COMMON FIXED POINTS OF A FINITE FAMILY OF ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPS

M. O. OSILIKE AND B. G. AKUCHU

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Convergence theorems for approximation of common fixed points of a finite family of asymptotically pseudocontractive mappings are proved in Banach spaces using an averaging implicit iteration process.

1. Introduction

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single-valued. In the sequel, we will denote the single-valued duality mapping by j.

Let K be a nonempty subset of E. A mapping $T : K \to K$ is said to be *asymptotically pseudocontractive* (see, e.g., [3]) if there exists a sequence $\{a_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} a_n = 1$ and

$$\langle T^n x - T^n y, j(x-y)\rangle \leq a_n\|x-y\|^2, \quad \forall n \geq 1, \tag{1.1}$$

for all $x, y \in K$, $j(x-y) \in J(x-y)$. In Hilbert spaces H, a self-mapping T of a nonempty subset K of H is asymptotically pseudocontractive if it satisfies the simpler inequality

$$\left\|T^n x - T^n y\right\|^2 \leq a_n\|x-y\|^2 + \left\|x-y-(T^n x - T^n y)\right\|^2, \quad \forall n \geq 1 \tag{1.2}$$

for all $x, y \in K$ and for some sequence $\{a_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} a_n = 1$. The class of asymptotically pseudocontractive mappings contains the important class of *asymptotically nonexpansive mappings* (i.e., mappings $T : K \to K$ such that

$$\left\|T^n x - T^n y\right\| \leq a_n\|x-y\|, \quad \forall n \geq 1,\ \forall x, y \in K, \tag{1.3}$$

and for some sequence $\{a_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} a_n = 1$). T is called *asymptotically quasi-nonexpansive* if $F(T) = \{x \in K : Tx = x\} \neq \varnothing$ and (1.3) is satisfied for all $x \in K$ and for all $y \in F(T)$. If there exists $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x-y\|$ for

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all $n \geq 1$ and for all $x, y \in K$, then T is said to be *uniformly L-Lipschitzian*. A mapping $T : K \to K$ is said to be *semicompact* (see, e.g., [4]) if for any sequence $\{x_n\}_{n=1}^{\infty}$ in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{x_{n_j}\}_{j=1}^{\infty}$ converges strongly to some $x^* \in K$.

In [5], Xu and Ori introduced an implicit iteration process and proved weak convergence theorem for approximation of common fixed points of a finite family of *nonexpansive mappings* (i.e., a subclass of asymptotically nonexpansive mappings for which $\|Tx - Ty\| \leq \|x - y\|\ \forall x, y \in K$).

In [4], Sun modified the implicit iteration process of Xu and Ori and applied the modified averaging iteration process for the approximation of fixed points of asymptotically quasi-nonexpansive maps. If K is a nonempty closed convex subset of E, and $\{T_i\}_{i=1}^{N}$ is N asymptotically quasi-nonexpansive self-maps of K, then for $x_0 \in K$ and $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0,1)$, the iteration process is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1) T_1 x_1, \\
x_2 &= \alpha_2 x_1 + (1-\alpha_2) T_2 x_2, \\
&\ \vdots \\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1^2 x_{N+1}, \\
x_{N+2} &= \alpha_{N+2} x_{N+1} + (1-\alpha_{N+2}) T_2^2 x_{N+2}, \\
&\ \vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N}) T_N^2 x_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1}) T_1^3 x_{2N+1}, \\
&\ \vdots
\end{aligned} \tag{1.4}$$

The iteration process can be expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_i^k x_n, \quad n \geq 1, \tag{1.5}$$

where $n = (k-1)N + i$, $i \in I = \{1, 2, \ldots, N\}$.

Assuming that the implicit iteration process is defined in K, Sun proved the following theorem.

Theorem 1.1. *Let E be a Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N}$ be N asymptotically quasi-nonexpansive self-maps of K (i.e., $\|T_i^n x - p_i\| \leq [1 + u_{in}]\|x - p_i\|$ for all $n \geq 1$, for all $x \in K$, and for all $p_i \in F(T_i)$, $i \in I$). Let $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and let $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$. Let $x_0 \in K$, $s \in (0,1)$, and $\{\alpha_n\}_{n=1}^{\infty} \subset (s, 1-s)$. Then the implicit iteration process (1.5) converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^{N}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{p\in F} \|x_n - p\|$.*

Theorem 1.2. *Let E be a real uniformly convex Banach space and K a nonempty closed convex bounded subset of E. Let $\{T_i\}_{i=1}^{N}$ be N uniformly Lipschitzian asymptotically quasi-nonexpansive self-maps of K such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$. Let $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$*

and let one member of the family $\{T_i\}_{i=1}^N$ *be semicompact. Let* $x_0 \in K$ *and let* s *and* $\{\alpha_n\}_{n=1}^\infty$ *be as in Theorem 1.1. Then the iteration process (1.5) converges strongly to a common fixed point of the family* $\{T_i\}_{i=1}^N$.

Observe that if $T : K \to K$ is a uniformly L-Lipschitzian asymptotically pseudocontractive map with sequence $\{a_n\}_{n=1}^\infty \subseteq [1,\infty)$ such that $\lim_{n\to\infty} a_n = 1$, then for every fixed $u \in K$ and $t \in (L/(1+L),1)$, the operator $S_{t,n} : K \to K$ defined for all $x \in K$ by

$$S_{t,n}x = tu + (1-t)T^n x \tag{1.6}$$

satisfies

$$||S_{t,n}x - S_{t,n}y|| \le (1-t)L\|x-y\|, \quad \forall x,y \in K. \tag{1.7}$$

Since $(1-t)L \in (0,1)$, it follows that $S_{t,n}$ is a contraction map and hence has a unique fixed point $x_{t,n}$ in K. This implies that there exists a unique $x_{t,n} \in K$ such that

$$x_{t,n} = tu + (1-t)T^n x_{t,n}. \tag{1.8}$$

Thus the implicit iteration process (1.5) is defined in K for the family $\{T_i\}_{i=1}^N$ of N uniformly L_i-Lipschitzian asymptotically pseudocontractive self-mappings of a nonempty convex subset K of a Banach space provided that $\alpha_n \in (\alpha,1)$ for all $n \ge 1$, where $\alpha = L/(1+L)$ and $L = \max_{1\le i\le N}\{L_i\}$.

It is our purpose in this paper to first extend Theorem 1.1 to the class of uniformly L-Lipschitzian asymptotically pseudocontractive mappings. The condition $\sum_{n=1}^\infty (a_{in}-1) < \infty$ for all $i \in I = \{1,2,\dots,N\}$ which is equivalent to the condition $\sum_{n=1}^\infty u_{in} < \infty$ for all $i \in I$ assumed in Theorems 1.1 and 1.2 is not imposed in our theorem. We do not want to make the general assumption that the iteration process is defined. If one assumes that the iteration process is always defined, our result will hold for even the more general class of asymptotically hemicontractive maps (i.e., mappings for which $F(T) \neq \emptyset$ and (1.1) holds for all $x \in K$ and $y \in F(T)$). If $E = H$, a Hilbert space, we obtain a strong convergence theorem similar to Theorem 1.2 for the class of uniformly L-Lipschitzian asymptotically pseudocontractive maps.

In the sequel we will need the following lemma.

Lemma 1.3 [1, page 80]. *Let* $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, *and* $\{\delta_n\}_{n=1}^\infty$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \quad n \ge 1. \tag{1.9}$$

If $\sum_{n=1}^\infty \delta_n < \infty$ *and* $\sum_{n=1}^\infty b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists. If in addition* $\{a_n\}_{n=1}^\infty$ *has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty} a_n = 0$.

Throughout the remaining part of this paper, $\{T_i\}_{i=1}^N$ is a finite family of uniformly L_i-Lipschitzian asymptotically pseudocontractive self-maps of a nonempty closed convex subset K of a Banach space so that

$$\langle T_i^n x - T_i^n y, j(x-y)\rangle \le a_{in}\|x-y\|^2, \quad \forall n \ge 1,\ \forall i \in I = \{1,2,\dots,N\},\ \forall x,y \in K, \tag{1.10}$$

and for some sequences $\{a_{in}\}_{n=1}^{\infty}$, $i \in I$, with $\lim_{n\to\infty} a_{in} = 1$, for all $i \in I$; $\|T_i^n x - T_i^n y\| \le L_i\|x-y\|$ for all $n \ge 1$, for all $i \in I$, for all $x,y \in K$, and for some $L_i > 0$, $i \in I$. $L = \max_{1\le i\le N}\{L_i\}$.

Theorem 1.4. *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L_i-Lipschitzian asymptotically pseudocontractive self-maps of K such that $F = \cap_{i=1}^N F(T_i) \ne \emptyset$. Let $x_0 \in K$ and let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $(\alpha,1)$ satisfying the condition $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, where $\alpha = (1+L)/(2+L)$ (so that $2\alpha - 1 > 0$). Then the implicit iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.5) exists in K and converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$, where $d(x_n,F) = \inf_{p\in F}\|x_n - p\|$.*

Proof. We will use the well-known inequality

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle \tag{1.11}$$

which holds for all $x,y \in E$ and for all $j(x-y) \in J(x-y)$ and which was first proved in [2].

Let $p \in F$, then using (1.1), (1.5), and (1.11), we obtain

$$\begin{aligned}
||x_n - p||^2 &= ||\alpha_n(x_{n-1}-p) + (1-\alpha_n)(T_i^k x_n - p)||^2 \\
&\le \alpha_n^2||x_{n-1}-p||^2 + 2(1-\alpha_n)\langle T_i^k x_n - p, j(x_n-p)\rangle \\
&\le \alpha_n^2||x_{n-1}-p||^2 + 2(1-\alpha_n)a_{ik}||x_n-p||^2.
\end{aligned} \tag{1.12}$$

Observe that since $\lim_{k\to\infty} a_{ik} = 1$ for all $i \in I$, then there exists N_0 such that for all $k > N_0/N + 1$ (i.e., for all $n \ge N_0$), we have $a_{ik} \le 1 + (2\alpha-1)/4(1-\alpha)$ for all $i \in I$. Consequently, for all $k > N_0/N+1$ (for all $n \ge N_0$), we have $1 - 2(1-\alpha_n)a_{ik} \ge (1/2)(2\alpha-1) > 0$.

Let $a = \max_{1\le i\le N}\{\sup_{k\ge1}\{a_{1k}\}, \sup_{k\ge1}\{a_{2k}\},\dots,\sup_{k\ge1}\{a_{Nk}\}\}$. Then for all $k > N_0/N+1$ (for all $n \ge N_0$), it follows from the last inequality in (1.12) that

$$\begin{aligned}
||x_n - p||^2 &\le \left[\frac{\alpha_n^2}{(1-2(1-\alpha_n)a_{ik})}\right]||x_{n-1}-p||^2 \\
&= \left[1 + \frac{2(1-\alpha_n)(a_{ik}-1)}{(1-2(1-\alpha_n)a_{ik})} + \frac{(1-\alpha_n)^2}{(1-2(1-\alpha_n)a_{ik})}\right]||x_{n-1}-p||^2 \\
&\le \left[1 + 4a[2\alpha-1]^{-1}(1-\alpha_n) + 2[2\alpha-1]^{-1}(1-\alpha_n)^2\right]||x_{n-1}-p||^2 \\
&= [1+\sigma_n]||x_{n-1}-p||^2,
\end{aligned} \tag{1.13}$$

where $\sigma_n = 4a[2\alpha-1]^{-1}(1-\alpha_n) + 2[2\alpha-1]^{-1}(1-\alpha_n)^2$. Since $\sum_{n=1}^{\infty}\sigma_n < \infty$, it follows from the last equality in (1.13) and Lemma 1.3 that $\lim_{n\to\infty}\|x_n - p\|$ exists so that there exists $M > 0$ such that $\|x_n - p\| \le M$ for all $n \ge 1$. Consequently, we obtain from the last equality in (1.13) that

$$||x_n - p|| \le [1+\sigma_n]^{1/2}||x_{n-1} - p|| \le [1+\sigma_n]||x_{n-1} - p|| \le ||x_{n-1} - p|| + M\sigma_n. \tag{1.14}$$

It follows from (1.14) that

$$d(x_n, F) \le [1+\sigma_n]d(x_{n-1}, F), \tag{1.15}$$

so that it again follows from Lemma 1.3 that $\lim_{n\to\infty} d(x_n, F)$ exists.

If $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point p of the family $\{T_i\}_{i=1}^{N}$, then $\lim_{n\to\infty} \|x_n - p\| = 0$. Since

$$0 \le d(x_n, F) \le ||x_n - p||, \tag{1.16}$$

we have $\liminf d(x_n, F) = 0$.

Conversely suppose $\liminf_{n\to\infty} d(x_n, F) = 0$, then we have $\lim_{n\to\infty} d(x_n, F) = 0$. Thus for arbitrary $\epsilon > 0$, there exists a positive integer N_1 such that

$$d(x_n, F) < \frac{\epsilon}{4}, \quad \forall n \ge N_1. \tag{1.17}$$

Furthermore, $\sum_{n=1}^{\infty} \sigma_n < \infty$ implies that there exists a positive integer N_2 such that $\sum_{j=n}^{\infty} \sigma_j < \epsilon/4M$ for all $n \ge N_2$. Choose $N = \max\{N_0, N_1, N_2\}$.

Then $d(x_N, F) \le \epsilon/4$ and $\sum_{j=N}^{\infty} \sigma_j < \epsilon/4M$. It follows from (1.14) that for all $n, m \ge N$ and for all $p \in F$, we have

$$\begin{aligned} ||x_n - x_m|| &\le ||x_n - p|| + ||x_m - p|| \\ &\le ||x_N - p|| + M \sum_{j=N+1}^{n} \sigma_j + ||x_N - p|| + M \sum_{j=N+1}^{m} \sigma_j \\ &\le 2||x_N - p|| + 2M \sum_{j=N}^{\infty} \sigma_j. \end{aligned} \tag{1.18}$$

Taking infimum over all $p \in F$, we obtain

$$||x_n - x_m|| \le 2d(x_N, F) + 2M \sum_{j=N}^{\infty} \sigma_j < \epsilon. \tag{1.19}$$

Thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Suppose $\lim_{n\to\infty} x_n = u$. Then $u \in K$ since K is closed. Furthermore, since $F(T_i)$ is closed for all $i \in I$, we have that F is closed. Since $\lim_{n\to\infty} d(x_n, F) = 0$, we have that $u \in F$. □

Remark 1.5. Prototype for the iteration parameter $\{\alpha_n\}$ in Theorem 1.4 is $\alpha_n = \alpha + n^2(1 - \alpha)/(n^2 + 1)$ $n \ge 1$.

THEOREM 1.6. *Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N$ be N uniformly L_i-Lipschitzian asymptotically pseudocontractive self-maps of K such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and $\sum_{n=1}^{\infty}(a_{in}-1) < \infty$ for all $i \in I$. Let one member of the family $\{T_i\}_{i=1}^N$ be semicompact. Let $x_0 \in K$ and let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ such that $0 < \alpha \leq \alpha_n \leq \beta < 1$ for all $n \geq 1$, where $\alpha = L/(1+L)$. Then the implicit iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.5) exists in K and converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^N$.*

Proof. We will use the well-known identity

$$\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 \tag{1.20}$$

which holds in Hilbert spaces H for all $x, y \in H$ and for all $t \in [0,1]$. Let $p \in F$, then using (1.2) and (1.20), we obtain

$$\begin{aligned}
\|x_n - p\|^2 &= \|\alpha_n(x_{n-1}-p) + (1-\alpha_n)(T_i^k x_n - p)\|^2 \\
&= \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\|T_i^k x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1} - T_i^k x_n\|^2 \\
&\leq \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\left[a_{ik}\|x_n - p\|^2 + \|x_n - T_i^k x_n\|^2\right] \\
&\quad - \alpha_n(1-\alpha_n)\|x_{n-1} - T_i^k x_n\|^2 \\
&= \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)a_{ik}\|x_n - p\|^2 - \alpha_n(1-\alpha_n)^2\|x_{n-1} - T_i^k x_n\|^2.
\end{aligned} \tag{1.21}$$

Observe that since $\lim_{k\to\infty} a_{ik} = 1$ for all $i \in I$, then there exists N_0 such that for all $k > N_0/N + 1$ (i.e., for all $n \geq N_0$), we have $a_{ik} \leq 1 + \alpha^2/(1-\alpha)$ for all $i \in I$. Consequently, for all $k > N_0/N + 1$ (for all $n \geq N_0$), we have $1-(1-\alpha_n)a_{ik} \geq \alpha(1-\alpha) > 0$. Thus for all $k > N_0/N + 1$ (for all $n \geq N_0$), it follows from (1.21) that

$$\begin{aligned}
\|x_n - p\|^2 &\leq \left[\frac{\alpha_n}{(1-(1-\alpha_n)a_{ik})}\right]\|x_{n-1}-p\|^2 - \alpha_n(1-\alpha_n)^2\|x_{n-1} - T_i^k x_n\|^2 \\
&= \left[1 + \frac{(1-\alpha_n)(a_{ik}-1)}{(1-(1-\alpha_n)a_{ik})}\right]\|x_{n-1}-p\|^2 - \alpha_n(1-\alpha_n)^2\|x_{n-1} - T_i^k x_n\|^2 \\
&\leq \left[1 + [\alpha(1-\alpha)]^{-1}(1-\alpha_n)(a_{ik}-1)\right]\|x_{n-1}-p\|^2 \\
&\quad - \alpha_n(1-\alpha_n)^2\|x_{n-1} - T_i^k x_n\|^2 \\
&\leq \left[1 + [\alpha(1-\alpha)]^{-1}(a_{ik}-1)\right]\|x_{n-1}-p\|^2 - \alpha(1-\beta)^2\|x_{n-1} - T_i^k x_n\|^2 \\
&= [1+\sigma_{ik}]\|x_{n-1}-p\|^2 - \alpha(1-\beta)^2\|x_{n-1} - T_i^k x_n\|^2,
\end{aligned} \tag{1.22}$$

where $\sigma_{ik} = [\alpha(1-\alpha)]^{-1}(a_{ik}-1)$. Since $\sum_{k=1}^{\infty}\sigma_{ik} < \infty$, it follows from the last equality in (1.22) and Lemma 1.3 that $\lim_{n\to\infty}\|x_n - p\|$ exists. Furthermore, there exists $D > 0$ such that $\|x_n - p\| \leq D$ for all $n \geq 1$. Thus from the last equality in (1.22), we obtain

$$\|x_n - p\|^2 \leq \|x_{n-1}-p\|^2 - \alpha(1-\beta)^2\|x_{n-1} - T_i^k x_n\|^2 + D^2\sigma_{ik}, \tag{1.23}$$

from which it follows that $\lim_{n\to\infty}\|x_{n-1}-T_i^k x_n\|=0$. Thus $\lim_{n\to\infty}\|x_{n-1}-T_n^k x_n\|=0$. Furthermore,

$$\begin{aligned}||x_n-T_n^kx_n||&=\alpha_n||x_{n-1}-T_n^kx_n||\longrightarrow 0\quad\text{as } n\longrightarrow\infty,\\ ||x_n-x_{n-1}||&=(1-\alpha_n)||x_{n-1}-T_n^kx_n||\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\end{aligned}\tag{1.24}$$

Thus $\lim_{n\to\infty}\|x_n-x_{n+i}\|=0$ for all $i\in I$. For all $n>N$, we have $T_n=T_{n-N}$ so that

$$\begin{aligned}||x_{n-1}-T_nx_n||&\le||x_{n-1}-T_n^kx_n||+||T_n^kx_n-T_nx_n||\\ &\le||x_{n-1}-T_n^kx_n||+L||T_n^{k-1}x_n-x_n||\\ &\le||x_{n-1}-T_n^kx_n||+L\big[||T_n^{k-1}x_n-T_{n-N}^{k-1}x_{n-N}||+||T_{n-N}^{k-1}x_{n-N}-x_{(n-N)-1}||\\ &\qquad\qquad+||x_{(n-N)-1}-x_n||\big]\\ &\le||x_{n-1}-T_n^kx_n||+L^2||x_n-x_{n-N}||+L||x_{(n-N)-1}-T_{n-N}^{k-1}x_{n-N}||\\ &\quad+L||x_n-x_{(n-N)-1}||\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\end{aligned}\tag{1.25}$$

Hence,

$$||x_n-T_nx_n||\le||x_n-x_{n-1}||+||x_{n-1}-T_nx_n||\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\tag{1.26}$$

Consequently, for all $i\in I$, we have

$$\begin{aligned}||x_n-T_{n+i}x_n||&\le||x_n-x_{n+i}||+||x_{n+i}-T_{n+i}x_{n+i}||+L||x_{n+i}-x_n||\\ &=(1+L)||x_{n+i}-x_n||+||x_{n+i}-T_{n+i}x_{n+i}||\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\end{aligned}\tag{1.27}$$

It follows that $\lim_{n\to\infty}\|x_n-T_ix_n\|=0$ for all $i\in I$. Since one member of $\{T_i\}_{i=1}^N$ is semi-compact, then there exists a subsequence $\{x_{n_j}\}_{j=1}^\infty$ of the sequence $\{x_n\}_{n=1}^\infty$ such that $\{x_{n_j}\}_{j=1}^\infty$ converges strongly to u. Since K is closed, $u\in K$, and furthermore,

$$||u-T_iu||=\lim_{j\to\infty}||x_{n_j}-T_ix_{n_j}||=0\quad\forall i\in I.\tag{1.28}$$

Thus $u\in F$. Since $\{x_{n_j}\}_{j=1}^\infty$ converges strongly to u and $\lim_{n\to\infty}\|x_n-u\|$ exists, it follows from Lemma 1.3 that $\{x_n\}_{n=1}^\infty$ converges strongly to u. □

Remark 1.7. Prototype for the iteration parameter $\{\alpha_n\}$ in Theorem 1.6 is $\alpha_n=\alpha+n(1-\alpha)/2(n+1)$ $n\ge 1$, for which $0<\alpha<\alpha+(1-\alpha)/4\le\alpha_n<\alpha+(1-\alpha)/2<1$.

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M. O. Osilike: Department of Mathematics, University of Nigeria, Nsukka, Nigeria
E-mail address: osilike@yahoo.com

B. G. Akuchu: Department of Mathematics, University of Nigeria, Nsukka, Nigeria
E-mail address: akuchubg@yahoo.com