POSITIVE SOLUTIONS OF SOME THREE-POINT BOUNDARY VALUE PROBLEMS VIA FIXED POINT INDEX FOR WEAKLY INWARD A-PROPER MAPS

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We use the theory of fixed point index for weakly inward A-proper maps to establish the existence of positive solutions of some second-order three-point boundary value problems in which the highest-order derivative occurs nonlinearly.

1. Introduction

In the present paper, we discuss the existence of positive solutions of the nonlinear three-point boundary value problem (BVP)

$$-u''(t) = f(t,u,u',u''), \quad t \in (0,1), \tag{1.1}$$

with the nonlocal boundary conditions (BCs)

$$u(0) = 0, \qquad \alpha u(\eta) = u(1), \quad 0 < \eta < 1,\ \alpha\eta < 1, \tag{1.2}$$

in which the second derivative may occur nonlinearly.

Positive solutions for the case $f(t,u,u',u'') = g(t)h(u)$ have been studied by Ma [15] and Webb [20, 21], when $f(t,u,u',u'') = h(t,u)$ by He and Ge [5] and also by Lan [11]. The case $f(t,u,u',u'') = g(t)h(u,u')$ has been studied by Feng [4]. The results in [4, 15] are obtained by means of Krasnosel'skiĭ's theorem [8], the ones in [5] use Leggett and Williams' theorem [14] and the results in [11, 20, 21] are achieved via the classical fixed point index for compact maps, see for example [1].

Lafferriere and Petryshyn [9] and Cremins [2] studied existence of positive solutions of the so-called Picard boundary value problem

$$-u''(t) = f(t,u,u',u''), \tag{1.3}$$

with BCs

$$u(0) = u(1) = 0, \tag{1.4}$$

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by means of fixed point index theory for A-proper maps. A key restriction in [2, 9] is that f must take positive values. Lan and Webb [13] improved the results of [2, 9] by allowing f to possibly take some negative values.

Here we will exploit Lan and Webb's theory [12] of fixed point index for weakly inward A-proper maps, to prove new results on the existence of positive solutions of the BVP (1.1)-(1.2).

We mention that with very little change, this technique may be applied to a variety of BCs, (e.g., other three-point BCs [6, 7], or m-point BCs [16]), but for brevity, we refrain from discussing other cases.

2. Preliminaries

Let X denote an infinite-dimensional Banach space endowed with a fixed projection scheme $\Gamma = \{X_n, P_n\}$, where $\{X_n\}$ is a sequence of finite-dimensional subspaces of X and $P_n : X \to X_n$ is a linear projection with $P_n x \to x$ for every $x \in X$. We recall below the concept of A-proper mapping, introduced by Petryshyn, and we refer to his book [18] for further information on projection schemes, properties, and applications of A-proper maps.

Definition 2.1. Given a map $T : D \subset X \to X$, T is said to be *A-proper* at a point $y \in X$ relative to Γ if

$$T_n := P_n T : D \cap X_n \longrightarrow X_n \tag{2.1}$$

is continuous for each $n \in \mathbb{N}$ and if $\{x_{n_j} | x_{n_j} \in X_{n_j}\}$ is a bounded sequence such that

$$||P_n T(x_{n_j}) - y|| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty, \tag{2.2}$$

there exists a subsequence $\{x_{n_{j(k)}}\}$ of $\{x_{n_j}\}$ and $x \in X$ such that $x_{n_{j(k)}} \to x$ and $T(x) = y$. T is A-proper on a set K if it is A-proper at all points of K. A-proper alone means A-proper on X.

In a similar way, for a fixed $\gamma \geq 0$, T is said to be *P_γ-compact* at a point $y \in X$ with respect to Γ if $\lambda I - T$ is A-proper at y for each λ dominating γ (i.e., $\lambda \geq \gamma$ if $\gamma > 0$ and $\lambda > 0$ if $\gamma = 0$). T is said to be P_γ-compact on a set K if it is P_γ-compact at all points of K.

We recall the definitions of weakly inward set and map, see for example [3].

Definition 2.2. Let K be a closed convex set in X. For $x \in K$ the set

$$I_K(x) = \{x + c(z - x) : z \in K,\ c \geq 0\} \tag{2.3}$$

is called the *inward* set of x relative to K. The closure of $I_K(x)$, $\overline{I}_K(x)$ is said to be the *weakly inward* set of x relative to K.

Geometrically, the inward set $I_K(x)$ is the union of all rays beginning at x and passing through some other point of K.

Recall that K is called a *wedge* if $\lambda x \in K$ for $x \in K$ and $\lambda \geq 0$. If, furthermore, $K \cap (-K) = \{0\}$, we say that K is a *cone*.

Definition 2.3. Given a map $T : \Omega \subset K \to K$, T is said to be *inward* on Ω relative to K if $Tx \in I_K(x)$ for $x \in \Omega$. If $Tx \in \overline{I}_K(x)$ for $x \in \Omega$, T is said to be *weakly inward*.

We recall the definition of k-semicontractive map.

Definition 2.4. Let D be a nonempty subset of X. A map $A : D \to X$ is said to be k-*semicontractive* map with constant $k \geq 0$ if there exists a map $V : D \times D \to X$ such that the following conditions hold.

(S_1) For each fixed $x \in D$, $V(x,\cdot) : D \to X$ is compact.

(S_2) For each $y \in D$, the map $V(\cdot,y) : D \to X$ is a Lipschitz map with Lipschitz constant k.

(S_3) $A(x) = V(x,x)$ for $x \in D$.

Lan and Webb [12] defined a fixed point index for weakly inward A-proper maps, which has the usual properties of the classical fixed point index, that is, existence, normalization, additivity, and homotopy invariance.

In this paper, we focus on some applications of this theory. Throughout the following, K is a cone. We set $K_r = \{x \in K : \|x\| < r\}$ and $\overline{K}_r = \{x \in K : \|x\| \leq r\}$.

First we state a lemma which implies that the fixed point index, $i_K(T,K_r)$, is 1. This uses the well-known Leray-Schauder condition.

LEMMA 2.5 (see [12]). *Assume that $T : \overline{K}_r \to X$ is weakly inward, P_1-compact on K, and satisfies*

(LS) $x \neq tT(x)$ *for* $\|x\| = r$ *and* $t \in [0,1)$.

Then T has a fixed point in $\overline{K}_r$. Furthermore, if $x \neq T(x)$ for $\|x\| = r$, then $i_K(T,K_r) = \{1\}$.

Now we give a condition which ensures that the fixed point index is 0.

LEMMA 2.6 (see [12]). *Assume that $T : \overline{K}_r \to X$ is weakly inward, P_1-compact on K, and $T(\overline{K}_r)$ is bounded. Suppose that $x \neq Tx$ for $\|x\| = r$, and*

(E) *there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for $\|x\| = r$ and $\lambda > 0$.*

Then $i_K(T,K_r) = \{0\}$.

These conditions imply the following theorem.

THEOREM 2.7 (see [12]). *Let $T : \overline{K}_r \to X$ be weakly inward, P_1-compact on K, with $T(\overline{K}_r)$ bounded. Suppose the following conditions are satisfied:*

(LS) *there exists $\rho \in (0,r)$ such that $x \neq tTx$ for $\|x\| = \rho$ and $0 \leq t < 1$,*

(E) *there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for $\|x\| = r$ and $\lambda > 0$.*

Then T has a fixed point in $\overline{K}_r \setminus K_\rho$. The same conclusion remains valid if (LS) holds for $\|x\| = r$ and (E) holds for $\|x\| = \rho$.

One benefit of such type or result, as compared with the well-known Krasnosel'skiĭ theorem, is that we do not require the cone to be sent into itself, but into a larger set.

3. Applications to three-point BVPs

In this section, we consider the existence of positive solutions of BVP

$$-u''(t) = f(t,u,u',u''), \quad t \in (0,1), \tag{3.1}$$

with boundary conditions

$$u(0) = 0, \qquad \alpha u(\eta) = u(1), \quad 0 < \eta < 1,\ \alpha\eta < 1. \tag{3.2}$$

We restrict our attention to the case $1 + \alpha\eta^2 \geq 2\alpha\eta$.

In order to apply the results of Section 2, we set

$$c_1 = \frac{1}{8}\left(\frac{1-\alpha\eta^2}{1-\alpha\eta}\right)^2, \qquad c_2 = \frac{1}{2}\left(\frac{1-2\alpha\eta+\alpha\eta^2}{1-\alpha\eta}\right), \qquad c_3 = \frac{1}{2}\left(\frac{1-\alpha\eta^2}{1-\alpha\eta}\right), \tag{3.3}$$

see (3.9) and (3.10) for the interpretation of these constants.

We make the following assumptions on f:

(C_1) there exists $r > 0$ such that $f : [0,1] \times [0,c_1 r] \times [-c_2 r, c_3 r] \times [-r,0] \to \mathbb{R}$ is a continuous function,

(C_2) there exists $k \in (0,1)$ such that $|f(t,u,v,-s_1) - f(t,u,v,-s_2)| \leq k|s_1 - s_2|$ for $t \in [0,1]$, $u \in [0,c_1 r]$, $v \in [-c_2 r,\ c_3 r]$, and $s_1, s_2 \in [0,r]$,

(C_3) $f(t,u,v,0) \geq 0$ for $t \in [0,1]$, $u \in [0,c_1 r]$, and $v \in [-c_2 r, c_3 r]$,

(C_4) $f(t,u,v,-r) \leq r$ for $t \in [0,1]$, $u \in [0,c_1 r]$, and $v \in [-c_2 r, c_3 r]$,

(C_5) there exists $\rho \in (0,r)$ such that $f(t,u,v,-\rho) \geq \rho$ for $t \in [0,1]$, $u \in [0,c_1 r]$ and $v \in [-c_2 r, c_3 r]$.

Remark 3.1. As in [13], we point out that condition (C_3) is weaker than the usual positivity requirement for $f(t,u,v,s)$. If f is not positive, the standard theory of fixed point index cannot be applied since it needs the cone to be sent back into itself (see, e.g., [18]).

Furthermore, we stress that weakly inward fixed point index only exists in nonreflexive spaces using A-proper theory, even for compact maps.

With respect to the alternative method of "solving" (1.1) for the highest-order derivative by means of the contractive hypothesis (C_2), the reader might find interesting comments in [19, 22].

For these reasons, we employ Lan and Webb's theory for weakly inward A-proper maps [12].

We work in $X = C[0,1]$, the space of continuous functions on $[0,1]$ with the usual maximum norm and use the projection scheme $\Gamma = \{X_n, P_n\}$ associated with the standard Schauder basis [17]. We use the cone of positive functions

$$K = \{u \in C[0,1] : u(t) \geq 0 \text{ for } t \in [0,1]\}. \tag{3.4}$$

It is known that $P_n K \subset K$.

We recall the following result which is a consequence of [3, Lemma 18.2].

Lemma 3.2 (see [10]). *Let $X = C[0,1]$ and K as above. Take $u \in K$ and define*

$$E(u) = \{t \in [0,1] : u(t) = 0\}. \tag{3.5}$$

Then,

(1) *if $E(u) = \emptyset$, that is, $u(t) > 0$ for every $t \in [0,1]$, or equivalently, u is an interior point of K, then $\overline{I}_K(u) = X$,*

(2) *if $E(u) \neq \emptyset$, that is, $u \in \partial K$, then the set $\{v \in X : v(t) \geq 0$ for $t \in E(u)\}$ is a subset of $\overline{I}_K(u)$, that is, if the values of v are nonnegative at all points at which the values of u are zero, then v belongs to the weakly inward set $\overline{I}_K(u)$ of u.*

We can now state a theorem for the positive solutions of (3.1)-(3.2).

Theorem 3.3. *Assume that the conditions (C_1)–(C_5) hold. Then (3.1)-(3.2) has a positive solution v with $\rho \leq \|v\| \leq r$.*

Proof. Let $U = \{u \in C^2[0,1] : u(0) = 0,\ \alpha u(\eta) = u(1)\}$. Define a map $L : U \to X$ by $Lu = -u''$. Then L is a linear isomorphism and

$$L^{-1}v(t) = \int_0^1 k(t,s)v(s)ds, \tag{3.6}$$

where

$$k(t,s) = \frac{1}{1-\alpha\eta}t(1-s) - \begin{cases} \dfrac{\alpha t}{1-\alpha\eta}(\eta - s), & s \leq \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t-s, & s \leq t, \\ 0, & s > t. \end{cases} \tag{3.7}$$

We define a continuous map $T : \overline{K}_r \to X$ by

$$Tv(t) = f\left(t, L^{-1}v, \frac{d}{dt}L^{-1}v, -v\right), \tag{3.8}$$

where $K_r = \{u \in K : \|u\| < r\}$. By direct calculation, it may be shown that

$$\max_{t\in[0,1]} \int_0^1 k(t,s)ds = c_1. \tag{3.9}$$

So if $v \in \overline{K}_r$, then $0 \leq L^{-1}v(t) \leq c_1 r$. Also by routine calculations, it may be shown that if $v \in \overline{K}_r$, then

$$-c_2 r \leq \frac{d}{dt}L^{-1}v(t) \leq c_3 r. \tag{3.10}$$

Therefore, T is well defined and (C_1) implies that T is continuous.

To show that T is P_1-compact, one studies the map $V : \overline{K}_r \times \overline{K}_r \to X$ defined by

$$V(u,v) = f\left(t, L^{-1}v, \frac{d}{dt}L^{-1}v, -u\right). \tag{3.11}$$

Then by (C_2), $V(u,\cdot)$ is Lipschitz with constant k and, since L^{-1} and $(d/dt)L^{-1}$ are compact, $V(\cdot,v)$ is compact. These conditions imply that V is a k-semicontraction with $k<1$, and hence $Tu = V(u,u)$ is P_γ-compact for every $\gamma \in (k,1)$. For the proof of this assertion, we refer to [18], see also [13].

To prove that T is weakly inward relative to K, let $v \in \partial K$, that is,

$$E(v) = \{t \in [0,1] : v(t) = 0\} \neq \varnothing. \tag{3.12}$$

Then $(Tv)(t) = f(t, L^{-1}v, (d/dt)L^{-1}v, 0)$ for every $t \in E(v)$. It follows from (C_3) that

$$(Tv)(t) \geq 0 \quad \text{for every } t \in E(v). \tag{3.13}$$

Using Lemma 3.2, we see that $Tv \in \bar{I}_K(v)$ and so T is weakly inward.

We show that T satisfies the condition (LS) in Theorem 2.7, that is, $v \neq \lambda Tv$ for $v \in \partial K_r$ and $\lambda \in (0,1)$. In fact, if not, there exist $v_0 \in \partial K_r$ and $\lambda_0 \in (0,1)$ such that $v_0 = \lambda Tv_0$. Let $t_0 \in [0,1]$ be such that $v_0(t_0) = r$. Then by (C_4), we have

$$r = v_0(t_0) = \lambda_0 f\left(t_0, L^{-1}v(t_0), \frac{d}{dt}L^{-1}v(t_0), -r\right) \leq \lambda_0 r < r, \tag{3.14}$$

a contradiction.

Finally, we prove that T satisfies the condition (E) in Theorem 2.7 with $e(t) \equiv 1$ for $t \in [0,1]$, that is, $v \neq Ty + \beta e$ for $v \in \partial K_\rho$ and $\beta > 0$. In fact, if not, there exist $v_0 \in \partial K_\rho$ and $\beta_0 > 0$ such that $v_0 = Tv_0 + \beta_0 e$. Let $t_0 \in [0,1]$ be such that $v_0(s) = \|v_0\| = \rho$. Then we have

$$\rho = f\left(t_0, L^{-1}v(t_0), \frac{d}{dt}L^{-1}v(t_0), \rho\right) + \beta_0 e \geq \rho + \beta_0 e > \rho, \tag{3.15}$$

a contradiction.

It follows from Theorem 2.7 that T has a fixed point $v \in K$ satisfying $\rho \leq \|v\| \leq r$. Take $u = L^{-1}v$, then u is a positive solution of (3.1)-(3.2). □

Example 3.4. The function $f(t,u,u',u'') \equiv 3/4\cos(u'')$ with $r = \pi$ and $\rho = \pi/6$ shows that the class of maps that satisfies the conditions (C_1)–(C_5) is nonempty.

Remark 3.5. In order to show the existence of two solutions via Theorem 2.7, one would be tempted to require the following (this is a standard argument in fixed point index theory):

(C_6) there exists $\tilde{\rho} \in (0,\rho)$ such that $f(t,u,v,-\tilde{\rho}) \leq \tilde{\rho}$ for $t \in [0,1]$, $u \in [0,c_1 r]$ and $v \in [0,r]$.

This would provide the existence of $\tilde{v} \in K$ satisfying $\tilde{\rho} \leq \|\tilde{v}\| \leq \rho$. However, as noted in [13, Remark 4.3], it is impossible to simultaneously satisfy (C_2), (C_5), and (C_6).

This error occurred in [2, 9], when the authors discussed the existence of one positive solution of the Picard BVP.

Remark 3.6. For the case $1+\alpha\eta^2 < 2\alpha\eta$, which occurs only when $\alpha > 1$, the value of the constant c_1 given in (3.9) has to be replaced by

$$\max_{t\in[0,1]} \int_0^1 k(t,s)\,ds = \frac{1}{2}\left(\frac{\alpha\eta(1-\eta)}{1-\alpha\eta}\right). \tag{3.16}$$

This is because the constant m on [21, page 914] should read

$$m = \begin{cases} \dfrac{8(1-\alpha\eta)^2}{\left(1-\alpha\eta^2\right)^2} & \text{if } 1+\alpha\eta^2 \geq 2\alpha\eta, \\ \dfrac{2(1-\alpha\eta)}{\alpha\eta(1-\eta)} & \text{if } 1+\alpha\eta^2 < 2\alpha\eta. \end{cases} \tag{3.17}$$

A similar result to Theorem 3.3 holds in this case for (the new) c_1.

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