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Research Article

Oscillatory Solutions of Neutral Equations with Polynomial Nonlinearities

Vasil G. Angelov[1] and Dafinka Tz. Angelova[2]

[1] *Department of Mathematics, University of Mining and Geology "St. I. Rilski", 1700 Sofia, Bulgaria*
[2] *Department of Mathematics and Physics, Higher School of Civil Engineering "L. Karavelov", 1373 Sofia, Bulgaria*

Correspondence should be addressed to Vasil G. Angelov, angelov@mgu.bg

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Existence uniqueness of an oscillatory solution for nonlinear neutral equations by fixed point method is proved.

1. Introduction

In [1, 2], we have considered a lossless transmission line terminated by a nonlinear resistive load and parallel connected capacitance (cf. Figure 1). The nonlinear boundary condition is caused by the polynomial type *V-I* characteristics of the nonlinear load at the second end of the transmission line (cf. Figure 1).

The voltage and current $u(x,t)$, $i(x,t)$ of the lossless transmission line can be found by solving the following mixed problem for the hyperbolic partial differential system:

$$C\frac{\partial u(x,t)}{\partial t} + \frac{\partial i(x,t)}{\partial x} = 0, \qquad L\frac{\partial i(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} = 0,$$
$$E(t) - u(0,t) = R_0 i(0,t), \quad t \geq 0, \tag{1.1}$$

$$C_0\frac{du(\Lambda,t)}{dt} = i(\Lambda,t) - f(u(\Lambda,t)), \quad t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad i(x,0) = i_0(x), \quad x \in [0,\Lambda], \tag{1.3}$$

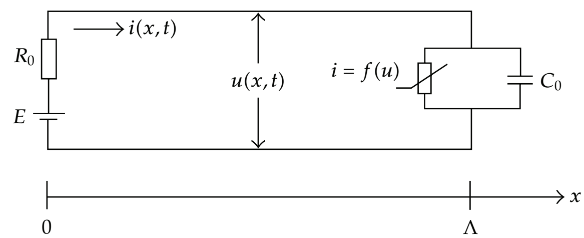

Figure 1

where $u_0(x)$ and $i_0(x)$ are prescribed initial functions, Λ is the length of the line, C is the per-unit length capacitance, and L is per-unit length inductance (cf. [3–10]). Here, the *V-I* characteristic of the nonlinear resistive load is $i = f(u) = \sum_{n=1}^{p} r_n u^n$, where r_n are real numbers, C_0 is parallel connected capacitance, E is the source voltage, R_0 is the source resistance, and $Z_0 = \sqrt{L/C}$ is the line characteristic impedance.

The above formulated mixed problem can be reduced (cf. [1, 2, 11]) to an equivalent initial value problem for a neutral functional differential equation (cf. [12]). Here, we consider the problem of an existence uniqueness of oscillatory solutions of the equation

$$\begin{aligned}\frac{du(t)}{dt} &= \frac{2E}{C_0(Z_0+R_0)} - \frac{u(t)}{C_0Z_0} - \frac{1}{C_0}\sum_{n=1}^{p} r_n[u(t)]^n - \frac{(Z_0-R_0)u(t-2T)}{Z_0C_0(Z_0+R_0)} \\ &\quad + \frac{Z_0-R_0}{C_0(Z_0+R_0)}\sum_{n=1}^{p} r_n[u(t-2T)]^n + \frac{Z_0-R_0}{Z_0+R_0}\frac{du(t-2T)}{dt}, \quad t \ge T, \\ u(t) &= v_0(t), \quad \frac{du(t)}{dt} = \frac{dv_0(t)}{dt}, \quad t \in [-T,T],\end{aligned} \tag{1.4}$$

where $(x,t) \in \Pi = \{(x,t) \in R^2 : (x,t) \in [0,\Lambda] \times [0,\infty)\}$, $\kappa = |Z_0 - R_0|/(Z_0 + R_0) < 1$, $u(t) = u(\Lambda,t)$. In fact, (1.4) is differential difference equation, and the initial function should be prescribed on an interval with length $2T$. Let us note that the initial function $v_0(t)$ can be obtained shifting the initial function $u_0(x)$ from (1.3) along the characteristics $x - \nu t =$ const., ($\nu = 1/\sqrt{LC}$) on $[0,T]$ and along the characteristics $x + \nu t =$ const. on $[-T,0]$ (cf. [1, 2]). So, we obtain an initial function $v_0(t)$ on $[-T,T]$.

Now, we are able to formulate the main problem: to find a solution of (1.4) with advanced prescribed zeros on the interval $[t_0,\infty)$, $T = t_0$.

Let $S_T = \{\tau_k\}_{k=0}^{n}$, $n \in N$ be the set of zeros of the initial function; that is, $v_0(\tau_k) = 0$ such that $\tau_0 = -T$, $\tau_n = T \equiv t_0$.

Let $S = \{t_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of real numbers satisfying the following conditions (C):

(C1) $\lim_{k\to\infty} t_k = \infty$,

(C2) $0 < l_0 = \inf\{t_{k+1} - t_k : k = 0,1,2,\ldots\} \le \sup\{t_{k+1} - t_k : k = 0,1,2,\ldots\} = T_0 < \infty$,

(C3) for every k there is $s < k$ such that $t_k - T = t_s$ where $t_s \in S_T \cup S$.

Introduce the sets: $C^1[t_0,\infty)$ consisting of all continuous and bounded functions differentiable with bounded derivatives on every interval (t_k, t_{k+1}) (the derivatives at t_k do not necessary exist), $M_S = \{u(\cdot) \in C^1[t_0, \infty) : u(t_k) = 0\ (k = 0, 1, 2, \ldots)\}$, $M_{SU} = \{u(\cdot) \in M_S : |u(t)| \le U_0 e^{\mu(t-t_k)},\ t \in [t_k, t_{k+1}]\}$, where U_0, μ are positive constants prescribed below.

We assume that $|v_0(t)| \le U_0 e^{\mu(t-\tau_k)},\ t \in [\tau_k, \tau_{k+1}]$, $(k = 0, 1, 2, \ldots, n-1)$.

The set M_{SU} turns out into a complete uniform space with respect to the family of pseudometrics $\rho_\mu^{(k)}(f, g) = \max\{\rho_k(f, g), \rho_k(\dot{f}, \dot{g})\}$, $(k = 0, 1, 2, \ldots)$, where $\rho_k(f, g) = \max\{e^{-\mu(t-t_k)}|f(t) - g(t)| : t \in [t_k, t_{k+1}]\}$, $\rho_k(\dot{f}, \dot{g}) = \max\{e^{-\mu(t-t_k)}|\dot{f}(t) - \dot{g}(t)| : t \in [t_k, t_{k+1}]\}$.

One can verify that M_{SU} is closed subset of $C^1[t_0, \infty)$ with respect to the above metric.

Remark 1.1. The functions from M_S are not necessary differentiable at t_k $(k = 0, 1, 2, \ldots)$. That is why we consider a space with a countable family of pseudometrics, and then, we have to apply the fixed point theory from [13].

Define the operator $B : M_{SU} \to M_{SU}$ by

$$B(u)(t) := \int_{t_k}^{t} U(u)(s)ds - \left(\frac{t - t_k}{t_{k+1} - t_k}\right) \int_{t_k}^{t_{k+1}} U(u)(s)ds, \quad t \in [t_k, t_{k+1}],\ (k = 0, 1, 2, \ldots), \tag{1.5}$$

where

$$\begin{aligned} U(u)(t) = {} & \frac{2E}{C_0(Z_0 + R_0)} - \frac{u(t)}{C_0 Z_0} - \frac{1}{C_0}\sum_{n=1}^{p} r_n [u(t)]^n - \frac{\kappa (K_T u)(t)}{Z_0 C_0} \\ & + \frac{\kappa}{C_0}\sum_{n=1}^{p} r_n [(K_T u)(t)]^n + \kappa \frac{d(K_T u)(t)}{dt}, \quad t \ge T, \end{aligned} \tag{1.6}$$

and $(K_T u)(t) = u(t - 2T)$ is M. A. Krasnoselskii operator (cf. [14]).

Remark 1.2. The operator K_T is well defined, because the initial function is defined on the interval $[-T, T]$. We notice that K_T maps M_S into itself. Indeed, consider the set $C^1[-T, \infty)$ consisting of all continuous and bounded functions differentiable with bounded derivatives on every interval (t_k, t_{k+1}). Introduce the set $M_S^{v_0} = \{u(\cdot) \in C^1[-T, \infty) : u(t) = v_0(t), t \in [-T, T]\ \}$. Then, K_T assigns to every function $u(\cdot) \in M_S$ the function $\tilde{u}(\cdot) \in M_S^{v_0}$ translated to the right on the interval $[T, \infty)$. So, the function $(K_T u)(t)$ coincides with $v_0(t)$ on $[t_0, t_0 + 2T]$. Besides $t_k - 2T = t_s$, and then

$$(K_T u)(t_k) = \begin{cases} u(t_k - 2T) = v_0(t_s) = 0, & t_k \in [T,\ 3T], \\ u(t_k - 2T) = u(t_n) = 0, & t \in (3T, \infty), \end{cases} \tag{1.7}$$

that is, $(K_T u)(\cdot) \in M_S$.

2. Main Results

Lemma 2.1. *If $E \le U_0$, problem* (1.4) *has a solution $u(\cdot) \in M_{SU}$ iff the operator B has a fixed point in M_{SU}, that is,*

$$u(t) = B(u)(t). \tag{2.1}$$

Proof. Let $u(\cdot) \in M_{SU}$ be a solution of (1.4). Then, integrating (1.4) on the interval $[t_k, t] \subset [t_k, t_{k+1}]$ $(k = 0, 1, 2 \ldots)$, we obtain $u(t) - u(t_k) = \int_{t_k}^{t} U(u)(s)ds \Leftrightarrow u(t) = \int_{t_k}^{t} U(u)(s)ds$, and then,

$$u(t) = \int_{t_k}^{t} U(u)(s)ds \implies 0 = u(t_{k+1}) = \int_{t_k}^{t_{k+1}} U(u)(s)ds \Longrightarrow \int_{t_k}^{t_{k+1}} U(u)(s)ds = 0. \tag{2.2}$$

Therefore, $u(t)$ satisfies

$$u(t) = \int_{t_k}^{t} U(u)(s)ds \Longleftrightarrow u(t) = \int_{t_k}^{t} U(u)(s)ds - \left(\frac{t - t_k}{t_{k+1} - t_k}\right) \int_{t_k}^{t_{k+1}} U(u)(s)ds, \tag{2.3}$$

that is, $u(\cdot)$ is a fixed point of B.

Conversely, let $u(\cdot) \in M_{SU}$ be a solution of $u = B(u)$; that is,

$$u(t) = \int_{t_k}^{t} U(u)(s)ds - \left(\frac{t - t_k}{t_{k+1} - t_k}\right) \int_{t_k}^{t_{k+1}} U(u)(s)ds. \tag{2.4}$$

Then, introducing $\mu_0 = \mu T_0$, we obtain

$$\begin{aligned}
&\left|\int_{t_k}^{t_{k+1}} U(u)(s)ds\right| \\
&\quad\le \frac{2E}{C_0(Z_0 + R_0)} \int_{t_k}^{t_{k+1}} e^{\mu(t-t_k)}dt + \frac{1}{C_0 Z_0} \int_{t_k}^{t_{k+1}} |u(t)|dt \\
&\qquad + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| \int_{t_k}^{t_{k+1}} |u(t)|^n dt + \frac{\kappa}{Z_0 C_0} \int_{t_k}^{t_{k+1}} |u(t - 2T)|dt \\
&\qquad + \frac{\kappa}{C_0} \sum_{n=1}^{p} |r_n| \int_{t_k}^{t_{k+1}} |u(t - 2T)|^n dt + \kappa \left|\int_{t_k}^{t_{k+1}} \dot{u}(t - 2T)dt\right| \\
&\quad\le \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| U_0^n \int_{t_k}^{t_{k+1}} e^{n\mu(t-t_k)}dt \\
&\qquad + \frac{\kappa U_0 e^{-2\mu T}}{Z_0 C_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + \frac{\kappa}{C_0} \sum_{n=1}^{p} |r_n| U_0^n e^{-2n\mu T} \\
&\qquad \times \int_{t_k}^{t_{k+1}} e^{n\mu(t-t_k)}dt + \kappa |u(t_{k+1} - 2T) - u(t_k - 2T)| \\
&\quad\le \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} \frac{e^{\mu T_0} - 1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu T_0} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^{p} |r_n| U_0^n \frac{e^{n\mu T_0} - 1}{n\mu} \\
&\qquad + \frac{U_0 \kappa e^{-2\mu T}}{C_0 Z_0} \frac{e^{\mu T_0} - 1}{\mu} + \frac{\kappa}{C_0} \sum_{n=1}^{p} |r_n| U_0^n e^{-2n\mu T} \frac{e^{n\mu T_0} - 1}{n\mu}
\end{aligned}$$

$$\leq \frac{e^{\mu_0}-1}{\mu C_0}\left(\frac{2U_0e^{-\mu T}}{Z_0+R_0}+\frac{U_0\left(1+\kappa e^{-2\mu T}\right)}{Z_0}\right)+\frac{1}{\mu C_0}\sum_{n=1}^{p}\frac{|r_n|U_0^n\left(1+\kappa e^{-2n\mu T}\right)\left(e^{n\mu_0}-1\right)}{n}$$
$$\equiv M(\mu). \tag{2.5}$$

Let us assume that $|\int_{t_k}^{t_{k+1}}U(u)(t)dt| = \beta > 0$. We have just obtained that $\beta \leq M(\mu)$. Then, for sufficiently large $\mu > 0$ (and sufficiently small $T_0 > 0$), one can reach the inequality $M(\mu) < \beta$. Consequently, $\int_{t_k}^{t_{k+1}}U(u)(t)dt = 0$. It follows that $u(t) = \int_{t_k}^{t}U(u)(s)ds$ and, after a differentiation, we obtain (1.4).

Lemma 2.1 is thus proved. □

Theorem 2.2. *Let $S_T = \{\tau_k\}_{k=0}^{n}$, $n \in N$ be the set of zeros of the initial function; that is, $v_0(\tau_k) = 0$ and $v_0(\cdot) \in C^1[-T,T]$. If $E \leq U_0$, $|v_0(t)| \leq U_0e^{\mu(t-\tau_k)}$, $t \in [\tau_k,\tau_{k+1}]$, $v_0(t_0) = 0$, then, there exists a unique oscillatory solution of the initial value problem* (1.4)*, belonging to M_{SU}.*

Proof. We show that B maps M_{SU} into itself; that is, $u \in M_{SU} \Rightarrow B(u) \in M_{SU}$.

Indeed, for every $u(\cdot) \in M_{SU}$, the function $B(u)(t)$ is continuous on $[t_0,\infty)$ and differentiable on every (t_k,t_{k+1}). We have also $B(u)(t_k) = 0$ and $B(u)(t_{k+1}) = 0$.

We show that $|(Bu)(t)| \leq U_0e^{\mu(t-t_k)}$, $t \in [t_k,t_{k+1}]$. (The last inequalities imply that $B(u)(t)$ is bounded because $e^{\mu(t-t_k)} \leq e^{\mu T_0}, t \in [T,\infty)$.)

We notice that $|(t-t_k)/(t_{k+1}-t_k)| \leq 1$, $t \in [t_k,t_{k+1}]$. For sufficiently large μ, we obtain for $t \in [t_k,t_{k+1}]$

$$|(Bu)(t)| \leq \left|\int_{t_k}^{t}U(u)(s)ds\right| + \left|\int_{t_k}^{t_{k+1}}U(u)(s)ds\right| \equiv B_1+B_2. \tag{2.6}$$

We have

$$B_1 \leq \left[\frac{2}{C_0(Z_0+R_0)}\int_{t_k}^{t}|E(s-T)|ds+\frac{1}{C_0Z_0}\int_{t_k}^{t}|u(s)|ds+\frac{1}{C_0}\sum_{n=1}^{p}|r_n|\int_{t_k}^{t}|u(s)|^n ds\right.$$
$$\left.+\frac{\kappa}{Z_0C_0}\int_{t_k}^{t}|u(s-2T)|ds+\frac{\kappa}{C_0}\sum_{n=1}^{p}|r_n|\int_{t_k}^{t}|u(s-2T)|^n ds\right]+\kappa\left|\int_{t_k}^{t}\dot{u}(s-2T)ds\right|$$
$$\leq \left[\frac{2U_0e^{-\mu T}}{C_0(Z_0+R_0)}\frac{e^{\mu(t-t_k)}-1}{\mu}+\frac{U_0}{C_0Z_0}\frac{e^{\mu(t-t_k)}-1}{\mu}+\frac{1}{C_0}\sum_{n=1}^{p}|r_n|U_0^n\int_{t_k}^{t}e^{n\mu(s-t_k)}ds\right.$$
$$\left.+\frac{\kappa U_0e^{-2\mu T}}{Z_0C_0}\frac{e^{\mu(t-t_k)}-1}{\mu}+\frac{\kappa}{C_0}\sum_{n=1}^{p}|r_n|U_0^ne^{-2n\mu T}\int_{t_k}^{t}e^{n\mu(s-t_k)}ds\right]+\kappa|u(t-2T)|$$
$$\leq e^{\mu(t-t_k)}U_0\left[\frac{1}{\mu C_0}\left(\frac{2e^{-\mu T}}{Z_0+R_0}+\frac{1+\kappa e^{-2\mu T}}{Z_0}+\sum_{n=1}^{p}\frac{|r_n|U_0^{n-1}\left(e^{(n-1)\mu T_0}-1\right)\left(1+\kappa e^{-2n\mu T}\right)}{n}\right)\right.$$
$$\left.+\kappa e^{-2\mu T}\right],$$

$$
\begin{aligned}
B_2 \le{}& \left[\frac{2U_0 e^{-\mu T}}{C_0(Z_0+R_0)} \frac{e^{\mu(t_{k+1}-t_k)}-1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu(t_{k+1}-t_k)}-1}{\mu} + \frac{1}{C_0}\sum_{n=1}^{p} |r_n| U_0^n \int_{t_k}^{t_{k+1}} e^{n\mu(s-T)} ds \right.\\
&\left. + \frac{\kappa U_0 e^{-2\mu T}}{Z_0 C_0} \frac{e^{\mu(t_{k+1}-t_k)}-1}{\mu} + \frac{\kappa}{C_0}\sum_{n=1}^{p} |r_n| U_0^n e^{-2n\mu T} \int_{t_k}^{t_{k+1}} e^{n\mu(s-T)} ds \right]\\
&+ \kappa |u(t_{k+1}-2T) - u(t_k-2T)|\\
\le{}& \left[\frac{2U_0 e^{-\mu T}}{C_0(Z_0+R_0)} \frac{e^{\mu T_0}-1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu T_0}-1}{\mu} + \frac{1}{C_0}\sum_{n=1}^{p} |r_n| U_0^n \frac{e^{n\mu T_0}-1}{n\mu} \right.\\
&\left. + \frac{\kappa U_0 e^{-2\mu T}}{C_0 Z_0} \frac{e^{\mu T_0}-1}{\mu} + \frac{\kappa}{C_0}\sum_{n=1}^{p} |r_n| U_0^n e^{-2n\mu T} \frac{e^{n\mu T_0}-1}{n\mu} \right]\\
\le{}& e^{\mu(t-t_k)} \frac{U_0}{\mu C_0} \left(\frac{2e^{-\mu T}\left(e^{\mu T_0}-1\right)}{Z_0+R_0} + \frac{\left(e^{\mu T_0}-1\right)\left(1+\kappa e^{-2\mu T}\right)}{Z_0} \right.\\
&\left. + \sum_{n=1}^{p} \frac{|r_n| U_0^{n-1}\left(1+\kappa e^{-2n\mu T}\right)\left(e^{n\mu T_0}-1\right)}{n} \right).
\end{aligned}
\tag{2.7}
$$

Therefore, for sufficiently large $\mu > 0$, we obtain

$$
\begin{aligned}
&|(Bu)(t)|\\
&\le e^{\mu(t-t_k)} U_0 \left[\frac{1}{\mu C_0} \left(\frac{2e^{-\mu T}}{Z_0+R_0} + \frac{1+\kappa e^{-2\mu T}}{Z_0} + \sum_{n=1}^{p} \frac{|r_n| U_0^{n-1}\left(e^{(n-1)\mu T_0}-1\right)\left(1+\kappa e^{-2n\mu T}\right)}{n} \right) \right.\\
&\qquad \left. + \kappa e^{-2\mu T} \right]\\
&\quad + e^{\mu(t-t_k)} U_0 \frac{1}{\mu C_0} \left(\frac{2e^{-\mu T}\left(e^{\mu T_0}-1\right)}{Z_0+R_0} + \frac{\left(e^{\mu T_0}-1\right)\left(1+\kappa e^{-2\mu T}\right)}{Z_0} \right.\\
&\qquad \left. + \sum_{n=1}^{p} |r_n| U_0^{n-1} \frac{e^{n\mu T_0}-1}{n} \left(1+\kappa e^{-2n\mu T}\right) \right)\\
&\le e^{\mu(t-t_k)} U_0 \left[\frac{1}{\mu C_0} \left(\frac{2e^{-\mu T} e^{\mu T_0}}{Z_0+R_0} + \frac{e^{\mu T_0}\left(1+\kappa e^{-2\mu T}\right)}{Z_0} \right. \right.\\
&\qquad \left. \left. + \sum_{n=1}^{p} \frac{|r_n| U_0^{n-1}\left(e^{n\mu T_0} + e^{(n-1)\mu T_0} - 2\right)\left(1+\kappa e^{-2n\mu T}\right)}{n} \right) + \kappa e^{-2\mu T} \right]\\
&\le e^{\mu(t-t_k)} U_0.
\end{aligned}
\tag{2.8}
$$

Consequently, the operator B maps M_{SU} into itself.

We show that B is a contractive operator. Indeed,

$$\begin{aligned}|B(u)(t)-B(\overline{u})(t)| \leq & \left|\int_{t_k}^{t}[U(u)(s)-U(\overline{u})(s)]ds\right| + \left|\int_{t_k}^{t_{k+1}}[U(u)(s)-U(\overline{u})(s)]ds\right| \\ & \equiv B_1+B_2, \quad t\in[t_k,t_{k+1}].\end{aligned} \tag{2.9}$$

We have

$$\begin{aligned}
B_1 \leq & \left[\frac{1}{C_0Z_0}\int_{t_k}^{t}|u(s)-\overline{u}(s)|ds + \frac{1}{C_0}\sum_{n=1}^{p}|r_n|\int_{t_k}^{t}\left|u^n(s)-\overline{u}^n(s)\right|ds\right. \\
& \left. +\frac{\kappa}{Z_0C_0}\int_{t_k}^{t}|u(s-2T)-\overline{u}(s-2T)|ds + \frac{\kappa}{C_0}\sum_{n=1}^{p}|r_n|\int_{t_k}^{t}\left|u^n(s-2T)-\overline{u}^n(s-2T)\right|ds\right] \\
& +\kappa\left|\int_{t_k}^{t}\left(\dot{u}(s-2T)-\dot{\overline{u}}(s-2T)\right)ds\right| \\
\leq & \left[\frac{\rho_k(u,\overline{u})}{C_0Z_0}\frac{e^{\mu(t-t_k)}-1}{\mu} + \frac{1}{C_0}\sum_{n=1}^{p}n|r_n|\operatorname{ess\,sup}\left\{\left|u^{n-1}(s)\right| : s\in[t_k,t_{k+1}]\right\}\int_{t_k}^{t}|u(s)-\overline{u}(s)|ds\right. \\
& + \frac{\kappa}{Z_0C_0}\rho_k(u,\overline{u})e^{-2\mu T}\ \frac{e^{\mu(t-t_k)}-1}{\mu} \\
& \left.+\frac{\kappa}{C_0}\sum_{n=1}^{p}n\,|r_n|\operatorname{ess\,sup}\left\{u^{n-1}(s-2T) : s\in[t_k,t_{k+1}]\right\}\int_{t_k}^{t}|u(s-2T)-\overline{u}(s-2T)|ds\right] \\
& +\kappa\rho_k\left(\dot{u},\dot{\overline{u}}\right)e^{-2\mu T}\frac{e^{\mu(t-t_k)}-1}{\mu} \\
\leq & e^{\mu(t-t_k)}\left[\frac{\rho_k(u,\overline{u})}{\mu C_0Z_0} + \frac{\rho_k(u,\overline{u})}{\mu C_0}\sum_{n=1}^{p}n|r_n|U_0^{n-1}e^{(n-1)\mu(t_{k+1}-t_k)} + \frac{\kappa\rho_k(u,\overline{u})e^{-2\mu T}}{\mu Z_0C_0}\right. \\
& \left.+\frac{\kappa\rho_k(u,\overline{u})e^{-2\mu T}}{\mu C_0}\sum_{n=1}^{p}n|r_n|U_0^{n-1}e^{-2(n-1)\mu T}e^{(n-1)\mu(t_{k+1}-t_k)}\right] + e^{\mu(t-t_k)}\frac{\kappa\rho_k\left(\dot{u},\dot{\overline{u}}\right)e^{-2\mu T}}{\mu} \\
\leq & e^{\mu(t-t_k)}\rho_k\left(\dot{u},\dot{\overline{u}}\right)\left[\frac{1}{\mu^2}\left(\frac{1+\kappa e^{-2\mu T}}{C_0Z_0} + \frac{1}{C_0}\sum_{n=1}^{p}n|r_n|U_0^{n-1}\left(1+\kappa e^{-2n\mu T}\right)e^{(n-1)\mu T_0}\right) + \frac{\kappa e^{-2\mu T}}{\mu}\right] \\
\leq & e^{\mu(t-t_k)}\rho_\mu^{(k)}(u,\overline{u})\left[\frac{1}{\mu^2}\left(\frac{1+\kappa e^{-2\mu T}}{C_0Z_0} + \frac{1}{C_0}\sum_{n=1}^{p}n|r_n|U_0^{n-1}\left(1+\kappa e^{-2n\mu T}\right)e^{(n-1)\mu T_0}\right) + \frac{\kappa e^{-2\mu T}}{\mu}\right],
\end{aligned}$$

$$\begin{aligned}
B_2 \leq & \left[\frac{1}{C_0Z_0}\int_{t_k}^{t_{k+1}}|u(s)-\overline{u}(s)|ds + \frac{1}{C_0}\sum_{n=1}^{p}|r_n|\int_{t_k}^{t_{k+1}}\left|u^n(s)-\overline{u}^n(s)\right|ds\right. \\
& \left.+\frac{\kappa}{Z_0C_0}\int_{t_k}^{t_{k+1}}|u(s-2T)-\overline{u}(s-2T)|ds + \frac{\kappa}{C_0}\sum_{n=1}^{p}|r_n|\int_{t_k}^{t_{k+1}}\left|u^n(s-2T)-\overline{u}^n(s-2T)\right|ds\right]
\end{aligned}$$

$$
\begin{aligned}
&+\kappa\left|\int_{t_k}^{t_{k+1}}(\dot{u}(s-2T)-\dot{u}(s-2T))ds\right| \\
&\leq \left[\frac{\rho_k(u,\overline{u})}{C_0Z_0}\frac{e^{\mu(t_{k+1}-t_k)}-1}{\mu}\right. \\
&\quad+\frac{1}{C_0}\sum_{n=1}^{p}|r_n|n.\ \operatorname{ess\,sup}\left\{\left|u^{n-1}(s)\right|: s\in[t_k,t_{k+1}]\right\}\int_{t_k}^{t_{k+1}}|u(s)-\overline{u}(s)|ds \\
&\quad+\frac{\kappa}{Z_0C_0}\rho_k(u,\overline{u})e^{-2\mu T}\ \frac{e^{\mu(t_{k+1}-t_k)}-1}{\mu} \\
&\quad\left.+\frac{\kappa}{C_0}\sum_{n=1}^{p}|r_n|\ n.\ \operatorname{ess\,sup}\left\{u^{n-1}(s-2T): s\in[t_k,t_{k+1}]\right\}\int_{t_k}^{t_{k+1}}|u(s-2T)-\overline{u}(s-2T)|ds\right] \\
&\leq \left[\frac{\rho_k(u,\overline{u})}{C_0Z_0}\frac{e^{\mu(t_{k+1}-t_k)}-1}{\mu}+\frac{\rho_k(u,\overline{u})}{C_0}\frac{e^{\mu(t_{k+1}-t_k)}-1}{\mu}\sum_{n=1}^{p}|r_n|nU_0^{n-1}e^{(n-1)\mu(t_{k+1}-t_k)}\right. \\
&\quad+\frac{\kappa\rho_k(u,\overline{u})e^{-2\mu T}}{Z_0C_0}\ \frac{e^{\mu(t_{k+1}-t_k)}-1}{\mu} \\
&\quad\left.+\frac{\kappa\rho_k(u,\overline{u})e^{-2\mu T}}{C_0}\frac{e^{\mu(t_{k+1}-t_k)}-1}{\mu}\sum_{n=1}^{p}n|r_n|U_0^{n-1}e^{(n-1)(\mu T_0-2\mu T)}\right] \\
&\leq \rho_k\left(\dot{u},\dot{\overline{u}}\right)\frac{e^{\mu T_0}-1}{\mu^2}\left(\frac{1+\kappa e^{-2\mu T}}{C_0Z_0}+\frac{1}{C_0}\sum_{n=1}^{p}|r_n|nU_0^{n-1}e^{(n-1)\mu T_0}\left(1+\kappa e^{-2n\mu T}\right)\right) \\
&\leq \rho_u^{(k)}(u,\overline{u})\frac{e^{\mu T_0}-1}{\mu^2C_0}\left(\frac{1+\kappa e^{-2\mu T}}{Z_0}+\sum_{n=1}^{p}|r_n|nU_0^{n-1}e^{(n-1)\mu T_0}\left(1+\kappa e^{-2n\mu T}\right)\right).
\end{aligned}
\tag{2.10}
$$

Consequently,

$$
\begin{aligned}
&|B(u)(t)-B(\overline{u})(t)| \\
&\quad\leq e^{\mu(t-t_k)}\rho_\mu^{(k)}(u,\overline{u})\left[\frac{1}{\mu^2}\left(\frac{1+\kappa e^{-2\mu T}}{C_0Z_0}+\frac{1}{C_0}\sum_{n=1}^{p}n|r_n|U_0^{n-1}\left(1+\kappa e^{-2n\mu T}\right)e^{(n-1)\mu T_0}\right)+\frac{\kappa e^{-2\mu T}}{\mu}\right] \\
&\quad\quad+\rho_u^{(k)}(u,\overline{u})\frac{e^{\mu T_0}-1}{\mu^2}\left(\frac{1+\kappa e^{-2\mu T}}{C_0Z_0}+\frac{1}{C_0}\sum_{n=1}^{p}|r_n|nU_0^{n-1}e^{(n-1)\mu T_0}\left(1+\kappa e^{-2n\mu T}\right)\right) \\
&\quad\leq \rho_u^{(k)}(u,\overline{u})\left[\frac{e^{\mu T_0}}{\mu^2}\left(\frac{1+\kappa e^{-2\mu T}}{C_0Z_0}+\frac{1}{C_0}\sum_{n=1}^{p}|r_n|nU_0^{n-1}e^{(n-1)\mu T_0}\left(1+\kappa e^{-2n\mu T}\right)\right)+\frac{\kappa e^{-2\mu T}}{\mu}\right]. \\
&\quad\equiv e^{\mu(t-t_k)}K_U\rho_\mu^{(k)}(u,\overline{u}).
\end{aligned}
\tag{2.11}
$$

Therefore, $\rho_k(Bu, B\overline{u}) \leq K_U \rho_\mu^{(k)}(u,\overline{u})$.

It remains to estimate the derivative of B.

We have

$$\left|\dot{B}(u)(t) - \dot{B}(\overline{u})(t)\right| \leq |\, U(u)(s) - U(\overline{u})(s)| \\ + \frac{1}{t_{k+1}-t_k}\left|\int_{t_k}^{t_{k+1}} [U(u)(s) - U(\overline{u})(s)\ ]ds\right| \equiv \dot{B}_1 + \dot{B}_2. \tag{2.12}$$

We have

$$\begin{aligned}
\dot{B}_1 \leq{}& \frac{1}{C_0 Z_0}|u(t)-\overline{u}(t)| + \frac{1}{C_0}\sum_{n=1}^{p}|r_n|\left|u^n(t)-\overline{u}^n(t)\right| + \frac{\kappa}{C_0 Z_0}|u(t-2T)-\overline{u}(t-2T)| \\
&+ \frac{\kappa}{C_0}\sum_{n=1}^{p}|r_n|\left|u^n(t-2T)-\overline{u}^n(t-2T)\right| + \kappa\left|\dot{u}(t-2T)-\dot{\overline{u}}(t-2T)\right| \\
\leq{}& \frac{e^{\mu(t-t_k)}\rho_k(u,\overline{u})}{C_0 Z_0} + \frac{1}{C_0}\sum_{n=1}^{p}|r_n| n \operatorname{ess\,sup}\left\{u^{n-1}(t) : t \in [t_k, t_{k+1}]\right\}|u(t)-\overline{u}(t)| \\
&+ \frac{\kappa e^{\mu(t-t_k)}\rho_k(u,\overline{u})e^{-2\mu T}}{C_0 Z_0} \\
&+ \frac{\kappa}{C_0}\sum_{n=1}^{p} n|r_n| \operatorname{ess\,sup}\left\{u^{n-1}(t-2T) : t \in [t_k, t_{k+1}]\right\}|u(t-2T)-\overline{u}(t-2T)| \\
&+ \kappa\left|\dot{u}(t-2T)-\dot{\overline{u}}(t-2T)\right| \\
\leq{}& \frac{e^{\mu(t-t_k)}\rho_k\left(\dot{u},\dot{\overline{u}}\right)}{\mu C_0 Z_0} + \frac{e^{\mu(t-t_k)}\rho_k\left(\dot{u},\dot{\overline{u}}\right)}{\mu C_0}\sum_{n=1}^{p}|r_n| n U_0^{n-1} e^{(n-1)\mu(t_{k+1}-t_k)} + e^{\mu(t-t_k)}\frac{\kappa\rho_k\left(\dot{u},\dot{\overline{u}}\right)e^{-2\mu T}}{\mu C_0 Z_0} \\
&+ \frac{e^{\mu(t-t_k)}\rho_k\left(\dot{u},\dot{\overline{u}}\right)\kappa e^{-2\mu T}}{\mu C_0}\sum_{n=1}^{p} n|r_n| U_0^{n-1} e^{(n-1)\mu(t_{k+1}-t_k)} + e^{\mu(t-t_k)}\kappa\rho_k\left(\dot{u},\dot{\overline{u}}\right)e^{-2\mu T} \\
\leq{}& e^{\mu(t-t_k)}\rho_\mu^{(k)}(u,\overline{u})\left[\frac{1+\kappa e^{-2\mu T}}{\mu C_0}\left(\frac{1}{Z_0} + \sum_{n=1}^{p}|r_n| n U_0^{n-1} e^{(n-1)\mu T_0}\right) + \kappa e^{-2\mu T}\right], \\
\dot{B}_2 \leq{}& \frac{1}{t_{k+1}-t_k}\left|\int_{t_k}^{t_{k+1}} (U(u)(s)-U(\overline{u})(s))ds\right| \leq \frac{1}{l_0}\left|\int_{t_k}^{t_{k+1}} (U(u)(s)-U(\overline{u})(s))ds\right| \\
\leq{}& \rho_u^{(k)}(u,\overline{u})\frac{e^{\mu T_0}-1}{\mu^2 C_0 l_0}\left(\frac{1+\kappa e^{-2\mu T}}{Z_0} + \sum_{n=1}^{p}|r_n| n U_0^{n-1} e^{(n-1)\mu T_0}\left(1+\kappa e^{-2n\mu T}\right)\right).
\end{aligned} \tag{2.13}$$

Therefore,

$$
\begin{aligned}
&\left|\dot{B}(u)(t)-\dot{B}(\overline{u})(t)\right| \\
&\quad\leq e^{\mu(t-t_k)}\rho_\mu^{(k)}(u,\overline{u})\left[\frac{1+\kappa e^{-2\mu T}}{\mu C_0}\left(\frac{1}{Z_0}+\sum_{n=1}^{p}|r_n|n\,U_0^{n-1}e^{(n-1)\mu T_0}\right)+\kappa e^{-2\mu T}\right] \\
&\qquad+\rho_u^{(k)}(u,\overline{u})\frac{e^{\mu T_0}-1}{\mu^2 C_0 l_0}\left(\frac{1+\kappa e^{-2\mu T}}{Z_0}+\sum_{n=1}^{p}|r_n|nU_0^{n-1}e^{(n-1)\mu T_0}\left(1+\kappa e^{-2n\mu T}\right)\right) \\
&\quad\leq \rho_\mu^{(k)}(u,\overline{u})\left[\frac{\left(e^{\mu T_0}+\mu\tau_0-1\right)\left(1+\kappa e^{-2\mu T}\right)}{\mu^2 C_0 l_0}\left(\frac{1}{Z_0}+\sum_{n=1}^{p}|r_n|nU_0^{n-1}e^{(n-1)\mu T_0}\right)+\kappa e^{-2\mu T}\right] \\
&\quad\equiv e^{\mu(t-t_k)}\dot{K}_U\;\rho_\mu^{(k)}(u,\overline{u}).
\end{aligned}
\tag{2.14}
$$

It follows $\rho_k(\dot{B}(u),\dot{B}(\overline{u}))\leq e^{\mu(t-t_k)}\dot{K}_U\rho_\mu^{(k)}(u,\overline{u})$.

Then $\rho_\mu^{(k)}(B(u),B(\overline{u}))\leq\max\{K_U,\dot{K}_U\}\rho_\mu^{(k)}(u,\overline{u})$.

Consequently,

$$
\rho_\mu^{(k)}(Bu,B\overline{u})\leq\overline{K}\rho_\mu^{(k)}(u,\overline{u})\quad(k=0,1,2,\ldots),
\tag{2.15}
$$

where $\overline{K}=\max\{K_U,\dot{K}_U\}<1$ does not depend on u and k.

We have to verify that M_{SU} is j-bounded. Indeed, since j is an identity mapping,

$$
\rho_u^{j^n(k)}(u,\overline{u})\leq\rho_u^{(k)}(u,\overline{u})<\infty\quad(n=0,1,2,\ldots).
\tag{2.16}
$$

Therefore, in view of the fixed point theorem for contractive mappings in uniform spaces (cf. [13]), the operator B has a unique fixed point, and it is an oscillatory solution of (1.4).

Theorem 2.2 is thus proved. □

3. Numerical Example

Finally, we summarize all inequalities needed for the applications:

$$
\begin{aligned}
&\frac{1}{\mu C_0}\left(\frac{2e^{-\mu T}e^{\mu T_0}}{Z_0+R_0}+\frac{e^{\mu T_0}\left(1+\kappa e^{-2\mu T}\right)}{Z_0}\right. \\
&\qquad\left.+\sum_{n=1}^{p}\frac{|r_n|{U_0}^{n-1}\left(e^{n\mu T_0}+e^{(n-1)\mu T_0}-2\right)\left(1+\kappa e^{-2n\mu T}\right)}{n}\right)+\kappa e^{-2\mu T}\leq 1, \\
&K_U=\frac{e^{\mu_0}}{\mu^2}\left(\frac{1+\kappa e^{-2\mu T}}{C_0 Z_0}+\frac{1}{C_0}\sum_{n=1}^{p}|r_n|nU_0^{n-1}e^{(n-1)\mu_0}\left(1+\kappa e^{-2n\mu T}\right)\right)+\frac{\kappa e^{-2\mu T}}{\mu}<1, \\
&\dot{K}_U=\frac{\left(e^{\mu_0}+\mu\tau_0-1\right)\left(1+\kappa e^{-2\mu T}\right)}{\mu^2 C_0 l_0}\left(\frac{1}{Z_0}+\sum_{n=1}^{p}|r_n|nU_0^{n-1}e^{(n-1)\mu_0}\right)+\kappa e^{-2\mu T}<1.
\end{aligned}
\tag{3.1}
$$

Consider a line with the following specific parameters:

$$\Lambda = 1\,\text{m}, \qquad L = 0,2\,\mu\text{H/m}, \qquad C = 80\,\text{pF/m},$$
$$\nu = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0,2\cdot 10^{-6}\cdot 80\cdot 10^{-12}}} = \frac{1}{4\cdot 10^{-9}} = 2,5\cdot 10^{8},$$
$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0,2\cdot 10^{-6}}{80\cdot 10^{-12}}} = 50\,\Omega, \qquad R_0 = 45\,\Omega, \qquad C_0 = 8\,\text{pF} = 8\cdot 10^{-12}\,\text{F}. \tag{3.2}$$

Then, $T = \Lambda\sqrt{LC} = 4.10^{-9}\text{s}$; $\kappa = (Z_0 - R_0)/(Z_0 + R_0) = 1/19 = 0,0526$.
Let us check the propagation of millimeter waves $\lambda_0 = 10^{-3}$ m. We have

$$f_0 = \frac{1}{\lambda_0\sqrt{LC}} = \frac{1}{10^{-3}\cdot\ 4\cdot 10^{-9}} = 2,5\cdot 10^{11}\,\text{Hz}$$
$$\Longrightarrow T_0 = \frac{1}{f_0} = \frac{1}{2,5\cdot 10^{11}} = 4\cdot 10^{-12}\,\text{sec.}; \qquad l = 2\cdot 10^{-12}\,\text{sec.} \tag{3.3}$$

If we choose $\mu = (1/4)10^{12}$, then $\mu T_0 = \mu_0 = 1$, $\mu\tau_0 = (1/2)$, and $T = 4\cdot 10^{-9}\cdot(1/4)\cdot 10^{12}T_0 = 1000\cdot T_0$.

Consequently, $\mu T = (1/4)10^{12}\cdot 2\cdot 10^{-8} = (1/2)10^{4}$, $\mu C_0 = (1/4)10^{12}\cdot 8\cdot 10^{4}10^{-12} = 2$, and $\mu^2 C_0 = (1/2)\cdot 10^{12}$.

Since $e^{-\mu T} = e^{-5000} = 0$, then the above inequalities (omitting the second one) become

$$\frac{e}{100} + \sum_{n=1}^{p}|r_n|U_0^{n-1}\frac{e^n + e^{n-1} - 2}{2n} \leq 1,$$
$$\dot{K}_U = 2\left(e - \frac{1}{2}\right)\left(\frac{1}{50} + \sum_{n=1}^{p}|r_n|nU_0^{n-1}e^{n-1}\right) < 1. \tag{3.4}$$

If the V-I characteristic of the nonlinear resistive element is $f(u) = -0,12u + 0,8u^3$, then $U_0 \leq 0,41$; $\dot{K}_U = U_0 < 0,06$. It follows that $U_0 < 0,06$.

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