = 0$ are equivalent to the equations $\mathcal{L}(v) - \mathcal{L}(u_0) = 0$ and $\mathcal{A}(v) - f(z) = 0$, respectively. Thus, a monotonous change of parameter p from zero to one corresponds to a continuous change of the trivial problem $\mathcal{L}(v) - \mathcal{L}(u_0) = 0$ to the original problem. Operator $\mathcal{H}(v,p)$ is called a homotopy map. Next, we assume that the solution of equation $\mathcal{H}(v,p)$ can be written as a power series in embedding parameter p, as follows:

$$v = v_0 + pv_1. \tag{2.6}$$

Now let us write (2.5) in the following form:

$$\mathcal{L}(v) = u_0(z) + p(f - \mathcal{N}(v) - u_0(z)). \tag{2.7}$$

By applying the inverse operator, $\mathcal{L}^{-1}$, to both sides of (2.7), we have

$$v = \mathcal{L}^{-1}\ u_0(z) + p\left(\mathcal{L}^{-1}\ f - \mathcal{L}^{-1}\ \mathcal{N}(v) - \mathcal{L}^{-1}\ u_0(z)\right). \tag{2.8}$$

Suppose that the initial approximation of (2.1) has the form

$$u_0(z) = \sum\nolimits_{n=0}^{\infty} a_n P_n(z), \tag{2.9}$$

where a_n, $n = 0, 1, 2, \ldots.$ are unknown coefficients and $P_n(z)$, $n = 0, 1, 2, \ldots.$ are specific functions on the problem. By substituting (2.6) and (2.9) into (2.8), we get

$$v_0 + pv_1 = \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right) + p\left(\mathcal{L}^{-1} f - \mathcal{L}^{-1}\left(\sum_{n=0}^{1} v_n p^n\right) - \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right)\right). \tag{2.10}$$

Equating the coefficients of like powers of p, we get following set of equations:

$$\begin{aligned} p^0 : v_0 &= \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right), \\ p : v_1 &= \mathcal{L}^{-1}(f) + \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} v_n p^n\right) - \mathcal{L}^{-1}\mathcal{N}(v_0). \end{aligned} \tag{2.11}$$

Now, if we solve these equations in such a way that $v_1(z) = 0$. Therefore, the approximate solution may be obtained as

$$u(z) = v_0(z) = \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right). \tag{2.12}$$

3. Implementation of the Method

To obtain the solution of (1.1) by NHPM, we construct the following homotopy:

$$(1-p)\left(\ddot{U} - u_0(t)\right) + p\left(\ddot{U} + \left(\alpha + \beta U^2\right)\dot{U} - \gamma U + \lambda U^3\right) = 0. \tag{3.1}$$

Applying the inverse operator, $\mathcal{L}^{-1}(\bullet) = \int_0^t \int_0^s (\bullet) d\xi\, ds$ to both sides of (3.1), we obtain

$$U(t) = U(0) + t\dot{U}(0) + \int_0^t \int_0^s u_0(\xi) d\xi\, ds - p \int_0^t \int_0^s \left(u_0(\xi) + \left(\alpha + \beta U^2\right)\dot{U} - \gamma U + \lambda U^3\right) d\xi\, ds. \tag{3.2}$$

The solution of (1.1) is to have the following form:

$$U(t) = U_0(t) + pU_1(t). \tag{3.3}$$

Substituting (3.3) in (3.2) and equating the coefficients of like powers of p, we get following set of equations:

$$\begin{aligned} U_0(t) &= U(0) + t\dot{U}(0) + \int_0^t \int_0^s u_0(\xi) d\xi\, ds, \\ U_1(t) &= \int_0^t \int_0^s \left(-u_0(\xi) - \left(\alpha + \beta U_0^2\right)\dot{U}_0 + \gamma U_0 - \lambda U_0^3\right) d\xi\, ds. \end{aligned} \tag{3.4}$$

Assuming $u_0(t) = \sum_{n=0}^{8} a_n P_n, P_k = t^k$, as well as solving the above equation for $U_1(t)$, leads to the following result:

$$U_1(t) = \left(\frac{-a_0}{2} - \frac{b\alpha}{2} - \frac{b\beta a^2}{2} + \frac{a\gamma}{2} - \frac{\lambda a^3}{2}\right)t^2 + \left(\frac{-a_1}{6} - \frac{a_0\alpha}{6} - \frac{a_0\beta a^2}{6} - \frac{\beta ab^2}{3} + \frac{b\gamma}{6} - \frac{\lambda a^2 b}{2}\right)t^3 + \left(\frac{-a_2}{12} - \frac{a_1\alpha}{24} - \frac{a_1\beta a^2}{24} - \frac{\beta a a_0 b}{4} - \frac{\beta b^3}{12} + \frac{a_0\gamma}{24} - \frac{\lambda a_0 a^2}{8} - \frac{\lambda a b^2}{4}\right)t^4 + \cdots . \tag{3.5}$$

With vanishing $U_1(t)$, we have the following coefficients: a_i, $i = 0, 1, \ldots, 8$,

$$a_0 = -b\alpha - b\beta a^2 + a\gamma - \lambda a^3,$$
$$a_1 = b\alpha^2 - 2\beta ab^2 + 2\alpha\beta ba^2 + \beta^2 ba^4 + b\gamma - a\alpha\gamma - a^3\beta\gamma - 3\lambda a^2 b + a^3\alpha\lambda + a^5\beta\lambda, \ldots . \tag{3.6}$$

Therefore, we obtain the solutions of (3.4) as

$$u(t) = a + bt + a_0 t^2 + a_1 t^3 + a_2 t^4 + \cdots . \tag{3.7}$$

The solution of (1.1) does not exhibit behavior for a large region. In order to improve the accuracy of the two-component solution, we implement the modification as follows.

Applying the Laplace transform to the series solution (3.7) yields

$$L[u(t)] = \frac{a}{s} + \frac{b}{s^2} + \frac{a_0}{s^3} + \frac{a_1}{s^4} + \frac{a_2}{s^5} + \cdots . \tag{3.8}$$

For simplicity, let $s = 1/t$; then,

$$L[u(t)] = at + bt^2 + a_0 t^3 + a_1 t^3 + a_2 t^4 + \cdots . \tag{3.9}$$

On applying $[m, n]$ Padé approximation,

$$L[u(t)]_{[m,n]} = \frac{at + A_1 t^2 + A_2 t^3 + \cdots}{1 + B_1 t + B_2 t^2 + B_3 t^3 + \cdots}. \tag{3.10}$$

Recalling $t = 1/s$, we obtain $[m, n]$ Padé approximation in terms of s. By using the inverse Laplace transform to the $[m, n]$ Padé approximant, we obtain the desired approximate solution of the Duffing-van der pol equation.

4. Numerical Solutions

In order to verify the procedure of the method, we consider the following particular cases and comparison will be made with RK4 method as well as [16].

Table 1: Comparison between the RK4, DTM [16], and present solution.

t	u_{RK}	u_{Present}	u_{DTM}
0	−0.28868000000	−0.28868000000	−0.28868000000
0.01	−0.28748349253	−0.28748347499	−0.28748522585
0.02	−0.28629387437	−0.28629385687	−0.28630661206
0.03	−0.28511109904	−0.28511108168	−0.28514428684
0.04	−0.28393510355	−0.28393508631	−0.28399838456
0.05	−0.28276582565	−0.28044716121	−0.28286904593
0.06	−0.28160320408	−0.27929767140	−0.28175641828
0.07	−0.28044717833	−0.27814565867	−0.28066065588
0.08	−0.27929768854	−0.27701806494	−0.27951892013
0.09	−0.27815467572	−0.27815465866	−0.27852037991
0.1	−0.27701808173	−0.27701806494	−0.27747621187

Numerical Experiment 1

Consider the Duffing-van der pol equation [16] by taking $\beta = 3$, $\lambda = 1$:

$$\ddot{u} + \left(\frac{4}{3} + 3u^2\right)\dot{u} + \frac{1}{3}u + u^3 = 0. \tag{4.1}$$

With initial conditions,

$$u(0) = -0.28868, \qquad \dot{u}(0) = 0.12. \tag{4.2}$$

The approximate analytical solution of (4.1) with conditions (4.2) can be obtained by applying the procedure mentioned in previous section as

$$u(t) = -0.000026166e^{-4.09188t} - 0.00212777e^{-2.16516t} - 0.0266831e^{-1.06495t} - 0.259843e^{-0.334316t}. \tag{4.3}$$

In Table 1, the results of proposed method are compared to DTM and the fourth-order Runge-Kutta method. For comparison, the displacements of the oscillator corresponding to the four different methods are depicted in Figures 1(a)–1(d) for the same values of the parameters. It is clearly seen from Figure 1(d) that the DTM solution converges in a small region.

Numerical Experiment 2

Consider the Duffing-van der pol equation taking $\alpha = -0.1$, $\beta = 0.1$, $\gamma = 1$, and $\lambda = 0.4$:

$$\ddot{u} + 0.1\left(u^2 - 1\right)\dot{u} + u + 0.4u^3 = 0. \tag{4.4}$$

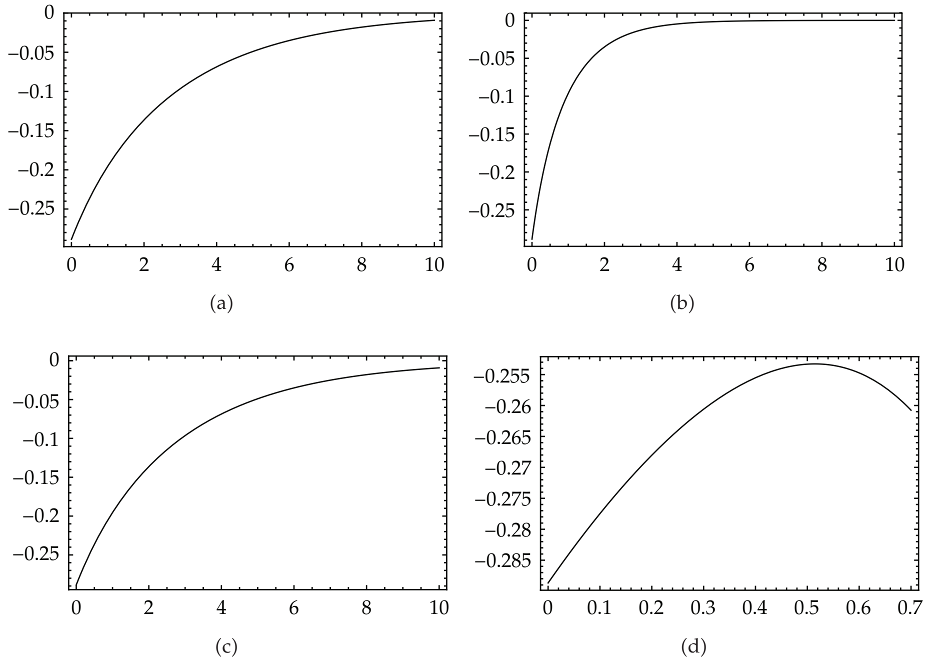

Figure 1: Plots of the Duffing-van der pol equation: (a) RK4, (b) Chandrasekar et al. [15], (c) present method, and (d) DTM [16].

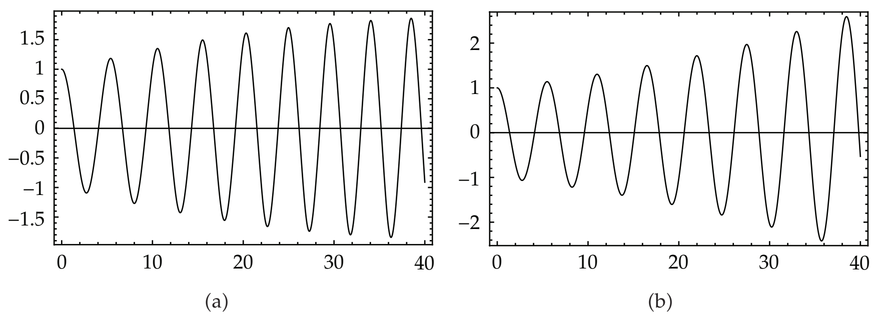

Figure 2: Plots of the Duffing-van der pol equation: (a) RK4, (b) present method.

With initial conditions,

$$u(0) = 1, \qquad \dot{u}(0) = 0. \tag{4.5}$$

The solution of (4.1) with conditions (4.5) exhibits the periodic behavior that is the characteristic of the oscillatory system. A comparison between the approximate solution and the solution that is obtained by the fourth-order Runge-Kutta method in Figures 2(a) and 2(b) shows that it converges in a wider region. Figures 3(a) and 3(b) represent approximate shape of the Duffing-van der pol limit cycle.

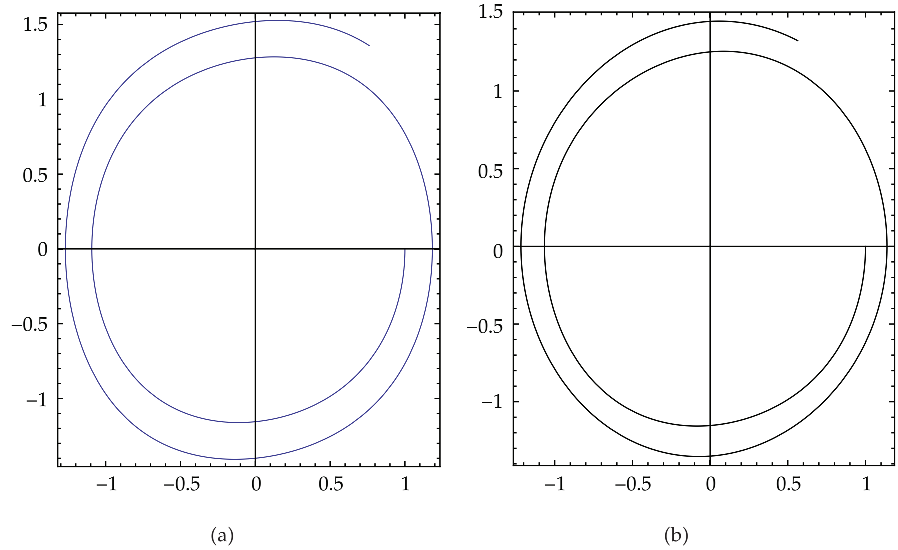

Figure 3: Comparisons for u versus $\dot{u}$ trajectory of the Duffing-van der pol equation (a) RK4, (b) present method.

5. Closing Remarks

In this work, the modified NHPM has been employed to analyze the force-free Duffing-van der pol oscillator with strong cubic nonlinearity. The results obtained from this method have been compared with those obtained from numerical method using RK algorithm and [15, 16]. This comparison shows excellent agreement between these methods. The presented scheme provides concise and straightforward solution to approach reliable results, and it overcomes the difficulties that have been arisen in conventional methods. Unlike the ADM, VIM, and HPM [17–20], the modified HPM [21] is free from the need to use the Adomian polynomials, the Lagrange multiplier, correction functional, stationary conditions, and calculating integrals. The present method is very simple method, leading to high accuracy of the obtained results.

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