Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2011, Article ID 989065, 10 pages
doi:10.1155/2011/989065

Research Article

Convergence of the New Iterative Method

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Received 7 May 2011; Accepted 21 August 2011

Academic Editor: Dexing Kong

A new iterative method introduced by Daftardar-Gejji and Jafari (2006) (DJ Method) is an efficient technique to solve nonlinear functional equations. In the present paper, sufficiency conditions for convergence of DJM have been presented. Further equivalence of DJM and Adomian decomposition method is established.

1. Introduction

A variety of problems in physics, chemistry, biology, and engineering can be formulated in terms of the nonlinear functional equation

$$u = f + N(u), \tag{1.1}$$

where f is a given function, and N is the nonlinear operator. Equation (1.1) represents integral equations, ordinary differential equations (ODEs), partial differential equations (PDEs), differential equations involving fractional order, systems of ODE/PDE, and so on. Various methods such as Laplace and Fourier transform and Green's function method have been used to solve linear equations. For solving nonlinear equations, however, one has to resort to numerical/iterative methods. Adomian decomposition method (ADM) has proved to be a useful tool for solving functional equation (1.1) [1–3], since it offers certain advantages over numerical methods. This method yields solutions in the form of rapidly converging infinite series which can be effectively approximated by calculating only first few terms. In the last two decades, extensive work has been done using ADM as it provides analytical approximate solutions for nonlinear equations without linearization, perturbation, or discretization. Though Adomian's technique is simple in principle, it involves tedious calculations of Adomian polynomials [4, 5]. Researchers have explored symbolic computational packages such as Mathematica for finding Adomian polynomials [6, 7].

As a pursuit of this, Daftardar-Gejji and Jafari [8] have introduced a new decomposition method (DJM) to solve (1.1) which is simple and easy to implement. It is economical in terms of computer power/memory and does not involve tedious calculations such as Adomian polynomials. DJM has been employed successfully to solve a variety of problems.

Present paper analyzes convergence of DJM in detail and establishes its equivalence to ADM.

2. Preliminaries

Let X, Y be Banach spaces and $F : X \to Y$ a map. $L(X, Y)$ denotes the set of all linear maps from X to Y. $L(X, Y)$ is also a Banach space.

Definition 2.1 (see [9]). F is said to be Fréchet differentiable at $x \in X$ if there exists a continuous linear map $A : X \to Y$ such that

$$F(x+h) - F(x) = Ah + w(x,h), \tag{2.1}$$

where

$$\lim_{\|h\| \to 0} \frac{\|w(x,h)\|}{\|h\|} = 0. \tag{2.2}$$

A is called the Fréchet derivative of F at x and is also denoted by $F'(x)$. Its value at h is denoted by $F'(x)(h)$.

Note that F' is a linear map from X to $L(X, Y)$.

Definition 2.2 (see [9]). F is said to be twice differentiable if the map $F' : X \to L(X, Y)$ is Fréchet differentiable. The second derivative of F is denoted by F'' and is a linear map from X to $L(X, L(X, Y))$.

Note that $L(X, L(X, Y))$ is isomorphic to $L(X \times X, Y)$.

Theorem 2.3 (see [9]). *The map $F''(x) \in L(X^2, Y)$ is symmetric, that is, $F''(x)(x_1, x_2) = F''(x)(x_2, x_1)$, $x_1, x_2 \in X$.*

In this manner, $F^{(3)}(x), F^{(4)}(x), \ldots$ are inductively defined and $F^{(n)}(x) \in L(X^n, Y)$ is multilinear and symmetric map.

Theorem 2.4 ([9, Taylor's theorem]). *Suppose that $F \in C^n(U)$, where U is an open subset of X containing the line segment from x_0 to h, then*

$$
\begin{aligned}
F(x_0+h) &= F(x_0) + F'(x_0)(h) + \frac{1}{2!}F''(x_0)(h,h) + \cdots + \frac{1}{(n-1)!}F^{(n-1)}(x_0)\underbrace{(h,\ldots,h)}_{n-1\ \text{times}} \\
&\quad + \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1}\, F^{(n)}(x_0+th)\underbrace{(h,\ldots,h)}_{n\ \text{times}}\, dt \\
&= \sum_{k=0}^{n} \frac{1}{k!} F^{(k)}(x_0)\underbrace{(h,\ldots,h)}_{k\ \text{times}} + q(x),
\end{aligned}
\tag{2.3}
$$

where $q(x)$ is such that $\|q(x)\| = O(\|x\|^n)$.

Since $F^{(k)}(x)$ is symmetric, we denote $\underbrace{(h,\ldots,h)}_{k\ \text{times}}$ by h^k.

3. An Iterative Method

Daftardar-Gejji and Jafari [8] have considered the following functional equation:

$$u = f + N(u), \tag{3.1}$$

where N is a nonlinear operator from a Banach space $B \to B$, and f is a given element of the Banach space B. u is assumed to be a solution of (3.1) having the series form

$$u = \sum_{i=0}^{\infty} u_i. \tag{3.2}$$

The nonlinear operator N is decomposed as

$$N(u) = N(u_0) + [N(u_0+u_1) - N(u_0)] + [N(u_0+u_1+u_2) - N(u_0+u_1)] + \cdots. \tag{3.3}$$

Let $G_0 = N(u_0)$ and

$$G_n = N\left(\sum_{i=0}^{n} u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), \quad n = 1, 2, \ldots. \tag{3.4}$$

Then $N(u) = \sum_{i=0}^{\infty} G_i$.

Set

$$u_0 = f \tag{3.5}$$

$$u_n = G_{n-1}, \quad n = 1, 2, \ldots, \tag{3.6}$$

then

$$u = \sum_{i=0}^{\infty} u_i \tag{3.7}$$

is a solution of (3.1).

3.1. Taylor Series and DJM

Using Theorem 2.4,

$$\begin{aligned} G_1 &= N(u_0 + u_1) - N(u_0) \\ &= N(u_0) + N'(u_0)u_1 + N''(u_0)\frac{u_1^2}{2!} + \cdots - N(u_0) \\ &= \sum_{k=1}^{\infty} N^{(k)}(u_0)\frac{u_1^k}{k!}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} G_2 &= N(u_0 + u_1 + u_2) - N(u_0 + u_1) = N'(u_0 + u_1)u_2 + N''(u_0 + u_1)\frac{u_2^2}{2!} + \cdots \\ &= \sum_{j=1}^{\infty}\left[\sum_{i=0}^{\infty} N^{(i+j)}(u_0)\frac{u_1^i}{i!}\right]\frac{u_2^j}{j!}, \end{aligned} \tag{3.9}$$

$$G_3 = \sum_{i_3=1}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty} N^{(i_1+i_2+i_3)}(u_0)\frac{u_3^{i_3}}{i_3!}\frac{u_2^{i_2}}{i_2!}\frac{u_1^{i_1}}{i_1!}. \tag{3.10}$$

In general,

$$G_n = \sum_{i_n=1}^{\infty}\sum_{i_{n-1}=0}^{\infty}\cdots\sum_{i_1=0}^{\infty}\left[N^{(\sum_{k=1}^{n} i_k)}(u_0)\left(\prod_{j=1}^{n}\frac{u_j^{i_j}}{i_j!}\right)\right]. \tag{3.11}$$

In view of (3.3)–(3.11),

$$\begin{aligned} N(u) &= G_0 + G_1 + G_2 + G_3 + \cdots \\ &= N(u_0) + \sum_{k=1}^{\infty} N^{(k)}(u_0)\frac{u_1^k}{k!} + \sum_{j=1}^{\infty}\left[\sum_{i=0}^{\infty} N^{(i+j)}(u_0)\frac{u_1^i}{i!}\right]\frac{u_2^j}{j!} \\ &\quad + \sum_{i_3=1}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty} N^{(i_1+i_2+i_3)}(u_0)\frac{u_3^{i_3}}{i_3!}\frac{u_2^{i_2}}{i_2!}\frac{u_1^{i_1}}{i_1!}\cdots \end{aligned}$$

$$
\begin{aligned}
&= N(u_0) + N'(u_0)[u_1 + u_2 + u_3 + \cdots] \\
&\quad + N''(u_0)\left[\frac{u_1^2}{2!} + \left(u_1u_2 + \frac{u_2^2}{2!}\right) + \left(u_3u_2 + u_3u_1 + \frac{u_3^2}{2!}\right) + \cdots\right] \\
&\quad + N^{(3)}(u_0)\left[\frac{u_1^3}{3!} + \left(u_2\frac{u_1^2}{2!} + \frac{u_2^2}{2!}u_1 + \frac{u_2^3}{3!}\right) + \cdots\right] + \cdots .
\end{aligned}
\tag{3.12}
$$

Using (3.22),

$$
\begin{aligned}
N(u) = N(u_0) + N'(u_0)[u_1 + u_2 + u_3 + \cdots] + \frac{N''(u_0)}{2!}[u_1 + u_2 + u_3 + \cdots]^2 \\
+ \frac{N^{(3)}(u_0)}{3!}[u_1 + u_2 + u_3 + \cdots]^3 + \cdots .
\end{aligned}
\tag{3.13}
$$

Equation (3.13) is Taylor series expansion of $N(u)$ around u_0. Thus, DJM is equivalent to Taylor series expansion around u_0. In Adomian decomposition method (ADM) [1], right hand side of (3.13) is written as

$$
\underbrace{N(u_0)}_{A_0} + \underbrace{N'(u_0)u_1}_{A_1} + \underbrace{\left(N''(u_0)\frac{u_1^2}{2!} + N'(u_0)u_2\right)}_{A_2} + \underbrace{\left(N^{(3)}(u_0)\frac{u_1^3}{3!} + N''(u_0)u_1u_2 + N'(u_0)u_3\right)}_{A_3} + \cdots ,
\tag{3.14}
$$

where $A_0, A_1, \ldots$ are Adomian polynomials, and $u_{n+1} = A_n,\ n \geq 0$.

3.2. Convergence of DJM

Now, we present the condition for convergence of DJM.

Theorem 3.1. *If N is $C^{(\infty)}$ in a neighborhood of u_0 and*

$$
\left\| N^{(n)}(u_0) \right\| = \operatorname{Sup}\left\{ N^{(n)}(u_0)(h_1, \ldots, h_n) : \|h_i\| \leq 1,\ 1 \leq i \leq n \right\} \leq L,
\tag{3.15}
$$

for any n and for some real $L > 0$ and $\|u_i\| \leq M < 1/e$, $i = 1, 2, \ldots$, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent, and moreover,

$$
\|G_n\| \leq LM^n e^{n-1}(e-1), \qquad n = 1, 2, \ldots .
\tag{3.16}
$$

Proof. In view of (3.11),

$$
\|G_n\| \leq LM^n \sum_{i_n=1}^{\infty} \sum_{i_{n-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} \left(\prod_{j=1}^{n} \frac{1}{i_j!} \right) = LM^n e^{n-1}(e-1).
\tag{3.17}
$$

Thus, the series $\sum_{n=1}^{\infty} \|G_n\|$ is dominated by the convergent series $LM(e-1)\sum_{n=1}^{\infty}(Me)^{n-1}$, where $M < 1/e$. Hence, $\sum_{n=0}^{\infty} G_n$ is absolutely convergent, due to the comparison test.

As it is difficult to show boundedness of $\|u_i\|$, for all i, a more useful result is proved in the following theorem, where conditions on $N^{(k)}(u_0)$ are given which are sufficient to guarantee convergence of the series. □

Theorem 3.2. *If N is $C^{(\infty)}$ and $\|N^{(n)}(u_0)\| \leq M \leq e^{-1}$, for all n, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent.*

Proof. Consider the recurrence relation

$$\xi_n = \xi_0 \exp(\xi_{n-1}), \quad n = 1, 2, 3, \ldots, \tag{3.18}$$

where $\xi_0 = M$. Define $\eta_n = \xi_n - \xi_{n-1}$, $n = 1, 2, 3, \ldots$. Using (3.6), (3.11), and the hypothesis of Theorem 3.2, we observe that

$$\|G_n\| \leq \eta_n, \quad n = 1, 2, 3, \cdots. \tag{3.19}$$

Let

$$\sigma_n = \sum_{i=1}^{n} \eta_i = \xi_n - \xi_0. \tag{3.20}$$

Note that $\xi_0 = e^{-1} > 0$, $\xi_1 = \xi_0 \exp(\xi_0) > \xi_0$, and $\xi_2 = \xi_0 \exp(\xi_1) > \xi_0 \exp(\xi_0) = \xi_1$. In general, $\xi_n > \xi_{n-1} > 0$. Hence, $\sum \eta_n$ is a series of positive real numbers. Note that

$$\begin{gathered} 0 < \xi_0 = M = e^{-1} < 1, \\ 0 < \xi_1 = \xi_0 \exp(\xi_0) < \xi_0 e^1 = e^{-1} e^1 = 1, \\ 0 < \xi_2 = \xi_0 \exp(\xi_1) < \xi_0 e^1 = 1. \end{gathered} \tag{3.21}$$

In general, $0 < \xi_n < 1$. Hence, $\sigma_n = \xi_n - \xi_0 < 1$. This implies that $\{\sigma_n\}_{n=1}^{\infty}$ is bounded above by 1, and hence convergent. Therefore, $\sum G_n$ is absolutely convergent by comparison test. □

3.3. Illustrative Example

Consider the nonlinear IVP,

$$y'(t) = \frac{1}{2} + \frac{1}{8} y^2(t), \qquad y(0) = \frac{1}{2}, \quad t \in [0, 1]. \tag{3.22}$$

Integrating (3.22), we get

$$y(t) = \frac{1}{2} + \frac{t}{2} + \frac{1}{8} \int_0^t y^2(s) ds. \tag{3.23}$$

Note that $y_0 = (1+t)/2$ and $N(y)(t) = (1/8)\int_0^t y^2(s)ds$ then [10]

$$(N'(y)z)(t) = \frac{1}{8}\int_0^t \frac{\partial}{\partial y}\left(y^2 z\right)ds \tag{3.24}$$

$$= \frac{1}{8}\int_0^t 2y(s)z(s)ds. \tag{3.25}$$

Similarly, $(N''(y)(z_1, z_2)(t) = (t/4)z_1(t)z_2(t)$ and $N^{(k)}(y) = 0$ for $k \geq 3$. Since $t \in [0,1]$,

$$\begin{aligned}
\|N(y_0)(t)\| &= \left\|\frac{1}{8}\int_0^t \frac{(1+s)^2}{4}ds\right\| \leq \frac{7}{64} < \frac{1}{e}, \\
\|N'(y_0)(t)\| &= \left\|\frac{1}{4}\int_0^t \frac{(1+s)}{2}ds\right\| \leq \frac{3}{16} < \frac{1}{e}, \\
\|N''(y_0)(t)\| &= \left\|\frac{t}{4}\right\| \leq \frac{1}{4} < \frac{1}{e}, \\
\left\|N^{(k)}(y_0)(t)\right\| &= 0, \quad k \geq 3.
\end{aligned} \tag{3.26}$$

As the conditions of Theorem 3.2 are satisfied, the solution series $y = \sum y_i$ obtained by DJM is convergent for $t \in [0,1]$. The terms of the series are given by

$$\begin{aligned}
y_1 &= N(y_0) = \frac{1}{32}\left(t + t^2 + \frac{t^3}{3}\right), \\
y_2 &= N(y_0 + y_1) - N(y_0) \\
&= \frac{t^2}{512} + 65\frac{t^3}{24576} + 67\frac{t^4}{49152} + 37\frac{t^5}{122880} + \frac{t^6}{73728} + \frac{t^7}{516096},
\end{aligned} \tag{3.27}$$

and so on. In Figure 1, a seven-term approximate solution obtained by DJM (dashed line) is compared with exact solution (solid line) $[8\sin(t/4) + 2\cos(t/4)\ ]/[4\cos(t/4) - \sin(t/4)]$.

3.4. Applications of DJM

DJM has been further explored by many researchers. Several numerical methods with higher-order convergence can be generated using DJM. M. A. Noor and K. I. Noor [11, 12] have developed a three-step predictor-corrector method for solving nonlinear equation $f(x) = 0$. Further, they have shown that this method has fourth-order convergence [13]. Some new methods [14, 15] are proposed by these authors using DJM.

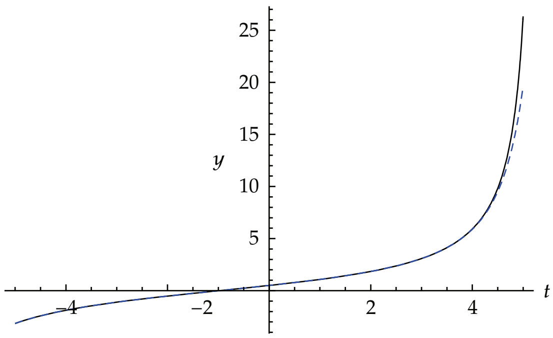

Figure 1: Solution of (3.22): dashed = DJM, solid = exact.

Mohyud-Din et al. [16] solved Hirota-Satsuma coupled KdV system

$$\begin{aligned} u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3(vw)x &= 0, \\ v_t - v_{xxx} - 3uv_x &= 0, \\ w_t + w_{xxx} - 3uw_x &= 0, \end{aligned} \tag{3.28}$$

using DJM. These authors [17] also have applied DJM in solutions of some fifth-order boundary value problems

$$y^{(v)}(x) = g(x)y + q(x), \tag{3.29}$$

with boundary conditions $y(a) = A_1,\ y'(a) = A_2,\ y''(a) = A_3,\ y(b) = B_1,\ y'(b) = B_2$. Noor and Din [18] have used DJM to solve Helmholtz equations.

A variety of fractional-order differential equations such as diffusion-wave equations [19], boundary value problems [20], partial differential equations [21, 22], and evolution equations [23] are solved successfully by Daftardar-Gejji and Bhalekar using DJM. Further Jafari et al. [24] have solved nonlinear diffusion-wave equations using DJM. Fard and Sanchooli [25] have used DJM for solving linear fuzzy Fredholm integral equations.

Recently, Srivastava and Rai [26] have proposed a new mathematical model for oxygen delivery through a capillary to tissues in terms of multiterm fractional diffusion equation. They have solved the multiterm fractional diffusion equation using DJM and ADM and have shown that the results are in perfect agreement.

Acknowledgment

V. Daftardar-Gejji acknowledges the Department of Science and Technology, New Delhi, India for the Research Grants (Project no. SR/S2/HEP-024/2009).

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