COINCIDENCE POINT AND INVARIANT APPROXIMATION FOR MAPPINGS SATISFYING GENERALIZED WEAK CONTRACTIVE CONDITION

ISMAT BEG AND MUJAHID ABBAS

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We prove the existence of coincidence point and common fixed point for mappings satisfying generalized weak contractive condition. As an application, related results on invariant approximation are derived. Our results generalize various known results in the literature.

1. Introduction and preliminaries

Sessa [15] introduced the notion of weakly commuting maps in metric spaces. Jungck [8] coined the term of compatible mappings in order to generalize the concept of weak commutativity. Jungck and Rhoades [9] then defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points. In recent years, several authors used these concepts to obtain coincidence point results of various classes of mappings on a metric space. For a survey of coincidence point theory, its applications, and related results, we refer to [1, 4, 5, 10, 13]. Meinardus [12] introduced the notion of invariant approximation. Brosowski [6] initiated the study of invariant approximation using fixed point theory and subsequently various interesting and valuable results applying fixed point theorems to obtain invariant approximation appeared in the literature of approximation theory (see [3, 7, 16–18]).

The aim of this paper is to present coincidence point result for two mappings which satisfy generalized weak contractive condition. Common fixed point theorem for a pair of weakly compatible maps, which is more general than R-weakly commuting and compatible maps, has also been proved. We also construct modified iterative procedures which converge to the common fixed points of the mappings mentioned afore. As an application, we obtain some results on the existence of common fixed points from the set of best approximations.

The following definitions and results will be needed in the sequel.

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Let M be a subset of a metric space X. The set $P_M(u) = \{x \in M : d(x,u) = \operatorname{dist}(u,M)\}$ is called the set of *best approximations* to u in X out of M, where $\operatorname{dist}(u,M) = \inf\{d(y,u) : y \in M\}$.

Definition 1.1. Let X be a metric space. A mapping $T : X \to X$ is called *weakly contractive* with respect to $f : X \to X$ if for each x, y in X,

$$d(Tx,Ty) \leq d(fx,fy) - \phi(d(fx,fy)), \tag{1.1}$$

where $\phi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing such that ϕ is positive on $(0,\infty)$, $\phi(0) = 0$ and $\lim_{t\to\infty}\phi(t) = \infty$.

Definition 1.2. A point x in X is a *coincidence point* (*common fixed point*) of f and T if $f(x) = T(x)$ ($f(x) = T(x) = x$).

Definition 1.3 (see [8]). Two mappings f and g are *compatible* if and only if

$$\lim_{n\to\infty} d(fg(x_n), gf(x_n)) = 0, \tag{1.2}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \in X$.

We will also need the following lemma from [11].

Lemma 1.4. *Let f, g be two compatible mappings on X. If $f(x) = g(x)$ for some x in X, then $fg(x) = gf(x)$.*

Note that every pair of R-weakly commuting self-maps is compatible and each pair of compatible self-maps is weakly compatible but the converse is not true in general.

Definition 1.5 (*modified Mann iterative scheme*). Let X be a Banach space and let T be a weakly contractive map with respect to f on X. Assume that $T(X) \subseteq f(X)$ and $f(X)$ is a convex subset of X. Define a sequence $\{y_n\}$ in $f(X)$ as

$$y_n = f(x_{n+1}) = (1-\alpha_n)f(x_n) + \alpha_n T(x_n), \quad x_0 \in X,\ n \geq 0, \tag{1.3}$$

where $0 \leq \alpha_n \leq 1$ for each n. The sequence thus obtained is *modified Mann iterative scheme*.

2. Coincidence and common fixed point

Alber and Guerre-Delabriere [2] coined the concept of weakly contractive maps and obtained fixed point results in the setting of Hilbert spaces. Rhoades [14] extended some of their work to Banach spaces. In this section, results regarding coincidence and common fixed point for two mappings, one is weakly contractive with respect to other, are presented.

Theorem 2.1. *Let (X,d) be a metric space and let T be a weakly contractive mapping with respect to f. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then f and T have coincidence point in X.*

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $T(x_0) = f(x_1)$. This can be done, since the range of f contains the range of T. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $T(x_n) = f(x_{n+1})$. Consider

$$\begin{aligned} d(f(x_{n+1}), f(x_{n+2})) &= d(T(x_n), T(x_{n+1})) \\ &\leq d(f(x_n), f(x_{n+1})) - \phi(d(f(x_n), f(x_{n+1}))) \\ &\leq d(f(x_n), f(x_{n+1})), \end{aligned} \tag{2.1}$$

which shows that $\{d(f(x_n), f(x_{n+1}))\}$ is a nonincreasing sequence of positive real numbers and therefore tends to a limit $l \geq 0$. If $l > 0$, then we have

$$d(f(x_{n+1}), f(x_{n+2})) \leq d(f(x_n), f(x_{n+1})) - \phi(l). \tag{2.2}$$

Thus,

$$d(f(x_{n+N}), f(x_{n+N+1})) \leq d(f(x_n), f(x_{n+1})) - N\phi(l), \tag{2.3}$$

which is a contradiction for N large enough. Therefore, $\lim_{n\to\infty} d(f(x_n), f(x_{n+1})) = 0$. Furthermore, for $m > n$

$$\begin{aligned} d(f(x_n), f(x_m)) \leq d(f(x_n), f(x_{n+1})) + d(f(x_{n+1}), f(x_{n+2})) \\ + \cdots + d(f(x_{m-1}), f(x_m)). \end{aligned} \tag{2.4}$$

Now using (2.4) and $\lim_{n\to\infty} d(f(x_n), f(x_{n+1})) = 0$ along with weak contractivity of T with respect to f we obtain $d(f(x_n), f(x_m)) \to 0$ as $m, n \to \infty$. As $f(X)$ is a complete subspace of X, therefore $\{f(x_{n+1})\}$ has a limit q in $f(X)$. Consequently, we obtain p in X such that $f(p) = q$. Thus,

$$\begin{aligned} d(f(x_{n+1}), T(p)) &= d(T(x_n), T(p)) \\ &\leq d(f(x_n), f(p)) - \phi(d(f(x_n), f(p))). \end{aligned} \tag{2.5}$$

Taking limit as $n \to \infty$, we obtain

$$d(q, T(p)) \leq d(q, f(p)) - \phi(q, f(p)). \tag{2.6}$$

Hence, p is a solution of the functional equation $f(x) = T(x)$. □

Remark 2.2. If $f(X) = X$ and $f = id_x$ (the identity map of X), then we conclude from Theorem 2.1 that the sequence $\{x_n\}$ converges to a fixed point of T. Thus, our Theorem 2.1 is a generalization of the corresponding theorem of Rhoades [14, Theorem 1].

Remark 2.3. If we define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t - r(t)$, where $r : [0,\infty) \to [0,\infty)$ is a continuous function such that $r(t) < t$ for each $t > 0$, we obtain the similar contractive condition as given in [13, Theorem 1].

Example 2.4. Let $X = R$ with usual metric and let T and f be given by

$$\begin{aligned} &T(x) = ax, \quad a \neq 0, \\ &f(x) = b + cx, \quad c > 0, b \neq 0, 1, (c-1) \geq a, \end{aligned} \tag{2.7}$$

for all $x \in X$. Define $\phi : [0,\infty) \to [0,\infty)$ as

$$\phi(x) = \frac{1}{c}x. \tag{2.8}$$

As

$$\begin{aligned} d(fx, fy) - \phi(d(fx, fy)) &= (c-1)|x-y| \\ &\geq a|x-y| = d(Tx, Ty), \end{aligned} \tag{2.9}$$

therefore T is a weakly contractive mapping with respect to f. However, T and f are not commuting on R. Also if we take $a > c$, then T is not f-nonexpansive map. Moreover, T and f have coincidence fixed point.

Theorem 2.5. *Let (X, d) be a metric space and let T be a weakly contractive mapping with respect to f. If T and f are weakly compatible and $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then f and T have common fixed point in X.*

Proof. By Theorem 2.1, we obtain a point p in X such that $T(p) = f(p) = q$ (say) which further implies $fT(p) = Tf(p)$. Obviously, $T(q) = f(q)$. Now we show $f(q) = q$. If it is not so, then consider

$$\begin{aligned} d(f(q), q) &= d(T(p), T(q)) \\ &\leq d(f(p), f(q)) - \phi(d(q, f(q))) < d(q, f(q)). \end{aligned} \tag{2.10}$$

This contradiction leads to the result. □

Theorem 2.6. *Let X be a normed space and let T be a weakly contractive mapping with respect to f. If T and f are weakly compatible and $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then modified Mann iterative scheme with $\sum \alpha_n = \infty$ converges to a common fixed point of f and T.*

Proof. From Theorem 2.5, we obtain a common fixed point q of T and f. Consider

$$\begin{aligned} ||y_n - q|| &= ||(1-\alpha_n)f(x_n) + \alpha_n T(x_n) - f(p)|| \\ &= ||(1-\alpha_n)(f(x_n) - f(p)) + \alpha_n(T(x_n) - T(p))|| \\ &\leq (1-\alpha_n)||f(x_n) - f(p)|| + \alpha_n||T(x_n) - T(p)|| \\ &\leq ||f(x_n) - f(p)|| - \alpha_n\phi(||f(x_n) - f(p)||) \leq ||y_{n-1} - q||, \end{aligned} \tag{2.11}$$

which gives $\lim_{n\to\infty} \|y_n - q\| = r \geq 0$. Now if $r > 0$, then for any fixed positive integer N we have

$$\begin{aligned} \sum_{n=N}^{\infty} \alpha_n\phi(r) &\leq \sum_{n=N}^{\infty} \alpha_n\phi(||y_n - q||) \\ &\leq \sum_{n=N}^{\infty} (||y_{n-1} - q|| - ||y_n - q||) < ||y_N - q||, \end{aligned} \tag{2.12}$$

which contradicts the choice of α_n. Therefore, the modified Mann iterative scheme converges to a common fixed point of T and f. □

Theorem 2.7. *Let T be a weakly contractive mapping with respect to f on a normed space X. If T and f are weakly compatible and $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, suppose two sequences of mappings $\{y_n\}$ and $\{z_n\}$ are defined as*

$$\begin{aligned} z_n &= f(x_{n+1}) = (1-\alpha_n) f(x_n) + \alpha_n T(v_n), \\ y_n &= f(v_n) = (1-\beta_n) f(x_n) + \beta_n T(x_n), \quad n = 0,1,2,\dots, \end{aligned} \tag{2.13}$$

where $0 \le \alpha_n, \beta_n \le 1$, $\sum \alpha_n \beta_n = \infty$, and $x_0 \in X$, then the iterative sequence $\{z_n\}$ converges to a common fixed point of f and T.

Proof. Let q be a common fixed point of T and f; the existence of common fixed point of T and f follows from Theorem 2.5. Now

$$\begin{aligned} ||z_n - q|| &= ||(1-\alpha_n) f(x_n) + \alpha_n T(v_n) - q|| \\ &\le (1-\alpha_n)||f(x_n) - q|| + \alpha_n ||T(v_n) - T(p)|| \\ &\le (1-\alpha_n)||f(x_n) - q|| + \alpha_n (||f v_n - q|| - \phi(||f(v_n) - q||)) \\ &= (1-\alpha_n)||f(x_n) - q|| + \alpha_n (\|(1-\beta_n) f(x_n) + \beta_n T(x_n) - q|| \\ &\quad - \phi(||f(v_n) - q||)) \\ &\le (1-\alpha_n)||f(x_n) - q|| + \alpha_n ((1-\beta_n)||f(x_n) - q|| \\ &\quad + \beta_n ||T(x_n) - T(p)||) - \alpha_n \phi(||f(v_n) - q||) \\ &\le (1-\alpha_n)||f(x_n) - q|| + \alpha_n (1-\beta_n)||f(x_n) - q\| \\ &\quad + \beta_n \alpha_n [||f(x_n) - q|| - \phi(||f(x_n) - q||)] - \alpha_n \phi(||f(v_n) - q||) \\ &\le ||f(x_n) - q|| - \beta_n \alpha_n \phi(||f(x_n) - q||) - \alpha_n \phi(||f(v_n) - q||) \\ &\le ||f(x_n) - q||. \end{aligned} \tag{2.14}$$

Thus, $\{\|z_n - q\|\}$ is a nonnegative nonincreasing sequence which converges to the limit $r \ge 0$. Suppose that $r > 0$, then for any fixed integer N we have

$$\begin{aligned} \sum_{n=N}^{\infty} \alpha_n \beta_n \phi(r) &\le \sum_{n=N}^{\infty} \alpha_n \beta_n \phi(||z_n - q||) \\ &\le \sum_{n=N}^{\infty} (||z_n - q|| - ||z_{n+1} - q||) \le ||z_N - q||, \end{aligned} \tag{2.15}$$

which contradicts $\sum \alpha_n \beta_n = \infty$. Hence, the result follows. □

3. Invariant approximation

As an application of Theorem 2.5, we have the following results regarding invariant approximation.

Theorem 3.1. *Let* (X,d) *be a metric space and let* T *be a weakly contractive mapping with respect to a continuous map* f. *Assume that* T *leaves* f*-invariant compact subset* M *of closed subspace* $f(X)$ *as invariant. If* T *and* f *are weakly compatible and* $x_0 \in F(T) \cap F(f)$, *then* $P_M(x_0) \cap F(T) \cap F(f) \neq \phi$.

Proof. Since M is a compact subset of $f(X)$, therefore $P_M(x_0) \neq \phi$. Now we show that $T(P_M(x_0)) \subseteq f(P_M(x_0))$. Assume on contrary that there exists b in $P_M(x_0)$ with $T(b) \notin f(P_M(x_0))$. Consider

$$\begin{aligned} d(f(b),x_0) &= d(x_0,M) \leq d(x_0,T(b)) = d(T(x_0),T(b)) \\ &\leq d(f(x_0),f(b)) - \phi(d(f(x_0),f(b))) < d(f(b),x_0). \end{aligned} \tag{3.1}$$

This contradiction leads to $T(P_M(x_0)) \subseteq f(P_M(x_0))$. Now since $f(P_M(x_0))$ being closed subset of a complete space is complete, therefore T and f have a common fixed point in $P_M(x_0)$. Hence, the result follows. □

Theorem 3.2. *Let* (X,d) *be a metric space and let* T *be a weakly contractive mapping with respect to a continuous map* f. *Assume that* T *leaves* f*-invariant compact subset* M *of closed subspace* $f(X)$ *as invariant. Let* $u \in X$ *and for each* $b \in P_M(u)$, $d(x,T(b)) < d(x,f(b))$ *and* $f(b) \in P_M(u)$. *If* T *and* f *are weakly compatible, then* u *has a best approximation in* M *which is also a common fixed point of* f *and* T.

Proof. Since M is a compact subset of $f(X)$, therefore $P_M(x_0) \neq \phi$. Now we show $T(P_M(x_0)) \subseteq f(P_M(x_0))$. Assume on contrary that there exists b in $P_M(x_0)$ with $T(b) \notin f(P_M(x_0))$. Consider

$$d(f(b),u) = d(u,M) \leq d(u,T(b)) < d(u,f(b)) < d(u,M). \tag{3.2}$$

This contradiction leads to the assumption. Now $f(P_M(x_0))$ being closed subset of a complete space is complete. Hence, u has a best approximation in M which is also common fixed point of f and T. □

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Ismat Beg: Department of Mathematics and Centre for Advanced Studies in Mathematics, Lahore University of Management Sciences, 54792 Lahore, Pakistan.
E-mail address: ibeg@lums.edu.pk

Mujahid Abbas: Department of Mathematics and Centre for Advanced Studies in Mathematics, Lahore University of Management Sciences, 54792 Lahore, Pakistan.
E-mail address: mujahid@lums.edu.pk