

MULTIVALUED p -LIENARD SYSTEMS

MICHAEL E. FILIPPAKIS AND NIKOLAOS S. PAPAGEORGIU

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We examine p -Lienard systems driven by the vector p -Laplacian differential operator and having a multivalued nonlinearity. We consider Dirichlet systems. Using a fixed point principle for set-valued maps and a nonuniform nonresonance condition, we establish the existence of solutions.

1. Introduction

In this paper, we use fixed point theory to study the following multivalued p -Lienard system:

$$\begin{aligned} (||x'(t)||^{p-2}x'(t))' + \frac{d}{dt} \nabla G(x(t)) + F(t, x(t), x'(t)) \ni 0 \quad \text{a.e. on } T = [0, b], \\ x(0) = x(b) = 0, \quad 1 < p < \infty. \end{aligned} \tag{1.1}$$

In the last decade, there have been many papers dealing with second-order multivalued boundary value problems. We mention the works of Erbe and Krawcewicz [5, 6], Frigon [7, 8], Halidias and Papageorgiou [9], Kandilakis and Papageorgiou [11], Kyritsi et al. [12], Palmucci and Papalini [17], and Pruszko [19]. In all the above works, with the exception of Kyritsi et al. [12], $p = 2$ (linear differential operator), $G = 0$, and $g = 0$. Moreover, in Frigon [7, 8] and Palmucci and Papalini [17], the inclusions are scalar (i.e., $N = 1$). Finally we should mention that recently single-valued p -Lienard systems were studied by Mawhin [14] and Manásevich and Mawhin [13].

In this work, for problem (1.1), we prove an existence theorem under conditions of nonuniform nonresonance with respect to the first weighted eigenvalue of the negative vector ordinary p -Laplacian with Dirichlet boundary conditions [15, 20]. Our approach is based on the multivalued version of the Leray-Schauder alternative principle due to Bader [1] (see Section 2).

2. Mathematical background

In this section, we recall some basic definitions and facts from multivalued analysis, the spectral properties of the negative vector p -Laplacian, and the multivalued fixed point principles mentioned in the introduction. For details, we refer to Denkowski et al. [3] and Hu and Papageorgiou [10] (for multivalued analysis), to Denkowski et al. [2] and Zhang [20] (for the spectral properties of the p -Laplacian), and to Bader [1] (for the multivalued fixed point principle; similar results can also be found in O'Regan and Precup [16] and Precup [18]).

Let (Ω, Σ) be a measurable space and X a separable Banach space. We introduce the following notations:

$$\begin{aligned} P_{f(c)}(X) &= \{A \subseteq X : \text{nonempty, closed (and convex)}\}, \\ P_{(w)k(c)}(X) &= \{A \subseteq X : \text{nonempty, (weakly) compact (and convex)}\}. \end{aligned} \quad (2.1)$$

A multifunction $F : \Omega \rightarrow P_f(X)$ is said to be measurable if, for all $x \in X$, $\omega \rightarrow d(x, F(\omega)) = \inf [\|x - y\| : y \in F(\omega)]$ is measurable. A multifunction $F : \Omega \rightarrow 2^X \setminus \{\emptyset\}$ is said to be “graph measurable” if $\text{Gr}F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ -field of X . For $P_f(X)$ -valued multifunctions, measurability implies graph measurability and the converse is true if Σ is complete (i.e., $\Sigma = \hat{\Sigma}$ = the universal σ -field). Let μ be a finite measure on (Ω, Σ) , $1 \leq p \leq \infty$, and $F : \Omega \rightarrow 2^X \setminus \{\emptyset\}$. We introduce the set $S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega) \mu\text{-a.e.}\}$. This set may be empty. For a graph-measurable multifunction, it is nonempty if and only if $\inf [\|y\| : y \in F(\omega)] \leq \varphi(\omega) \mu\text{-a.e.}$ on Ω , with $\varphi \in L^p(\Omega)_+$.

Let Y, Z be Hausdorff topological spaces. A multifunction $G : Y \rightarrow 2^Z \setminus \{\emptyset\}$ is said to be “upper semicontinuous” (usc for short) if, for all $C \subseteq Z$ closed, $G^-(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$ is closed or equivalently for all $U \subseteq Z$ open, $G^+ \{y \in Y : G(y) \subseteq U\}$ is open. If Z is a regular space, then a $P_f(Z)$ -valued multifunction which is usc has a closed graph. The converse is true if the multifunction G is locally compact (i.e., for every $y \in Y$, there exists a neighborhood U of y such that $\overline{G(U)}$ is compact in Z). A $P_k(Z)$ -valued multifunction which is usc maps compact sets to compact sets.

Consider the following weighted nonlinear eigenvalue problem in \mathbb{R}^N :

$$\begin{aligned} -(\|x'(t)\|^{p-2} x'(t))' &= \lambda \theta(t) \|x(t)\|^{p-2} x(t) \quad \text{a.e. on } T = [0, b], \\ x(0) = x(b) &= 0, \quad 1 < p < \infty, \quad \theta \in L^\infty(T), \quad |\{\theta > 0\}|_1 > 0, \quad \lambda \in \mathbb{R}. \end{aligned} \quad (2.2)$$

Here by $|\cdot|_1$ we denote the 1-dimensional Lebesgue measure. The real parameters λ , for which problem (2.3) has a nontrivial solution, are called eigenvalues of the negative vector p -Laplacian with Dirichlet boundary conditions denoted by $(-\Delta_p, W_0^{1,p}(T, \mathbb{R}^N))$, with weight $\theta \in L^\infty(T)$. The corresponding nontrivial solutions are known as eigenfunctions. We know that the eigenvalues of problem (2.3) are the same as those of the corresponding scalar problem [13]. Then from Denkowski et al. [2] and Zhang [20], we know that there exist two sequences $\{\lambda_n(\theta)\}_{n \geq 1}$ and $\{\lambda_{-n}(\theta)\}_{n \geq 1}$ such that $\lambda_n(\theta) > 0$, $\lambda_n(\theta) \rightarrow +\infty$ and $\lambda_{-n}(\theta) < 0$, $\lambda_{-n}(\theta) \rightarrow -\infty$ as $n \rightarrow \infty$. Moreover, if $\theta(t) \geq 0$ a.e. on T with strict inequality on a set of positive Lebesgue measure, then we have only the positive

sequence $\{\lambda_n(\theta)\}_{n \geq 1}$. Also, for $\lambda_1(\theta) > 0$, we have the following variational characterization:

$$\lambda_1(\theta) = \inf \left[\frac{\|x'\|_p^p}{\int_0^b \theta(t) \|x(t)\|^p dt} : x \in W_0^{1,p}(T, \mathbb{R}^N), x \neq 0 \right]. \tag{2.3}$$

The infimum is attained at the normalized principal eigenfunction u_1 ($\lambda_1(\theta) > 0$ is simple) and $u_1(t) \neq 0$ a.e. on T . Also, $\lambda_1(\theta)$ is strictly monotone with respect to θ , namely, if $\theta_1(t) \leq \theta_2(t)$ a.e. on T with strict inequality on a set of positive measure, then $\lambda_1(\theta_2) < \lambda_1(\theta_1)$ (see (3.2)).

Finally we state the multivalued fixed point principle that we will use in the study of problem (1.1). So let Y, Z be two Banach spaces and $C \subseteq Y, D \subseteq Z$ two nonempty closed and convex sets. We consider multifunctions $G : C \rightarrow 2^C \setminus \{\emptyset\}$ which have a decomposition $G = K \circ N$, satisfying the following: $K : D \rightarrow C$ is completely continuous, namely, if $z_n \xrightarrow{w} z$ in D , then $K(z_n) \rightarrow K(z)$ in C and $N : C \rightarrow P_{wkc}(D)$ is usc from C , furnished with the strong topology into D , furnished with the weak topology.

THEOREM 2.1. *If C, D , and $G = K \circ N$ are as above, $0 \in C$, and G is compact (namely, G maps bounded subsets of C into relatively compact subsets of D), then one of the following alternatives holds:*

- (a) $S = \{y \in C : y \in \mu G(y) \text{ for some } \mu \in (0, 1)\}$ is unbounded or
- (b) G has a fixed point, that is, there exists $y \in C$ such that $y \in G(y)$.

Remark 2.2. Evidently this is a multivalued version of the classical Leray-Schauder alternative principle [2, page 206]. In contrast to previous multivalued extensions of the Leray-Schauder alternative principal [4, page 61], **Theorem 2.1** does not require G to have convex values, which is important when dealing with nonlinear problems such as (1.1).

3. Nonuniform nonresonance

In this section, we deal with problem (1.1) using a condition of nonuniform nonresonance with respect to the first eigenvalue $\lambda_1(\theta) > 0$. Our hypotheses on the multivalued nonlinearity $F(t, x, y)$ are as follows.

- $(H(F)_1)$ $F : T \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow P_{kc}(\mathbb{R}^N)$ is a multifunction such that
- (i) for all $x, y \in \mathbb{R}^N, t \rightarrow F(t, x, y)$ is graph measurable;
 - (ii) for almost all $t \in T, (x, y) \rightarrow F(t, x, y)$ is usc;
 - (iii) for every $M > 0$, there exists $\gamma_M \in L^1(T)_+$ such that, for almost all $t \in T$, all $\|x\|, \|y\| \leq M$, and all $u \in F(t, x, y)$, we have $\|u\| \leq \gamma_M(t)$;
 - (iv) there exists $\theta \in L^\infty(T), \theta(t) \geq 0$ a.e. on T , with strict inequality on a set of positive measure and

$$\limsup_{\|x\| \rightarrow +\infty} \frac{\sup [(u, x)_{\mathbb{R}^N} : u \in F(t, x, y), y \in \mathbb{R}^N]}{\|x\|^p} \leq \theta(t) \tag{3.1}$$

uniformly for almost all $t \in T$ and $\lambda_1(\theta) > 1$.

Remark 3.1. Hypothesis $(H(F)_1)(iv)$ is the nonuniform nonresonance condition. In the literature [15, 20], we encounter the condition $\theta(t) \leq \lambda_1$ a.e. on T with strict inequality on a set of positive measure. Here $\lambda_1 > 0$ is the principal eigenvalue corresponding to the unit weight $\theta = 1$ (i.e., $\lambda_1 = \lambda_1(1)$). Then by virtue of the strict monotonicity property, we have $\lambda_1(\lambda_1) = 1 < \lambda_1(\theta)$, which is the condition assumed in hypothesis $(H(F)_1)(iv)$.

$(H(G)_1)$ $G \in C^2(\mathbb{R}^N, \mathbb{R})$.

Given $h \in L^1(T, \mathbb{R}^N)$, we consider the following Dirichlet problem:

$$\begin{aligned} -(\|x'(t)\|^{p-2} x'(t))' &= h(t) \quad \text{a.e. on } T = [0, b], \\ x(0) &= x(b) = 0. \end{aligned} \quad (3.2)$$

From Manásevich and Mawhin [13, Lemma 4.1], we know that problem (3.3) has a unique solution $K(h) \in C_0^1(T, \mathbb{R}^N) = \{x \in C^1(T, \mathbb{R}^N) : x(0) = x(b) = 0\}$. So we can define the solution map $K : L^1(T, \mathbb{R}^N) \rightarrow C_0^1(T, \mathbb{R}^N)$.

PROPOSITION 3.2. $K : L^1(T, \mathbb{R}^N) \rightarrow C_0^1(T, \mathbb{R}^N)$ is completely continuous, that is, if $h_n \xrightarrow{w} h$ in $L^1(T, \mathbb{R}^N)$, then $K(h_n) \rightarrow K(h)$ in $C_0^1(T, \mathbb{R}^N)$.

Proof. Let $h_n \xrightarrow{w} h$ in $L^1(T, \mathbb{R}^N)$ and set $x_n = K(h_n)$, $n \geq 1$. We have

$$-(\|x'_n(t)\|^{p-2} x'_n(t))' = h_n(t) \quad \text{a.e. on } T, \quad x_n(0) = x_n(b) = 0, \quad n \geq 1. \quad (3.3)$$

Taking the inner product with $x_n(t)$, integrating over T , and performing integration by parts, we obtain

$$\|x'_n\|_p^p \leq \|h_n\|_1 \|x_n\|_\infty \leq c_1 \|x'_n\|_p \quad \text{for some } c_1 > 0 \text{ and all } n \geq 1. \quad (3.4)$$

Here we have used Hölder and Poincaré inequalities. It follows that

$$\begin{aligned} \{x'_n\}_{n \geq 1} &\subseteq L^p(T, \mathbb{R}^N) \text{ is bounded (since } p > 1) \\ \implies \{x_n\}_{n \geq 1} &\subseteq W_0^{1,p}(T, \mathbb{R}^N) \text{ is bounded (by the Poincaré inequality).} \end{aligned} \quad (3.5)$$

So from (3.22) we infer that

$$\begin{aligned} \{\|x'_n\|^{p-2} x'_n\}_{n \geq 1} &\subseteq W^{1,q}(T, \mathbb{R}^N) \left(\frac{1}{p} + \frac{1}{q} = 1 \right) \text{ is bounded} \\ \implies \{\|x'_n\|^{p-2} x'_n\}_{n \geq 1} &\subseteq C(T, \mathbb{R}^N) \text{ is relatively compact} \end{aligned} \quad (3.6)$$

(recall that $W^{1,q}(T, \mathbb{R}^N)$ is embedded compactly in $C(T, \mathbb{R}^N)$). The map $\varphi_p : \mathbb{R}^N \rightarrow \mathbb{R}^N$, defined by $\varphi_p(y) = \|y\|^{p-2} y$, $y \in \mathbb{R}^N \setminus \{\emptyset\}$, and $\varphi_p(0) = 0$, is a homeomorphism and so $\hat{\varphi}_p^{-1} : C(T, \mathbb{R}^N) \rightarrow C(T, \mathbb{R}^N)$, defined by $\hat{\varphi}_p^{-1}(y)(\cdot) = \varphi_p^{-1}(y(\cdot))$, is continuous and bounded. Thus it follows that

$$\begin{aligned} \{x'_n\}_{n \geq 1} &\subseteq C(T, \mathbb{R}^N) \text{ is relatively compact} \\ \implies \{x_n\}_{n \geq 1} &\subseteq C_0^1(T, \mathbb{R}^N) \text{ is relatively compact.} \end{aligned} \quad (3.7)$$

Therefore we may assume that $x_n \rightarrow x$ in $C_0^1(T, \mathbb{R}^N)$. Also $\{\|x'_n\|^{p-2}x'_n\}_{n \geq 1} \subseteq W^{1,q}(T, \mathbb{R}^N)$ is bounded and so we may assume that $\|x'_n\|^{p-2}x'_n \xrightarrow{w} u$ in $W^{1,q}(T, \mathbb{R}^N)$ and $\|x'_n\|^{p-2}x'_n \rightarrow u$ in $C(T, \mathbb{R}^N)$ (because $W^{1,q}(T, \mathbb{R}^N)$ is embedded compactly in $C(T, \mathbb{R}^N)$). It follows that $u = \|x'\|^{p-2}x'$. Hence if in (3.22) we pass to the limit as $n \rightarrow \infty$, we obtain

$$\begin{aligned} -(\|x'(t)\|^{p-2}x'(t))' &= h(t) \quad \text{a.e. on } T = [0, b], \quad x(0) = x(b) = 0 \\ \implies K(h) &= x. \end{aligned} \tag{3.8}$$

Since every subsequence of $\{x_n\}_{n \geq 1}$ has a further subsequence which converges to x in $C_0^1(T, \mathbb{R}^N)$, we conclude that the original sequence converges too. This proves the complete continuity of K . \square

Let $N_F : C_0^1(T, \mathbb{R}^N) \rightarrow 2^{L^1(T, \mathbb{R}^N)}$ be the multivalued Nemitsky operator corresponding to F , that is,

$$N_F(x) = \{u \in L^1(T, \mathbb{R}^N) : u(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T\}. \tag{3.9}$$

Also let $N : C_0^1(T, \mathbb{R}^N) \rightarrow 2^{L^1(T, \mathbb{R}^N)}$ be defined by

$$N(x) = \frac{d}{dx} \nabla G(x(\cdot)) + N_F(x). \tag{3.10}$$

This multifunction has the following structure.

PROPOSITION 3.3. *If hypotheses $(H(F)_1)$ and $(H(G)_1)$ hold, then N has values in $P_{wkc}(L^1(T, \mathbb{R}^N))$ and it is usc from $C_0^1(T, \mathbb{R}^N)$ with the norm topology into $L^1(T, \mathbb{R}^N)$ with the weak topology.*

Proof. Clearly N has closed, convex values which are uniformly integrable (see hypothesis $(H(F)_1)$ (iii)). Therefore for every $x \in C_0^1(T, \mathbb{R}^N)$, $N(x)$ is convex and w -compact in $L^1(T, \mathbb{R}^N)$. What is not immediately clear is that $N(x) \neq \emptyset$, since hypotheses $(H(F)_1)$ (i) and (ii) in general do not imply the graph measurability of $(t, x, y) \rightarrow F(t, x, y)$ [10, page 227]. To see that $N(x) \neq \emptyset$, we proceed as follows. Let $\{s_n\}_{n \geq 1}, \{r_n\}_{n \geq 1}$ be step functions such that $s_n \rightarrow x$ and $r_n \rightarrow x'$ a.e. on T and $\|s_n(t)\| \leq \|x(t)\|, \|r_n(t)\| \leq \|x'(t)\|$ a.e. on $T, n \geq 1$. Then by virtue of hypothesis $(H(F)_1)$ (i), for every $n \geq 1$, the multifunction $t \rightarrow F(t, s_n(t), r_n(t))$ is measurable and so by the Yankon-von Neumann-Aumann selection theorem [10, page 158], we can find $u_n : T \rightarrow \mathbb{R}^N$ a measurable map such that $u_n(t) \in F(t, s_n(t), r_n(t))$ for all $t \in T$. Note that $\|s_n\|_\infty, \|r_n\|_\infty \leq M_1$ for some $M_1 > 0$ and all $n \geq 1$. So $\|u_n(t)\| \leq \gamma_{M_1}(t)$ a.e. on T , with $\gamma_{M_1} \in L^1(T)_+$ (see hypothesis $(H(F)_1)$ (iii)). Thus by virtue of the Dunford-Pettis theorem, we may assume that $u_n \xrightarrow{w} u$ in $L^1(T, \mathbb{R}^N)$ as $n \rightarrow \infty$. From Hu and Papageorgiou [10, page 694], we have

$$u(t) \in \overline{\text{conv}} \limsup_{n \rightarrow \infty} F(t, s_n(t), r_n(t)) \subseteq F(t, x(t), x'(t)) \quad \text{a.e. on } T, \tag{3.11}$$

with the last inclusion being a consequence of hypothesis $(H(F)_1)$ (ii). So we have $u \in S_{F(\cdot, x(\cdot), x'(\cdot))}^q$, hence $N(x) \neq \emptyset$.

Next we check the upper semicontinuity of N into $L^1(T, \mathbb{R}^N)_w$ ($L^1(T, \mathbb{R}^N)_w$ equals the Banach space $L^1(T, \mathbb{R}^N)$ furnished with the weak topology). Because of hypothesis $(H(F)_1)(iii)$, N is locally compact into $L^1(T, \mathbb{R}^N)_w$ (recall that uniformly integrable sets are relatively compact in $L^1(T, \mathbb{R}^N)_w$). Also on weakly compact subsets of $L^1(T, \mathbb{R}^N)$, the relative weak topology is metrizable. Therefore to check the upper semicontinuity of N , it suffices to show that $\text{Gr}N$ is sequentially closed in $C_0^1(T, \mathbb{R}^N) \times L^1(T, \mathbb{R}^N)_w$ (see Section 2). To this end, let $(x_n, f_n) \in \text{Gr}N$, $n \geq 1$, and suppose that $x_n \rightarrow x$ in $C_0^1(T, \mathbb{R}^N)$ and $f_n \xrightarrow{w} f$ in $L^1(T, \mathbb{R}^N)$. For every $n \geq 1$, we have

$$f_n(t) = \frac{d}{dt} \nabla G(x_n(t)) + u_n(t) \quad \text{a.e. on } T, \text{ with } u_n \in S_{F(\cdot, x_n(\cdot), x'_n(\cdot))}^1. \tag{3.12}$$

Because of hypothesis $(H(F)_1)(iii)$, we may assume (at least for a subsequence) that $u_n \xrightarrow{w} u$ in $L^1(T, \mathbb{R}^N)$. As before, from Hu and Papageorgiou [10, page 694], we have

$$u(t) \in \overline{\text{conv}} \limsup_{n \rightarrow \infty} F(t, x_n(t), x'_n(t)) \subseteq F(t, x(t), x'(t)) \quad \text{a.e. on } T \tag{3.13}$$

(again the last inclusion follows from hypothesis $(H(F)_1)(ii)$). So $u \in S_{F(\cdot, x(\cdot), x'(\cdot))}^1$. Also by virtue of hypothesis $(H(G)_1)$, we have

$$\begin{aligned} \frac{d}{dt} \nabla G(x_n(t)) &= G''(x_n(t))x'_n(t) \rightarrow G''(x(t))x'(t) = \frac{d}{dt} \nabla G(x(t)), \quad \forall t \in T \\ \implies \frac{d}{dt} \nabla G(x_n(\cdot)) &\rightarrow \frac{d}{dt} \nabla G(x(\cdot)) \quad \text{in } L^1(T, \mathbb{R}^N) \\ &\text{(by the dominated convergence theorem).} \end{aligned} \tag{3.14}$$

So in the limit as $n \rightarrow \infty$, we have

$$\begin{aligned} f &= \frac{d}{dt} \nabla G(x(\cdot)) + u \quad \text{with } u \in N_F(x) \\ \implies (x, f) &\in \text{Gr}N. \end{aligned} \tag{3.15}$$

This proves the desired upper semicontinuity of N . □

PROPOSITION 3.4. *There exists $\xi > 0$ such that, for all $x \in W_0^{1,p}(T, \mathbb{R}^N)$,*

$$\|x'\|_p^p - \int_0^b \theta(t) \|x(t)\|^p dt \geq \xi \|x'\|_p^p. \tag{3.16}$$

Proof. Let $\eta : W_0^{1,p}(T, \mathbb{R}^N) \rightarrow \mathbb{R}$ be the functional defined by

$$\eta(x) = \|x'\|_p^p - \int_0^b \theta(t) \|x(t)\|^p dt. \tag{3.17}$$

From the variational characterization of $\lambda_1(\theta) > 1$, we see that $\eta(x) > 0$ for all $x \in W_0^{1,p}(T, \mathbb{R}^N)$, $x \neq 0$. Suppose that the proposition was not true. Then by virtue of the p -homogeneity of η , we can find $\{x_n\}_{n \geq 1} \subset W_0^{1,p}(T, \mathbb{R}^N)$ such that $\|x'_n\|_p = 1$ and $\eta(x_n) \downarrow 0$.

By the Poincaré inequality, the sequence $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(T, \mathbb{R}^N)$ is bounded and so we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_0^{1,p}(T, \mathbb{R}^N), \quad x_n \rightarrow x \quad \text{in } C_0(T, \mathbb{R}^N). \quad (3.18)$$

Also exploiting the weak lower semicontinuity of the norm functional in a Banach space, we obtain

$$\|x'\|_p^p \leq \int_0^b \theta(t) \|x(t)\|^p dt \implies \lambda_1(\theta) \leq 1, \quad (3.19)$$

a contradiction to our hypothesis that $\lambda_1(\theta) > 1$. \square

We introduce the set

$$S = \{x \in C_0^1(T, \mathbb{R}^N) : x \in \lambda KN(x), 0 < \lambda < 1\}. \quad (3.20)$$

PROPOSITION 3.5. *If hypotheses $(H(F)_1)$ and $(H(G)_1)$ hold, then $S \subseteq C_0^1(T, \mathbb{R}^N)$ is bounded.*

Proof. Let $x \in S$. We have

$$\begin{aligned} & \frac{1}{\lambda} x \in KN(x) \quad \text{with } 0 < \lambda < 1 \\ \implies & \frac{1}{\lambda^{p-1}} (\|x'(t)\|^{p-2} x'(t))' + \frac{d}{dt} \nabla G(x(t)) + u(t) = 0 \quad \text{a.e. on } T, \text{ with } u \in S_{F(\cdot, x(\cdot), x'(\cdot))}^1 \\ \implies & (\|x'(t)\|^{p-2} x'(t))' + \lambda^{p-1} \frac{d}{dt} \nabla G(x(t)) + \lambda^{p-1} u(t) = 0 \quad \text{a.e. on } T. \end{aligned} \quad (3.21)$$

Taking the inner product with $x(t)$, integrate over T , and perform integration by parts, we obtain

$$-\|x'\|_p^p - \lambda^{p-1} \int_0^b (\nabla G(x(t)), x'(t))_{\mathbb{R}^N} dt + \lambda^{p-1} \int_0^b (u(t), x(t))_{\mathbb{R}^N} dt = 0. \quad (3.22)$$

Remark that

$$\int_0^b (\nabla G(x(t)), x'(t))_{\mathbb{R}^N} dt = \int_0^b \frac{d}{dt} G(x(t)) dt = G(x(b)) - G(x(0)) = 0. \quad (3.23)$$

By virtue of hypotheses $(H(F)_1)$ (iii) and (iv), given $\varepsilon > 0$, we can find $\gamma_\varepsilon \in L^1(T)_+$ such that for almost all $t \in T$, all $x, y \in \mathbb{R}^N$, and all $u \in F(t, x, y)$, we have

$$(u, x)_{\mathbb{R}^N} \leq (\theta(t) + \varepsilon) \|x\|^p + \gamma_\varepsilon(t). \quad (3.24)$$

So we have

$$\int_0^b (u(t), x(t))_{\mathbb{R}^N} dt \leq \int_0^b \theta(t) \|x(t)\|^p dt + \varepsilon \|x\|_p^p + \|\gamma_\varepsilon\|_1. \quad (3.25)$$

Using (3.24) and (3.27) in (3.23), we obtain

$$\begin{aligned} \|x'\|_p^p &\leq \int_0^b \theta(t) \|x(t)\|^p dt + \varepsilon \|x\|_p^p + \|\gamma_\varepsilon\|_1 \\ \implies \xi \|x'\|_p^p - \frac{\varepsilon}{\lambda_1} \|x'\|_p^p &\leq \|\gamma_\varepsilon\|_1 \end{aligned} \tag{3.26}$$

(see Proposition 3.5 and recall that $\lambda_1 \|x\|_p^p \leq \|x'\|_p^p$, $\lambda_1 = \lambda_1(1)$).

Choose $\varepsilon > 0$ so that $\varepsilon < \lambda_1 \xi$. Then from the last inequality, we infer that

$$\begin{aligned} \{x'\}_{x \in S} &\subseteq L^p(T, \mathbb{R}^N) \text{ is bounded} \\ \implies S &\subseteq W_0^{1,p}(T, \mathbb{R}^N) \text{ is bounded (by Poincaré's inequality)} \\ \implies S &\subseteq C_0(T, \mathbb{R}^N) \text{ is relatively compact.} \end{aligned} \tag{3.27}$$

Also we have

$$\begin{aligned} &\|(\|x'(t)\|^{p-2} x'(t))'\| \\ &\leq \|G'(x(t))\|_{\mathcal{G}} \|x'(t)\| + \|u(t)\| \text{ a.e. on } T \\ &\leq M_2 (\|x'(t)\| + \theta(t) + \varepsilon + \gamma_\varepsilon(t)) \text{ a.e. on } T \text{ for some } M_2 > 0 \text{ (see (3.25))} \\ &\implies \{\|x'\|^{p-2} x'\}_{x \in S} \subseteq W^{1,1}(T, \mathbb{R}^N) \text{ is bounded} \\ &\implies \{\|x'\|^{p-2} x'\}_{x \in S} \subseteq C(T, \mathbb{R}^N) \text{ is bounded} \\ &\quad \text{(since } W^{1,1}(T, \mathbb{R}^N) \text{ is embedded continuously but not compactly in } C(T, \mathbb{R}^N)) \\ &\implies \{x'\}_{x \in S} \subseteq C(T, \mathbb{R}^N) \text{ is bounded.} \end{aligned} \tag{3.28}$$

From (3.28) and (3.29), we conclude that $S \subseteq C_0^1(T, \mathbb{R}^N)$ is bounded. □

Propositions 3.2, 3.3, and 3.5 permit the use of Theorem 2.1. So we obtain the following existence result for problem (1.1).

THEOREM 3.6. *If hypotheses $(H(F)_1)$ and $(H(G)_1)$ hold, then problem (1.1) has a solution $x \in C_0^1(T, \mathbb{R}^N)$ with $\|x'\|^{p-2} x' \in W^{1,1}(T, \mathbb{R}^N)$.*

As an application of this theorem, we consider the following system:

$$\begin{aligned} (\|x'(t)\|^{p-2} x'(t))' + \|x(t)\|^{p-2} Ax(t) + F(t, x(t)) &\ni e(t) \quad \text{a.e. on } T = [0, b], \\ x(0) = x(b) = 0, \quad e &\in L^1(T, \mathbb{R}^N). \end{aligned} \tag{3.29}$$

Our hypotheses on the data of problem (3.29) are the following.

$(H(A))$ A is an $N \times N$ matrix such that for all $x \in \mathbb{R}^N$ we have $(Ax, x)_{\mathbb{R}^N} \leq \theta \|x\|^2$ with $\theta < (\pi_p/b)^p$.

Remark 3.7. The quantity π_p is defined by $\pi_p = 2(p-1)^{1/p} \int_0^1 (1/(1-t)^{1/p}) dt = 2(p-1)^{1/p} ((\pi/p)/\sin(\pi/p))$. If $p = 2$, then $\pi_2 = \pi$. Recall that the eigenvalues of $(-\Delta_p, W_0^{1,p}(T, \mathbb{R}^N))$ are $\lambda_n = (n\pi_p/b)^p$, $n \geq 1$ [13]. So in hypothesis $(H(A))$, we have $\theta < \lambda_1$.

$(H(F)'_1)$ $F : T \times \mathbb{R}^N \rightarrow P_{kc}(\mathbb{R}^N)$ is a multifunction such that

- (i) for all $x \in \mathbb{R}^N$, $t \rightarrow F(t, x)$ is graph measurable;
- (ii) for almost all $t \in T$, $x \rightarrow F(t, x)$ is usc;
- (iii) for every $M > 0$, there exists $\gamma_M \in L^1(T)_+$ such that for almost all $t \in T$, all $\|x\| \leq M$, and all $u \in F(t, x)$, we have $\|u\| \leq \gamma_M(t)$;
- (iv) $\lim_{\|x\| \rightarrow \infty} ((u, x)_{\mathbb{R}^N} / \|x\|^p) = 0$ uniformly for almost all $t \in T$ and all $u \in F(t, x)$.

Invoking [Theorem 3.6](#), we obtain the following existence result for problem [\(3.29\)](#).

THEOREM 3.8. *If hypotheses $(H(A))$ and $(H(F)'_1)$ hold, then for every $e \in L^1(T, \mathbb{R}^N)$, problem [\(3.29\)](#) has a solution $x \in C_0^1(T, \mathbb{R}^N)$ with $\|x'\|^{p-2}x' \in W^{1,1}(T, \mathbb{R}^N)$.*

Remark 3.9. [Theorem 3.8](#) extends [Theorem 7.1](#) of Manásevich and Mawhin [\[13\]](#).

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Michael E. Filippakis: Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece

E-mail address: mfil@math.ntua.gr

Nikolaos S. Papageorgiou: Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece

E-mail address: npapg@math.ntua.gr