AN APPROXIMATION OF SOLUTIONS OF VARIATIONAL INEQUALITIES

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We use a Mann-type iteration scheme and the metric projection operator (the nearest-point projection operator) to approximate the solutions of variational inequalities in uniformly convex and uniformly smooth Banach spaces.

1. Introduction

Let $(B,\|\cdot\|)$ be a Banach space with the topological dual space B^*, and let $\rangle\varphi,x\langle$ denote the duality pairing of B^* and B, where $\varphi\in B^*$ and $x\in B$. Let $f:B\to B^*$ be a mapping and let K be a nonempty, closed, and convex subset of B. The (general) *variational inequality* defined by the mapping f and the set K is

$$\begin{aligned}&\mathrm{VI}(f,K):\text{find } x_*\in K \text{ such that}\\ &\qquad\langle f(x_*),x-x_*\rangle\geq 0 \text{ for every } x\in K.\end{aligned}\tag{1.1}$$

The *nonlinear complementarily problem* defined by f and K is by definition as follows:

$$\begin{aligned}&\mathrm{NCP}(f,K):\text{find } x_*\in K \text{ such that } \langle f(x_*),x\rangle\geq 0,\\ &\qquad\text{for every } x\in K \text{ and } \langle f(x_*),x_*\rangle=0.\end{aligned}\tag{1.2}$$

It is known (see [5, 6]) that when K is a closed convex cone, problems $\mathrm{NCP}(f,K)$ and $\mathrm{VI}(f,K)$ are equivalent.

To study the existence of solutions of the $\mathrm{NCP}(f,K)$ and $\mathrm{VI}(f,K)$ problems, many authors have used the techniques of KKM mappings, and the Fan-KKM theorem from fixed point theory (see [1, 5, 6, 7, 8, 9, 10]). In case B is a Hilbert space, Isac and other authors have used the notion of "exceptional family of elements" (EFE) and the Leray-Schauder alternative theorem (see [5, 6]).

In [1, 2], Alber generalized the metric projection operator P_K to a generalized projection operator $\pi_K:B^*\to K$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and Alber used this operator to study $\mathrm{VI}(f,K)$ problems and to

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approximate the solutions by an iteration sequence. In [7], the author used the generalized projection operator and a Mann-type iteration sequence to approximate the solutions of the VI(f,K) problems.

In case B is a uniformly convex and uniformly smooth Banach space, the continuity property of the metric projection operator P_K has been studied by Goebel, Reich, Roach, and Xu (see [4, 12, 13]). In this paper, we use the operator P_K and a Mann-type iteration scheme to approximate the solutions of NCP(f,K) problems.

2. Preliminaries

Let $(X,\|\cdot\|)$ be a normed linear space and let K be a nonempty subset of X. For every $x \in X$, the distance between a point x and the set K is denoted by $d(x,K)$ and is defined by the following minimum equation

$$d(x,K) = \inf_{y\in K} \|x-y\|. \tag{2.1}$$

The metric projection operator (or the nearest-point projection operator) P_K defined on X is a mapping from X to 2^K:

$$P_K(x) = \{z \in X : \|x-z\| = d(x,K),\ \forall x \in X\}. \tag{2.2}$$

If $P_K(x) \neq \emptyset$, for every $x \in X$, then K is called *proximal*. If $P_K(x)$ is a singleton for every $x \in X$, then K is said to be a *Chebyshev* set.

Theorem 2.1. *Let $(B,\|\cdot\|)$ be a reflexive Banach space. Then B is strictly convex if and only if every nonempty closed convex subset $K \subset B$ is a Chebyshev set.*

Since uniformly convex and uniformly smooth Banach spaces are reflexive and strictly convex, the above theorem implies that if $(B,\|\cdot\|)$ is a uniformly convex and uniformly smooth Banach space, then every nonempty closed convex subset $K \subset B$ is a Chebyshev set.

Let T be a uniformly convex Banach space. Its modulus of convexity is denoted by δ and is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| = \|y\| = 1,\ \|x-y\| = \epsilon\right\}. \tag{2.3}$$

It follows that δ is a strictly increasing, convex, and continuous function from $(0,2]$ to $[0,1]$, and it is known that $\delta(\epsilon)/\epsilon$ is nondecreasing on $(0,2]$.

If B is uniformly smooth, its modulus of smoothness is denoted by $\rho(\tau)$ and is defined by

$$\rho(\tau) = \sum\left\{\frac{1}{2}\|x+y\| + \frac{1}{2}\|x-y\| - 1 : \|x\| = 1 \cdot \|y\| \leq \tau\right\}. \tag{2.4}$$

It can be shown that ρ is a convex and continuous function from $[0,\infty)$ to $[0,\infty)$ with the properties that $\rho(\tau)/\tau$ is nondecreasing, $\rho(\tau) \leq \tau$ for all $\tau \geq 0$, and $\lim_{\tau\to 0^+} \rho(\tau)/\tau = 0$. For the details of the properties of δ and ρ, the reader is refereed to [10, 11].

Theorem 2.2 (Xu and Roach [13]). *Let M be a convex Chebyshev set of a uniformly convex and uniformly smooth Banach spaces X and let $P;X \to M$ be the metric projection. Then*

(i) *P is Lipschitz continuous mod M; namely, there exists a constant $k > 0$ such that*

$$\|P(x) - z\| \leq k\|x - z\|, \quad \text{for any } x \in X \text{ and } z \in M, \tag{2.5}$$

(ii) *P is uniformly continuous on every bounded subset of X and, furthermore, there exist positive constants k_r for every $B_r := \{x \in X : \|x\| \leq r\}$ such that*

$$\|P(x) - P(y)\| \leq \|x - y\| + k\delta^{-1}(\psi(\|x - y\|)), \quad \text{for any } x, y \in B_r, \tag{2.6}$$

where ψ is defined by

$$\psi(t) = \int_0^t \frac{\rho(s)}{s} ds. \tag{2.7}$$

Theorem 2.3 (Xu and Roach [13]). *If $X = L_p, \ell_p$, or W_m^p $(1 < p < \infty)$ in Theorem 2.2, then the metric projection P is Hölder continuous on every bounded subset of X, and, moreover, there exist positive k_r for every B_r such that*

$$\|P(x) - P(y)\| \leq k_r\|x - y\|^{\min(2,p)/\max(2,p)}, \quad \text{for any } x, y \in B_r. \tag{2.8}$$

The normalized duality mapping $J : B \to 2^{B^*}$ is defined by

$$J(x) = \{j(x) \in B^* : \langle j(x), x\rangle = \|j(x)\|\|x\| = \|x\|^2 = \|j(x)\|^2\}. \tag{2.9}$$

Clearly, $\|j(x)\|$ is the B^*-norm of $j(x)$ and $\|x\|$ is its B-norm. It is known that if B is uniformly convex and uniformly smooth, then J is a single-valued, strictly monotone, homogeneous, and uniformly continuous operator on each bounded set. Furthermore, J is the identity in Hilbert spaces; that is, $J = I_H$.

The following theorem provides a tool to solve a variational inequality by finding a fixed point of a certain operator.

Theorem 2.4 (Li [8]). *Let $(B, \|\cdot\|)$ be a reflexive and smooth Banach space and $K \subset B$ a nonempty closed convex subset. For any given $x \in B$, $x_0 \in P_K(x)$ if and only if*

$$\langle J(x - x_0), x_0 - y\rangle \geq 0, \quad \forall y \in K. \tag{2.10}$$

Let $F : K \to B$ be a mapping. The *locality variational inequality* defined by the mapping F and the set K is

$$\begin{aligned} \text{LVI}(F,K) &: \text{find } x_* \in K \text{ and } j(F(x_*)) \in J(F(x_*)) \\ &\text{such that } \langle j(F(x_*)), y - x_*\rangle \geq 0, \text{ for every } y \in K. \end{aligned} \tag{2.11}$$

The next theorem follows from Theorem 2.4.

Theorem 2.5 (Li [8]). *Let $(B, \|\cdot\|)$ be a reflexive and smooth Banach space and $K \subset B$ a nonempty closed convex subset. Let $F: B \to B$ be a mapping. Then an element $x_* \in KE$ is a solution of* LVI(F,K) *if and only if $x_* \in P_K(x_* - F(x_*))$.*

3. The compact case

Theorem 3.1. *Let $(B, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space and K a nonempty compact convex subset of B. Let $F: K \to B$ be a continuous mapping. Suppose that* LVI(F,K) *has a solution $x_* \in K$ and F satisfies the following condition:*

$$\begin{aligned} &||x - x_* - (F(X) - F(x_*))|| + k_r\delta^{-1}(\rho(||x - x_* - (F(X) - F(x_*)||)) \\ &\quad \le ||x - x_*||, \quad \text{for every } x \in K, \end{aligned} \tag{3.1}$$

where k_r is the positive constant given in Theorem 2.2 that depends on the bounded subset K. For any $x_0 \in K$, define the Mann iteration scheme as follows:

$$x_{n+1} = (1 - \alpha_n)x_n - \alpha_n P_K(x_n - F(x_n)), \quad n = 1, 2, 3, \ldots, \tag{3.2}$$

where $\{\alpha_n\}$ satisfies conditions (a) $0 \le \alpha_n \le 1$ for all n, (b) $\sum \alpha_n(1 - \alpha_n) = \infty$. Then there exists a subsequence $\{n(i)\} \subseteq \{n\}$ such that $\{x_{n(i)}\}$ converges to a solution x' of LVI(F,K).

Proof. Since B is uniformly convex and uniformly smooth, there exists a continuous strictly increasing and convex function $g: \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(0) = 0$ and, for all $x, y \in B_r(0) := \{x \in E : \|x\| \le r\}$ and for any $\alpha \in [0,1]$, we have

$$||\alpha x + (1 - \alpha)y||^2 \le \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)g(\|x - y\|), \tag{3.3}$$

where r is a positive number such that $K \subseteq B_r(0)$ (see [3]). Since x_* is a solution of LVI(F,K), from Theorems 2.1 and 2.5, $x_* = P_K(x_* - F(x_*))$. Using Theorem 2.2,

$$\begin{aligned} ||x_{n+1} - x_*||^2 &= ||(1 - \alpha_n)(x_n - x_*) + \alpha_n(P_K((x_n) - F(x_n)) - x_*)||^2 \\ &\le (1 - \alpha_n)||x_n - x_*||^2 + \alpha_n||P_K((x_n) - F(x_n)) - x_*||^2 \\ &\quad - \alpha_n(1 - \alpha_n)g(||(x_n - x_*) - (P_K((x_n)) - F(x_n)) - x_*)||) \\ &= (1 - \alpha_n)||x_n - x_*||^2 + \alpha_n(P_K((x_n) - F(x_n)) - x_*)||^2 \\ &\quad - \alpha_n(1 - \alpha_n)g(||P_K(x_n - F(x_n)) - x_n||) \\ &\le (1 - \alpha_n)||x_n - x_*||^2 + \alpha_n(||x_n - F(x_n)) - (x_* - F(x_*))|| \\ &\quad + k_r\delta^{-1}(\psi((||x_n - F(x_n)) - (x_* - F(x_*))||)^2 \\ &\quad - \alpha_n(1 - \alpha_n)g(||P_K(x_n - F(x_n)) - x_n||). \end{aligned} \tag{3.4}$$

Since $\rho(\tau)/\tau$ is nondecreasing and $\lim_{\tau\to 0^+}\rho(\tau)/\tau = 0$,

$$\begin{aligned}
&\psi(\|(x_n - F(x_n)) - (x_* - F(x_*))\|) \\
&\quad = \int_0^{\|(x_n-F(x_n))-(x_*-F(x_*))\|} \frac{\rho(s)}{s}\,ds \\
&\quad \le \frac{\rho(\|(x_n - F(x_n)) - (x_* - F(x_*))\|)}{\|(x_n - F(x_n)) - (x_* - F(x_*))\|}\|(x_n - F(x_n)) - (x_* - F(x_*))\| \\
&\quad = \rho(\|(x_n - F(x_n)) - (x_* - F(x_*))\|).
\end{aligned} \tag{3.5}$$

Applying condition (3.1) and the above inequality, we obtain

$$\begin{aligned}
\|x_{n+1} - x_*\|^2 &\le (1-\alpha_n)\|x_n - x_*\|^2 + \alpha_n(\|(x_n - F(x_n)) - (x_n - F(x_*))\| \\
&\qquad + k_r\delta^{-1}\rho(\|x_n - F(x_n)) - (x_* - F(x_*))\|)^2 \\
&\qquad - \alpha_n(1-\alpha_n)g(\|P_K(x_n - F(x_n)) - x_n\|) \\
&\le (1-\alpha_n)\|x_n - x_*\|^2 + \alpha_n\|x_n - x_*\|^2 \\
&\qquad - \alpha_n(1-\alpha_n)g(\|P_K(x_n - F(x_n)) - x_n\|) \\
&= \|x_n - x_*\|^2 - \alpha_n(1-\alpha_n)g(\|P_K(x_n - F(x_n)) - x_n\|).
\end{aligned} \tag{3.6}$$

Therefore

$$\begin{aligned}
&\alpha_n(1-\alpha_n)g(\|P_K(x_n - F(x_n)) - x_n\|) \\
&\qquad \le \|x_n - x_*\|^2 - \|x_{n+1} - x_*\|^2, \quad n = 1,2,\ldots.
\end{aligned} \tag{3.7}$$

For any positive integer m, taking the sum for $n = 1,2,3,\ldots,m$, we have

$$\begin{aligned}
&\sum_{n=1}^{m}\alpha_n(1-\alpha_n)g(\|P_K(x_n - F(x_n)) - x_n\|) \\
&\qquad \le \|x_1 - x_*\|^2 - \|x_{n+1} - x_*\|^2 \le \|x_1 - x_*\|^2,
\end{aligned} \tag{3.8}$$

which implies that

$$\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)g(\|P_K(x_n - F(x_n)) - x_n\|) \le \|x_1 - x_*\|^2. \tag{3.9}$$

From the condition $\sum\alpha_n(1-\alpha_n) = \infty$, there exists a subsequence $\{n(i)\} \subseteq \{n\}$ such that $g(\|P_K(x_{n(i)} - F(x_{n(i)})) - x_{n(i)}\|) \to 0$ as $i \to \infty$. Since g is continuous and strictly increasing such that $g(0) = 0$, we obtain

$$||P_K(x_{n(i)} - F(x_{n(i)})) - x_{n(i)}|| \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \tag{3.10}$$

From the compactness of K, there exists a subsequence of $\{x_{n(i)}\}$ which, without loss of generality, we may assume is the sequence $\{n(i)\}$, and an element $x' \in K$ such that $x_{n(i)} \to x'$ and $i \to \infty$. From the continuity of P_K and F, we have $P_K(x_{n(i)} - F(x_{n(i)})) \to P_K(x' - F(x'))$ as $i \to \infty$. Statement (3.10) implies that $P_K(x' - F(x')) = x'$. Applying Theorem 2.5, x' is a solution of the LVI(F,K) problem. □

Corollary 3.2. *Theorem 3.1 is still true if condition (3.1) is replaced by the following condition:*

$$\begin{aligned} ||x - x_* - (F(x) - F(x_*))|| + k\delta^{-1}(||x - x_* - (F(x) - F(x_*))||) \\ \leq ||x - x_*||, \quad \text{for every } x \in K. \end{aligned} \tag{3.11}$$

Proof. From the property $\rho(\tau) \leq \tau$ for all $\tau \geq 0$, and the nondecreasing property of δ (and δ^{-1}), condition (3.11) implies condition (3.1). The conclusion of the corollary then follows immediately from Theorem 3.1. □

One of the most important types of variational inequalities and complementarily problems deals with completely continuous field mappings. This type of variational inequality and complementarily problem has been studied by many authors in Hilbert spaces (see, e.g., [5, 6]). Recently, the first author and Isac have studied the existence of solutions of this type problem in uniformly convex and uniformly smooth Banach spaces.

Recall that a mapping $T : B \to B$ is completely continuous if T is continuous, and for any bounded set $D \subset B$, we have that $T(D)$ is relatively compact. A mapping $F : B \to B$ has a completely continuous field if F has a representation $F(x) = x - T(X)$ for all $x \in B$, where $T : B \to B$ is a completely continuous mapping.

As an application of Theorem 3.1 we have the following corollary.

Corollary 3.3. *Let $(B, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space, $K \subset B$ a closed convex cone, and $F : B \to B$ a completely continuous field with the representation $F(x) = x - T(x)$. Suppose that* LVI(F,K) *has a solution $x_* \in K$ and that T satisfies the condition*

$$||T(x) - T(x_*)|| + k\delta^{-1}(\rho(||T(x) - T(x_*)||)) \leq ||x - x_*|| \quad \text{for every } x \in K, \tag{3.12}$$

where k_r is the positive constant given in Theorem 2.2 that depends on the bounded subset K. Then there exists a subsequence $\{n(i)\}$ of the sequence defined by (3.2) such that $\{x_{n(i)}\}$ converges to a solution of LVI(F,K).

Proof. Replacing $F(x)$ by $x - T(x)$ in (3.1) of Theorem 3.1 yields the conclusion of the corollary. □

It is well known that L_p, ℓ_p, and W_m^p $(1 < p < \infty)$ are special uniformly convex and uniformly smooth Banach spaces. In [2], Alber and Notik provided formulas for the calculation of the modulus of convexity δ and the modulus of smoothness ρ for these spaces:

$$\delta(\epsilon) = \begin{cases} \frac{1}{8}(p-1)\epsilon^2 + o(\epsilon^2) \geq \frac{1}{8}(p-1)\epsilon^2, & 1 < p \leq 2, \\ 1 - \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{1/p} \geq \frac{1}{p}\left(\frac{\epsilon}{2}\right)^p, & 2 \leq p < \infty, \end{cases}$$
$$\rho(\tau) = \begin{cases} (1+\tau^p)^{1/p} - 1 \leq \frac{1}{p}\tau^p, & 1 < p \leq 2, \\ \frac{p-1}{2}\tau^2 + o(\tau^2) \leq \frac{p-1}{2}\tau^2, & 2 \leq p < \infty. \end{cases} \tag{3.13}$$

Applying the above formulas to Theorem 3.1, we can obtain more detailed applications and examples.

Corollary 3.4. *Let $B = L_p, \ell_p$ or W_m^p $(1 < p < \infty)$ and K a nonempty compact convex subset of B. Let $F : B \to B$ be a completely continuous field with the representation $F(x) = x - T(x)$. Suppose that* LVI(F,K) *has a solution $x_* \in K$ and, for every $x \in K$, T satisfies the following conditions:*

$$\begin{aligned} &||T(x) - T(x_*)|| \\ &\quad \leq \min\left\{\frac{1}{2}||x - x_*||, \left(\frac{1}{4k_r}\right)^{2/p}\left(\frac{p(p-1)}{2}\right)^{1/p}||x - x_*||^{2/p}\right\} \quad \text{if } 1 < p \leq 2, \\ &||T(x) - T(x_*)|| \\ &\quad \leq \min\left\{\frac{1}{2}||x - x_*||, \left(\frac{1}{2k_r}\right)^{p}\frac{4}{(p(p-1))^2}||x - x_*||^{p/2}\right\} \quad \text{if } 2 \leq p < \infty, \end{aligned} \tag{3.14}$$

where k_r is the positive constant given in Theorem 2.2. Then there exists a subsequence $\{n(i)\}$ of the sequence defined by (3.2) such that $\{x_{n(i)}\}$ converges to a solution x' of LVI(F,K).

Proof. Assume that $1 < p \leq 2$. From the inequality

$$\delta(\epsilon) \geq \frac{1}{8}(p-1)\epsilon^2, \tag{3.15}$$

we obtain

$$\delta^{-1}(\epsilon) \leq \left(\frac{8}{p-1}\epsilon\right)^{1/2}. \tag{3.16}$$

Noting that both δ and ρ are strictly increasing, and using the inequality $\rho(\tau) \leq \tau^p/p$, we have

$$\begin{aligned} &||T(x) - T(x_*)|| + k_r\delta^{-1}(\rho(||T(x) - T(x_*)||)) \\ &\quad \leq \frac{1}{2}||x - x_*|| + k_r\delta^{-1}\left(\frac{1}{p}||T(x) - T(x_*)||^p\right) \\ &\quad \leq \frac{1}{2}||x - x_*|| + k_r\left(\frac{8}{p(p-1)}||T(x) - T(x_*)||^p\right)^{1/2} \\ &\quad \leq ||x - x_*||. \end{aligned} \tag{3.17}$$

The last inequality follows from the condition of this corollary. Then this case can be obtained by using Corollary 3.4. The case for $2 \leq p < \infty$ can be proved similarly. □

Theorem 3.5. *Let B, K, F be as in Theorem 3.1. If inequality (3.1) holds for all solutions of* LVI(F,K), *then the sequence $\{x_n\}$ defined by (3.2) converges to a solution x' of the* LVI(F,K) *problem.*

Proof. From Theorem 3.1, $\{x_n\}$ has a subsequence $\{x_{n(i)}\}$ that converges to a solution x', as $i \to \infty$. In the proof of Theorem 3.1, replacing x_* by x', we obtain

$$\begin{aligned} ||x_n - x'||^2 &\leq ||x_n - x'||^2 - \alpha_n(1-\alpha_n)g(||P_K(x_n - F(x_n)) - x_n||) \\ &\leq ||x_n - x'||^2, \quad n = 1,2,3,\dots, \end{aligned} \tag{3.18}$$

which implies that $\{\|x_n - x'\|^2\}$ is a decreasing sequence. Since there exists a subsequence $\{x_{n(i)}\}$ such that $\|x_{n(i)} - x'\| \to 0$ as $i \to \infty$, we obtain the fact that $\|x_n - x'\| \to 0$ as $n \to \infty$. □

Corollary 3.6. *Let B, K, F be as in Theorem 3.1. If inequality (3.1) holds for all $y \in K$, then the sequence defined by (3.2) converges to a solution x' of the* LVI(F,K) *problem.*

Proof. This corollary follows immediately from Theorem 3.5. □

If we apply Theorem 2.3 to the special uniformly convex and uniformly smooth Banach spaces L_p, ℓ_p, and W_m^p $(1 < p < \infty)$, and apply the techniques of the proof of Theorem 3.1, we obtain the following.

Theorem 3.7. *Let $B = L_p, \ell_p$, or W_m^p $(1 < p < \infty)$ and K a nonempty compact convex subset of B. Let $F : K \to B$ be a continuous mapping. Suppose that* LVI(F,K) *as a solution $x_* \in K$ and F satisfies the following:*

$$k_r||(x - x_*) - (F(x) - F(x_*))||^{\min(2,p)/\max(2,p)} \leq ||x - x_*|| \quad \text{for any } x \in K, \tag{3.19}$$

where k_r is the positive constant given in Theorem 2.2 that depends on the bounded subset K. Then there exists a subsequence $\{x_{n(i)}\}$ of the sequence $\{x_n\}$ defined by (3.2) that converges to a solution x' of LVI(F,K).

Proof. Here we use Theorem 2.3 to obtain

$$\begin{aligned} &||P_K(x_n - F(x_n)) - P_K(x_* - F(x_*))|| \\ &\quad \le k_r||(x - F(x)) - (x_* - F(x_*))||^{\min(2,p)/\max(2,p)}. \end{aligned} \tag{3.20}$$

The rest of the proof is similar to that of Theorem 3.1. □

Corollary 3.8. *Let B, K, F be as in Theorem 3.5. If inequality (3.19) holds for all solutions of* LVI(F,K), *then the sequence $\{x_n\}$ defined by (3.2) converges to a solution x' of the* LVI(F,K) *problem.*

Corollary 3.9. *Let B, K, F be as in Theorem 3.1. If inequality (3.19) holds for all $y \in K$, then the sequence $\{x_n\}$ defined by (3.2) converges to a solution of the* LVI(F,K) *problem.*

4. The unbounded case

If K is unbounded, for example, if K is a closed convex cone, the following theorems are needed for estimation.

Theorem 4.1 (Xu and Roach [13]). *Let M be a convex Chebyshev set of a uniformly convex and uniformly smooth Banach space X and $P : X \to M$ be the metric projection. Then, for every $x, y \in X$,*

$$\begin{aligned} &||P(x) - P(y)|| \\ &\quad \le \|x - y\| + 4(||x - P(x)|| \vee ||P(x) - y||)\delta^{-1}\left(C_1\psi\left(\frac{\|x - y\|}{||x - P(y)|| \vee ||y - P(x)||}\right)\right), \end{aligned} \tag{4.1}$$

where C_1 is a fixed constant and ψ is as defined in Theorem 2.2.

Theorem 4.2. *Let $(B, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space and K a nonempty closed convex subset of B. Let $F : K \to B$ be a continuous mapping such that the* LVI(F,K) *problem has a solution $x_* \in K$. If there exist positive constants κ and λ satisfying the following conditions:*

(i) $\|x - x_* - (F(x) - F(x_*))\| \le \|x - x_*\|$ *for every $x \in K$;*

(ii) $t^{-1}\delta^{-1}(t) \le \lambda\ \forall t$;

(iii) $(\kappa + 4C_1\kappa\lambda) < 1$, *where C_1 is the constant given in Theorem 4.1,*

then the sequence $\{x_n\}$ defined by (3.2) converges to the solution x_* of the LVI(F,K) *problem.*

Proof. Using Theorem 4.1, similar to the proof of Theorem 3.1, we have

$$
\begin{aligned}
&||x_{n+1}-x_*||^2 \\
&\quad = ||(1-\alpha_n)(x_n-x_*)+\alpha_n(P_K(x_n-F(x_n))-x_n)||^2 \\
&\quad \leq (1-\alpha_n)||x_n-x_*||^2+\alpha_n||P_K(x_n-F(x_n))-P_K(x_*-F(x_*))||^2 \\
&\qquad -\alpha_n(1-\alpha_n)g(||P_K(x_n-F(x_n))-x_n||) \\
&\quad \leq (1-\alpha_n)||x_n-x_*||^2+\alpha_n(||x_n-F(x_n))-(x_*-F(x_*))|| \\
&\qquad +4(||(x_n-F(x_n))-P_K(x_*-F(x_*))||\vee||P_K(x_n-F(x_n))-(x_*-F(x_*))||) \\
&\qquad \times\delta^{-1}(C_1\psi(||(x_n-F(x_n))-(x_*-F(x_*))|| \\
&\qquad \div||(x_n-f(x_n))-P_K(x_*-F(x_*))||\vee||P_K(x_n-F(x_n))-(x_*-F(x_*))||))^2 \\
&\qquad -\alpha_n(1-\alpha_n)g(||P_K(x_n-F(x_n))-x_n||).
\end{aligned}
\tag{4.2}
$$

The property $\rho(\tau)\leq\tau$ for all $\tau\geq 0$ implies that $\rho(\tau)/\tau\leq 1$ for all $\tau\geq 0$. From the definition of ψ, we have $\psi(t)\leq t$ for all $t\geq 0$. Since δ^{-1} is a strictly increasing function, from conditions (i) and (ii), we obtain

$$
\begin{aligned}
&||x_{n+1}-x_*||^2 \\
&\leq (1-\alpha_n)||x_n-x_*||^2+\alpha_n(||x_n-F(x_n))-(x_*-F(x_*))|| \\
&\quad +4(||x_n-F(x_n)-P_K(x_*-F(x_*))||\vee||P_K(x_n-F(x_n))-(x_*-F(x_*))||) \\
&\quad \times\delta^{-1}\left(C_1\frac{||(x_n-F(x_n))-(x_*-F(x_*))||}{||(x_n-F(x_n))-P_K(x_*-F(x_*))||\vee||P_K(x_n-F(x_n))-(x_*-F(x_*))||}\right)^2 \\
&\quad -\alpha_n(1-\alpha_n)g(||P_K(x_n-F(x_n))-x_n||) \\
&\leq (1-\alpha_n)||x_n-x_*||^2+\alpha_n(||x_n-F(x_n))-(x_*-F(x_*))|| \\
&\quad +4(||x_n-F(x_n))-(x_*-F(x_*))|| \\
&\quad \times C_1\left(C_1\frac{||(x_n-F(x_n))-(x_*-F(x_*))||}{||(x_n-F(x_n))-P_K(x_*-F(x_*))||\vee||P_K(x_n-F(x_n))-(x_*-F(x_*))||}\right)^{-1} \\
&\quad \times\delta^{-1}\left(C_1\frac{||(x_n-F(x_n))-(x_*-F(x_*))||}{||(x_n-F(x_n))-P_K(x_*-F(x_*))||\vee||P_K(x_n-F(x_n))-(x_*-F(x_*))||}\right)^2 \\
&\quad -\alpha_n(1-\alpha_n)g(||P_K(x_n-F(x_n))-x_n||) \\
&\leq (1-\alpha_n)||x_n-x_*||^2+\alpha_n(\kappa||x_n-x_*||+4(\kappa||x_n-x_*||C_1\lambda))^2 \\
&\quad -\alpha_n(1-\alpha_n)g(||P_K(x_n-F(x_n))-x_n||)(1-\alpha_n)||x_n-x_*||^2 \\
&\quad +\alpha_n(\kappa+4C_1\kappa\lambda)^2||x_n-x_*||^2-\alpha_n(1-\alpha_n)g(||P_K(x_n-F(x_n))-x_n||) \\
&\leq ||x_n-x_*||^2-\alpha_n(1-\alpha_n)g(||P_K(x_n-F(x_n))-x_n||).
\end{aligned}
\tag{4.3}
$$

The last inequality is obtained by applying condition (iii) of this theorem. Thus we have

$$\|x_{n+1} - x_*\|^2 \le \|x_n - x_*\|^2 - \alpha_n(1-\alpha_n)g(\|P_K(x_n - F(x_n)) - x_n\|). \tag{4.4}$$

Similar to the proof of Theorem 3.1, we can show that there exists a subsequence $\{n(i)\} \subseteq \{n\}$ such that

$$\|P_K(x_{n(i)} - F(x_{n(i)})) - x_{n(i)}\| \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \tag{4.5}$$

For this subsequence, by applying Theorem 4.1 and using an argument similar to the proof of the first part of this theorem, we have

$$\begin{aligned}\|x_{n(i)} - x_*\| &= \|x_{n(i)} - P_K(x_{n(i)} - F(x_{n(i)})) + P_K(x_{n(i)} - F(x_{n(i)})) - x_*\| \\ &\le \|x_{n(i)} - P_K(x_{n(i)} - F(x_{n(i)}))\| + \|P_K(x_{n(i)} - F(x_{n(i)})) - x_*\| \\ &= \|x_{n(i)} - P_K(x_{n(i)} - F(x_{n(i)}))\| + \|P_K(x_{n(i)} - F(x_{n(i)})) - P_K(x_* - F(x_*))\| \\ &\le \|x_{n(i)} - P_K(x_{n(i)} - F(x_{n(i)}))\| + (\kappa + 4C_1\kappa\lambda)\|x_{n(i)} - x_*\|,\end{aligned} \tag{4.6}$$

which implies that

$$(1 - (\kappa + 4C_1\kappa\lambda))\|x_{n(i)} - x_*\| \le \|x_{n(i)} - P_K(x_{n(i)} - F(x_{n(i)}))\|. \tag{4.7}$$

Since $(1 - (\kappa + 4C_1\kappa\lambda)) > 0$ (condition (iii) of this theorem), from (3.10) it follows that

$$\|x_{n(i)} - x_*\| \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \tag{4.8}$$

From inequality (3.19), $\{\|x_n - x_*\|\}$ is a decreasing sequence. Thus

$$\|x_n - x_*\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.9}$$

□

Corollary 4.3. *Let $(B, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space and K a nonempty closed convex subset of B. Let $F : K \to B$ be a continuous mapping. If the Banach space B and the mapping F satisfy conditions (3.2) and (3.10) in Theorem 3.7 and F satisfies the condition*

$$\|x - y - (F(x) - F(y))\| \le \kappa\|x - y\| \quad \text{for every } x, y \in K, \tag{4.10}$$

then the LVI(F,K) *problem has at most one solution.*

Proof. From Theorem 3.7, every solution of the LVI(FmK) problem must be the limit of the sequence (3.2). □

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