

COINCIDENCE AND FIXED POINT THEOREMS FOR FUNCTIONS IN S -KKM CLASS ON GENERALIZED CONVEX SPACES

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We establish a coincidence theorem in S -KKM class by means of the basic defining property for multifunctions in S -KKM. Based on this coincidence theorem, we deduce some useful corollaries and investigate the fixed point problem on uniform spaces.

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1. Introduction

A multimap $T : X \rightarrow 2^Y$ is a function from a set X into the power set 2^Y of Y . If $H, T : X \rightarrow 2^Y$, then the coincidence problem for H and T is concerned with conditions which guarantee that $H(\hat{x}) \cap T(\hat{x}) \neq \emptyset$ for some $\hat{x} \in X$. Park [11] established a very general coincidence theorem in the class \mathbf{U}_c^k of admissible functions, which extends and improves many results of Browder [1, 2], Granas and Liu [6].

On the other hand, Huang together with Chang et al. [3] introduced the S -KKM class which is much larger than the class \mathbf{U}_c^k . A lot of interesting and generalized results about fixed point theory on locally convex topological vector spaces have been studied in the setting of S -KKM class in [3]. In this paper, we will at first construct a coincidence theorem in S -KKM class on generalized convex spaces by means of the basic defining property for multimaps in S -KKM class. And then based on this coincidence theorem, we deduce some useful corollaries and investigate the fixed point problem on uniform spaces.

2. Preliminaries

Throughout this paper, $\langle Y \rangle$ denotes the class of all nonempty finite subsets of a nonempty set Y . The notation $T : X \multimap Y$ stands for a multimap from a set X into $2^Y \setminus \{\emptyset\}$. For a multimap $T : X \rightarrow 2^Y$, the following notations are used:

- (a) $T(A) = \bigcup_{x \in A} T(x)$ for $A \subseteq X$;
- (b) $T^-(y) = \{x \in X : y \in T(x)\}$ for $y \in Y$;
- (c) $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ for $B \subseteq Y$.

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All topological spaces are supposed to be Hausdorff. Let X and Y be two topological spaces. A multimap $T : X \rightarrow 2^Y$ is said to be

- (a) upper semicontinuous (u.s.c.) if $T^{-1}(B)$ is closed in X for each closed subset B of Y ;
- (b) compact if $T(X)$ is contained in a compact subset of Y ;
- (c) closed if its graph $\text{Gr}(T) = \{(x, y) : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$.

LEMMA 2.1 (Lassonde [9, Lemma 1]). *Let X and Y be two topological spaces and $T : X \multimap Y$.*

- (a) *If Y is regular and T is u.s.c. with closed values, then T is closed. Conversely, if Y is compact and T is closed, then T is u.s.c. with closed values.*
- (b) *If T is u.s.c. and compact-valued, then $T(A)$ is compact for any compact subset A of X .*

Let X be a subset of a vector space and D a nonempty subset of X . Then (X, D) is called a convex space if the convex hull $\text{co}(A)$ of any $A \in \langle D \rangle$ is contained in X and X has a topology that induces the Euclidean topology on such convex hulls. A subset C of (X, D) is said to be D -convex if $\text{co}(A) \subseteq C$ for any $A \in \langle D \rangle$ with $A \subseteq C$. If $X = D$, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [9]. The concept of convexity is further generalized under an extra condition by Park and Kim [12]. Later, Lin and Park [10] give the following definition by removing the extra condition.

Definition 2.2. A generalized convex space or a G -convex space $(X, D; \Gamma)$ consists of a topological space X , a nonempty subset D of X and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with $|A| = n + 1$, there exists a continuous function $\varphi_A : \Delta_n \rightarrow \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\varphi_A(\Delta_J) \subseteq \Gamma(J)$, where Δ_J denotes the face of Δ_n corresponding to $J \in \langle A \rangle$.

A subset K of a G -convex space $(X, D; \Gamma)$ is said to be Γ -convex if for any $A \in \langle K \cap D \rangle$, $\Gamma(A) \subseteq K$.

In what follows we will express $\Gamma(A)$ by Γ_A , and we just say that (X, Γ) is a G -convex space provided that $D = X$.

The c -space introduced by Horvath [7] is an example of G -convex space.

For topological spaces X and Y , $\mathcal{C}(X, Y)$ denote the class of all continuous (single-valued) functions from X to Y .

Given a class \mathcal{L} of multimaps, $\mathcal{L}(X, Y)$ denotes the set of multimaps $T : X \rightarrow 2^Y$ belonging to \mathcal{L} , and \mathcal{L}_c the set of finite composites of multimaps in \mathcal{L} . Park and Kim [12] introduced the class \mathbf{U} to be the one satisfying

- (a) \mathbf{U} contains the class \mathcal{C} of (single-valued) continuous functions;
- (b) each $T \in \mathbf{U}_c$ is upper semicontinuous and compact-valued; and
- (c) for any polytope P , each $T \in \mathbf{U}_c(P, P)$ has a fixed point.

Further, Park defined the following

$$T \in \mathbf{U}_c^k(X, Y) \iff \text{for any compact subset } K \text{ of } X, \text{ there is a} \quad (2.1)$$

$$\Gamma \in \mathbf{U}_c(X, Y) \text{ such that } \Gamma(x) \subseteq T(x) \text{ for each } x \in K.$$

A uniformity for a set X is a nonempty family \mathcal{U} of subsets of $X \times X$ such that

- (a) each member of \mathcal{U} contains the diagonal Δ ;
- (b) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
- (c) if $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some V in \mathcal{U} ;
- (d) if U and V are members of \mathcal{U} , then $U \cap V \in \mathcal{U}$; and
- (e) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

If (X, \mathcal{U}) is a uniform space the topology \mathcal{T} induced by \mathcal{U} is the family of all subsets W of X such that for each x in W there is U in \mathcal{U} such that $U[x] \subseteq W$, where $U[x]$ is defined as $\{y \in X : (x, y) \in U\}$. For details of uniform spaces we refer to [8].

3. The results

The concept of S-KKM property of [3] can be extended to G -convex spaces.

Definition 3.1. Let X be a nonempty set, $(Y, D; \Gamma)$ a G -convex space and Z a topological space. If $S : X \multimap D$, $T : Y \multimap Z$ and $F : X \multimap Z$ are three multimaps satisfying

$$T(\Gamma_{S(A)}) \subseteq F(A) \tag{3.1}$$

for any $A \in \langle X \rangle$, then F is called a S-KKM mapping with respect to T . If the multimap $T : Y \multimap Z$ satisfies that for any S-KKM mapping F with respect to T , the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. The class $S\text{-KKM}(X, Y, Z)$ is defined to be the set $\{T : X \multimap Y : T \text{ has the S-KKM property}\}$.

When $D = Y$ is a nonempty convex subset of a linear space with $\Gamma_B = \text{co}(B)$ for $B \in \langle Y \rangle$, the $S\text{-KKM}(X, Y, Z)$ is just that as in [3]. In the case that $X = D$ and S is the identity mapping 1_D , $S\text{-KKM}(X, Y, Z)$ is abbreviated as $\text{KKM}(Y, Z)$, and a 1_D -KKM mapping with respect to T is called a KKM mapping with respect to T , and 1_D -KKM property is called KKM property. Just as [3, Propositions 2.2 and 2.3], for X a nonempty set, $(Y, D; \Gamma)$ a G -convex space, Z a topological space and any $S \multimap D$, one has $T \in \text{KKM}(Y, Z) \subseteq S\text{-KKM}(X, Y, Z)$. By the corollary to [13, Theorem 2], we have $U_c^k(Y, Z) \subseteq \text{KKM}(Y, Z)$, and so $U_c^k(Y, Z) \subseteq S\text{-KKM}(X, Y, Z)$.

Here we like to give a concrete multimap T having KKM property on a G -convex space. Let $X = [0, 1] \times [0, 1]$ be endowed with the Euclidean metric. For any $A = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \in \langle X \rangle$, define $\Gamma_A = \bigcup_{i=1}^n [0, \mathbf{x}_i]$, where $[0, \mathbf{x}_i]$ denotes the line segment joining $\mathbf{0}$ and \mathbf{x}_i . It is easy to see that (X, Γ) is a c -space, and so it is a G -convex space. Let $T : X \multimap X$ be defined by $T(\mathbf{x}) = [(0, 0), (0, 1)] \cup [(0, 0), (1, 0)]$. If $F : X \multimap X$ is any KKM mapping with respect to T , then for any $A = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \in \langle X \rangle$, since $T(\Gamma_A) \subseteq F(A)$ and $(0, 0) \in T(0, 0)$, we infer that $(0, 0) \in T(\mathbf{x}_i) \subseteq F(\mathbf{x}_i)$ for any $i = 1, \dots, n$, so $(0, 0) \in \bigcap_{i=1}^n F(\mathbf{x}_i)$. This shows that T has the KKM property.

A subset B of a topological space Z is said to be compactly open if for any compact subset K of Z , $K \cap B$ is open in K . We begin with the following coincidence theorem.

THEOREM 3.2. *Let X be any nonempty set, $(Y, D; \Gamma)$ a G -convex space and Z a topological space. Suppose $s : X \rightarrow D$, $W : D \rightarrow 2^Z$, $H : Y \rightarrow 2^Z$ and $T \in s\text{-KKM}(X, Y, Z)$ satisfy the*

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following conditions:

(3.2.1) T is compact;

(3.2.2) for any $y \in D$, $W(y) \subseteq H(y)$ and $W(y)$ is compactly open in Z ;

(3.2.3) for any $z \in T(Y)$, $M \in \langle W^-(z) \rangle$ implies that $\Gamma_M \subseteq H^-(z)$;

(3.2.4) $\overline{T(Y)} \subseteq \bigcup_{x \in X} W(s(x))$.

Then T and H have a coincidence point.

Proof. We prove the theorem by contradiction. Assume that $T(y) \cap H(y) = \emptyset$ for any $y \in Y$. Put $K = \overline{T(Y)}$. By (3.2.1), K is a compact subset of Z . Define $F : X \rightarrow 2^Z$ by

$$F(x) = K \setminus W(s(x)) \quad (3.2)$$

for $x \in X$. Since $W(s(x))$ is compactly open, $F(x)$ is closed for each $x \in X$. The assumption that $T(y) \cap H(y) = \emptyset$ for any $y \in Y$ implies that $T(s(x)) \cap H(s(x)) = \emptyset$ for any $x \in X$, so

$$\begin{aligned} \emptyset \neq T(s(x)) &\subseteq K \setminus H(s(x)) \\ &\subseteq K \setminus W(s(x)) \\ &= F(x). \end{aligned} \quad (3.3)$$

Hence F is a nonempty and compact-valued multimap. Since

$$\begin{aligned} \bigcap_{x \in X} F(x) &= \bigcap_{x \in X} (K \setminus W(s(x))) \\ &= K \setminus \bigcup_{x \in X} W(s(x)) \\ &\subseteq K \setminus K \quad \text{by (3.2.4)} \\ &= \emptyset, \end{aligned} \quad (3.4)$$

F is not a s -KKM mapping with respect to T . Hence there is $A = \{x_1, \dots, x_n\} \in \langle X \rangle$ such that

$$T(\Gamma_{\{s(x_1), \dots, s(x_n)\}}) \not\subseteq \bigcup_{i=1}^n F(x_i). \quad (3.5)$$

Choose $\hat{y} \in \Gamma_{\{s(x_1), \dots, s(x_n)\}}$ and $\hat{z} \in T(\hat{y})$ such that $\hat{z} \notin \bigcup_{i=1}^n F(x_i)$. It follows from

$$\begin{aligned} \hat{z} &\in K \setminus \bigcup_{i=1}^n F(x_i) \\ &= \bigcap_{i=1}^n (K \setminus F(x_i)) \\ &\subseteq \bigcap_{i=1}^n W(s(x_i)) \\ &\subseteq \bigcap_{i=1}^n H(s(x_i)) \end{aligned} \quad (3.6)$$

that $s(x_i) \in W^-(\hat{z}) \subseteq H^-(\hat{z})$ for any $i \in \{1, \dots, n\}$. Therefore by (3.2.3), $\Gamma_{\{s(x_1), \dots, s(x_n)\}} \subseteq H^-(\hat{z})$. In particular, $\hat{y} \in H^-(\hat{z})$, and so $\hat{z} \in H(\hat{y}) \cap T(\hat{y})$, a contradiction. This completes the proof. \square

COROLLARY 3.3. *Let (Y, D) be a convex space and Z a topological space. Suppose $H : Y \rightarrow 2^Z$ and $T \in \text{KKM}(Y, Z)$ satisfy the following conditions:*

- (3.3.1) *T is compact;*
- (3.3.2) *for any $z \in T(Y)$, $H^-(z)$ is D -convex;*
- (3.3.3) *$\overline{T(Y)} \subseteq \bigcup_{y \in D} \text{Int}(H(y))$.*

Then T and H have a coincidence point.

Proof. Putting $X = D$, $s : X \rightarrow D$ be the identity mapping 1_D and $W : D \rightarrow 2^Z$ be defined by $W(y) = \text{Int}(H(y))$ in the above theorem, the result follows immediately. \square

Here we like to mention that Corollary 3.3 is an improvement for Theorem 4 of Chang and Yen [4], where except the conditions (3.3.1) \sim (3.3.3), they require T be closed. For $\mathbf{U}_c^k(Y, Z)$ instead of $\text{KKM}(Y, Z)$, Corollary 3.3 is due to Park [11]. We now give a concrete example showing that Corollary 3.3 extends both of [4, Theorem 4] and [11, Theorem 2] properly. Let $X = [0, 1]$ and V be any convex open subset of 0 in \mathbb{R} . Define $T : X \rightarrow X$ by $T(x) = \{1\}$ for $x \in [0, 1)$; and $[0, 1)$ for $x = 1$, and $H : X \rightarrow X$ by $H(x) = (x + V) \cap X$. Then we have

- (a) T belongs to $\text{KKM}(X, X)$ and is compact;
- (b) $H^-(y)$ is convex for each $y \in X$, and
- (c) each $H(x)$ is open and $\overline{T(X)} \subseteq \bigcup_{x \in X} H(x)$.

Thus, Corollary 3.3 guarantees that $T(\hat{x}) \cap H(\hat{x}) \neq \emptyset$ for some $\hat{x} \in [0, 1]$. But, Theorem 4 of Chang and Yen [4] is not applicable in this case because T is not closed. On the other hand, if $T \in \mathbf{U}_c^k(X, X)$, then there would exist $\Gamma \in \mathbf{U}_c(X, X)$ such that $\Gamma(x) \subseteq T(x)$ for each $x \in [0, 1]$. Since X is a polytope, Γ must have a fixed point which is impossible by noting that T has no fixed point. Consequently, $T \notin \mathbf{U}_c^k(X, X)$, and hence we can not apply Theorem 2 of Park [11] to conclude that T and H have a coincidence point.

COROLLARY 3.4. *Let X be any nonempty set, (Y, D) a convex space and Z a topological space. Suppose $s : X \rightarrow D$, $H : Y \rightarrow 2^Z$ and $T \in s\text{-KKM}(X, Y, Z)$ satisfy the following conditions:*

- (3.4.1) *T is compact;*
- (3.4.2) *for any $z \in T(Y)$, $H^-(z)$ is D -convex;*
- (3.4.3) *$\overline{T(Y)} \subseteq \bigcup_{x \in X} \text{Int}(H(s(x)))$.*

Then T and H have a coincidence point.

Proof. In Theorem 3.2, putting $W : D \rightarrow 2^Z$ be $W(y) = \text{Int}(H(y))$ for each $y \in Y$, the result follows immediately. \square

LEMMA 3.5 (Lassonde [9, Lemma 2]). *Let Y be a nonempty subset of a topological vector space E , $T : Y \rightarrow 2^E$ a compact and closed multimap and $i : Y \rightarrow E$ the inclusion map. Then for each closed subset B of Y , $(T - i)(B)$ is closed in E .*

COROLLARY 3.6. *Let X be any nonempty set and Y, C be two nonempty convex subsets of a locally convex topological vector space E . Suppose $s : X \rightarrow Y$ and $T \in s\text{-KKM}(X, Y, Y + C)$ satisfy the following conditions (3.6.1), (3.6.2) and any one of (3.6.3), (3.6.3)' and (3.6.3)''.*

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(3.6.1) T is compact and closed.

(3.6.2) $\overline{T(Y)} \subseteq s(X) + C$.

(3.6.3) Y is closed and C is compact.

(3.6.3)' Y is compact and C is closed.

(3.6.3)'' $C = \{0\}$.

Then there is $\hat{y} \in Y$ such $(\hat{y} + C) \cap T(\hat{y}) \neq \emptyset$.

Proof. Let V be any convex open neighborhood of $0 \in E$ and $K = \overline{T(Y)}$. Define $H : Y \rightarrow 2^{Y+C}$ to be $H(y) = (y + C + V) \cap K$ for each $y \in Y$. Each $H(y)$ is open in K and $H^{-}(z) = (z - C - V) \cap Y$ is convex for any $z \in K$. Moreover,

$$\begin{aligned} \bigcup_{x \in X} H(s(x)) &= \bigcup_{x \in X} ((s(x) + C + V) \cap K) \\ &= (s(X) + C + V) \cap K \\ &= \overline{T(Y)} \quad \text{by (3.6.2).} \end{aligned} \tag{3.7}$$

Therefore, it follows from Corollary 3.4 that there are $y_V \in Y$ and $z_V \in K$ such that $z_V \in T(y_V) \cap H(y_V)$. Then in view of the definition of H , $z_V - y_V \in C + V$. Up to now, we have proved the assertion.

(*) For each convex open neighborhood V of 0 in E , $(T - i)(Y) \cap (C + V) \neq \emptyset$, where $i : Y \rightarrow E$ is the inclusion map.

Now take into account of conditions (3.6.3), (3.6.3)' and (3.6.3)''. Suppose (3.6.3) holds. Since Y is closed, so is $(T - i)(Y)$ by Lemma 3.5, and then the assertion (*) in conjunction with the compactness of C and the regularity of E implies that $(T - i)(Y) \cap C \neq \emptyset$, that is, there exists a $\hat{y} \in Y$ such that $T(\hat{y}) \cap (\hat{y} + C) \neq \emptyset$. In case that (3.6.3)' holds, since $(T - i)(Y)$ is compact by Lemma 2.1 and since C is closed, the conclusion follows as the previous case. Finally, assume that (3.6.3)'' holds. By (*), for every convex open neighborhood V of 0 , there are y_V and z_V in Y such that $z_V \in T(y_V)$ and $z_V - y_V \in V$. Since $\overline{T(Y)}$ is compact, we may assume that $z_V \rightarrow \hat{y}$ for some $\hat{y} \in \overline{T(Y)}$. Then we also have that $y_V \rightarrow \hat{y}$. The closedness of T implies that $\hat{y} \in T(\hat{y})$. This completes the proof. \square

The above corollary extends Park [11, Theorem 3], which in turn is a generalization to Lassonde [9, Theorem 1.6 and Corollary 1.18].

We now turn to investigate the fixed point problem on uniform spaces. At first we apply Theorem 3.2 to establish a useful lemma.

LEMMA 3.7. Let X be any nonempty set, $(Y, D; \Gamma)$ be a G -convex space whose topology is induced by a uniformity \mathcal{U} . Suppose $s : X \rightarrow D$ and $T \in s\text{-KKM}(X, Y, Y)$ satisfy that

(3.7.1) T is compact; and

(3.7.2) $\overline{T(Y)} \subseteq s(X)$.

If $V \in \mathcal{U}$ is symmetric and satisfies that $V[y]$ is Γ -convex for any $y \in Y$, then there is $y_V \in Y$ such that

$$V[y_V] \cap T(y_V) \neq \emptyset. \tag{3.8}$$

Proof. Define $H : Y \rightarrow 2^Y$ to be $H(y) = V[y]$ for any $y \in Y$. By symmetry of V it is easy to see that $H^-(z) = V[z]$ for any $z \in Y$, and so $H^-(z)$ is Γ -convex. Also, it follows from condition (3.6.2) that for any $z \in \overline{T(Y)}$, there is $x_0 \in s(X)$ such that $z = s(x_0)$. Then in view of $(s(x_0), s(x_0)) \in V$ we see that $z = s(x_0) \in V[s(x_0)] = H(s(x_0))$, and hence $z \in \bigcup_{x \in X} H(s(x))$, that is $\overline{T(Y)} \subseteq \bigcup_{x \in X} H(s(x))$. Finally, noting H is open-valued and putting $W : D \rightarrow 2^Y$ to be $W(y) = H(y)$ for any $y \in D$, we see that all the requirements of Theorem 3.2 are satisfied. Thus there is $y_V \in Y$ such that $H(y_V) \cap T(y_V) \neq \emptyset$, that is $V[y_V] \cap T(y_V) \neq \emptyset$. \square

Definition 3.8 [14]. A G -convex space $(X, D; \Gamma)$ is said to be a locally G -convex uniform space if the topology of X is induced by a uniformity 0u which has a base \mathcal{N} consisting of symmetric entourages such that for any $V \in \mathcal{N}$ and $x \in X$, $V[x]$ is Γ -convex.

Recall that the concepts of *l.c.* space and *l.c.* metric space in Horvath [7]. If $D = X$ and $\Gamma_x = \{x\}$ for any $x \in X$, then it is obvious that both of them are examples of locally G -convex uniform space.

THEOREM 3.9. *Let X be any nonempty set, $(Y, D; \Gamma)$ a locally G -convex space. Suppose $s : X \rightarrow D$ and $T \in s\text{-KKM}(X, Y, Y)$ satisfy that*

(3.9.1) *T is compact and closed;*

(3.9.2) $\overline{T(Y)} \subseteq s(X)$.

Then T has a fixed point.

Proof. By Lemma 3.7, for any $V \in \mathcal{N}$ there is $y_V \in Y$ such that $V[y_V] \cap T(y_V) \neq \emptyset$. Choose $z_V \in V[y_V] \cap T(y_V)$. Then $(y_V, z_V) \in V \cap \text{Gr}(T)$. Since T is compact, we may assume that $\{z_V\}_{V \in \mathcal{N}}$ converges to z_0 . For any $W \in \mathcal{N}$, choose $U \in \mathcal{N}$ such that $U \circ U \subseteq W$. Since $\{z_V\}_{V \in \mathcal{N}}$ converges to z_0 , there is $V_0 \in \mathcal{N}$ such that $V_0 \subseteq U$ and

$$z_V \in U[z_0], \quad \forall V \in \mathcal{N} \text{ with } V \subseteq V_0, \quad (3.9)$$

that is,

$$(z_V, z_0) \in U, \quad \forall V \in \mathcal{N} \text{ with } V \subseteq V_0. \quad (3.10)$$

Thus, for $V \in \mathcal{N}$ with $V \subseteq V_0$, it follows from

$$(y_V, z_V) \in V \subseteq U, \quad (z_V, z_0) \in U \quad (3.11)$$

that $(y_V, z_0) \in U \circ U \subseteq W$. Hence $y_V \in W[z_0]$. This shows that $\{y_V\}_{V \in \mathcal{N}}$ converges to z_0 . Since T is closed, we conclude that $z_0 \in T(z_0)$, completing the proof. \square

For a topological space X and locally G -convex uniform space (Y, Γ) , define

$$\begin{aligned} T \in \mathcal{H}(X, Y) &\iff T : X \longrightarrow Y \text{ is a Kakutani map, that is,} \\ &T \text{ is u.s.c. with nonempty compact } \Gamma\text{-convex values.} \end{aligned} \quad (3.12)$$

$\mathcal{H}_c(X, Y)$ denotes the set of finite composites of multimaps in \mathcal{H} of which ranges are contained in locally G -convex uniform spaces (Y_i, Γ_i) ($i = 0, \dots, n$) for some n .

LEMMA 3.10 (Watson [14]). *Let (X, Γ) be a compact locally G -convex uniform space. Then any u.s.c. $T : X \multimap X$ with closed Γ -convex values has a fixed point.*

By the above lemma, we see that, in the setting of locally G -convex uniform spaces, the class \mathcal{H} is an example of the Park's class \mathbf{U} . Therefore, for any locally G -convex uniform space (X, Γ) , $\mathcal{H}_c(X, X) \subseteq \text{KKM}(X, X)$, and so we have the following theorem.

THEOREM 3.11. *Suppose (X, Γ) is a locally G -convex uniform space. If $T \in \mathcal{H}_c(X, X)$ is compact, then it has a fixed point.*

Proof. Since X is regular by Kelley [8, Corollary 6.17 on page 188] and $T \in \mathcal{H}_c(X, X)$, it is u.s.c. and compact-valued, and so it is closed. Now due to that $\mathcal{H}_c(X, X) \subseteq \text{KKM}(X, X)$, we have $T \in \text{KKM}(X, X)$. Since T is compact and closed, it follows from Theorem 3.9 that T has a fixed point. \square

Since any metric space is regular, we infer that for any l.c. metric space (X, d) satisfying that $\Gamma_x = \{x\}$, if $T \in \mathcal{H}_c(X, X)$ is compact, then T has a fixed point. This generalizes the famous Fan-Glicksberg fixed point theorem [5].

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