STRONG CONVERGENCE THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS

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Strong convergence theorems are obtained from modified Halpern iterative scheme for asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups, respectively. Our results extend and improve the recent ones announced by Nakajo, Takahashi, Kim, Xu, and some others.

1. Introduction and preliminary

Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $T : C \to C$ a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C, \tag{1.1}$$

and T is asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers with $\lim_{n\to\infty} k_n = 1$ and such that

$$\left|\left|T^n x - T^n y\right|\right| \leq k_n \|x - y\| \quad \forall n \geq 1,\ x, y \in C. \tag{1.2}$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. Also, recall that a family $S = \{T(s) \mid 0 \leq s < \infty\}$ of mappings from C into itself is called an asymptotically nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
(iii) there exists a positive valued function $L : [0,\infty) \to [1,\infty)$ such that $\lim_{s\to\infty} L_s = 1$ and $\|T(s)x - T(s)y\| \leq L_s\|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

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We denote by $F(S)$ the set of all common fixed points of S, that is, $F(S) = \bigcap_{0\le s<\infty} F(T(s))$. It is known that $F(S)$ is closed and convex. Construction of fixed point of nonexpansive mapping is an important subject in the theory of nonexpansive mappings and finds applications in a number of applied areas, in particular, in image recovery and signal processing (see, e.g., [14, 15]). However, the sequence $\{T^n x\}_{n=0}^{\infty}$ of iterates of the mapping T at a point $x \in C$ may not converge in the weak topology. Thus averaged iterations prevail. In fact, Mann's iterations do have weak convergence. More precisely, Mann's iteration procedure is a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n, \tag{1.3}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily.

Reich [9] proved that if E is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.3) converges weakly to a fixed point of T. However we note that Mann's iterations have only weak convergence even in a Hilbert space [1].

Recently many authors want to modify the Mann iteration method (1.3) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [8] proposed the following modification of the Mann iteration (1.3) for a single nonexpansive mapping T in a Hilbert space:

$$\begin{gathered} x_0 \in C \quad \text{arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{gathered} \tag{1.4}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H and proved that sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

They also proposed the following iteration process for a nonexpansive semigroup $S = \{T(s) \mid 0 \le s < \infty\}$ in a Hilbert space H:

$$\begin{gathered} x_0 \in C \quad \text{arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s) x_n \, ds, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{gathered} \tag{1.5}$$

They proved that if the sequence $\{\alpha_n\}$ is bounded from one and if $\{t_n\}$ is a positive real divergent sequence, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(S)} x_0$.

Halpern [3] firstly studied iteration scheme as follows:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.6}$$

where $u, x_0 \in C$ are arbitrary (but fixed) and $\{\alpha_n\} \subset (0,1)$. He pointed out that the conditions $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$ are necessary in the sense that if the iteration scheme (1.6) converges to a fixed point of T, then these conditions must be satisfied. Ten years later, Lions [6] investigated the general case in Hilbert space under the conditions

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty, \qquad \lim_{n\to\infty}\frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0 \tag{1.7}$$

on the parameters. However, Lions' conditions on the parameters were more restrictive and did not include the natural candidate $\{\alpha_n = 1/n\}$. Reich [10] gave the iteration scheme (1.6) in the case when E is uniformly smooth and $\alpha_n = n^{-\delta}$ with $0 < \delta < 1$.

Wittmann [13] studied the iteration scheme (1.6) in the case when E is a Hilbert space and $\{\alpha_n\}$ satisfies

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty, \qquad \sum_{n=1}^{\infty}||\alpha_{n+1} - \alpha_n|| < \infty. \tag{1.8}$$

Reich [11] obtained a strong convergence of the iterates (1.6) with two necessary and decreasing conditions on parameters for convergence in the case when E is uniformly smooth with a weakly continuous duality mapping.

Recently, Martinez-Yanes and Xu [7] adapted the iteration (1.6) in Hilbert space as follows:

$$\begin{gathered} x_0 \in C \quad \text{arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : ||y_n - z||^2 \leq ||x_n - z||^2 + \alpha_n(||x_0||^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x_0. \end{gathered} \tag{1.9}$$

More precisely, they prove the following theorem.

Theorem 1.1 (Martinez-Yanes and Xu [7]). *Let H be a real Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\} \subset (0,1)$ is such that* $\lim_{n\to\infty}\alpha_n = 0$. *Then the sequence $\{x_n\}$ defined by (1.9) converges strongly to $P_{F(T)}x_0$.*

The purpose of this paper is to employ Nakajo and Takahashi's [8] idea to modify process (1.6) for asymptotically nonexpansive mappings and asymptotically nonexpansive semigroup to have strong convergence theorem in Hilbert space.

In the sequel, we need the following lemmas for the proof of our main results.

Lemma 1.2. *Let K be a closed convex subset of real Hilbert space H and let P_K be the metric projection from H onto K (i.e., for $x \in H$, P_k is the only point in K such that $\|x - P_k x\| = \inf\{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relations*

$$\langle x - z, y - z \rangle \le 0 \quad \forall y \in K. \tag{1.10}$$

Lemma 1.3 (Lin et al. [5]). *Let T be an asymptotically nonexpansive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the properties* (i) $x_n \rightharpoonup p$ *and* (ii) $Tx_n - x_n \to 0$. *Then $p \in F(T)$.*

Lemma 1.4 (Kim and Xu [4]). *Let C be a nonexpansive bounded closed convex subset of H and let $S = \{T(t) : 0 \le t < \infty\}$ be an asymptotically nonexpansive semigroup on C. Then it holds that*

$$\limsup_{s\to\infty} \limsup_{n\to\infty} \sup_{x\in C} \left\| T(s)\left(\frac{1}{t}\int_0^t T(u)x_n du\right) - \frac{1}{t}\int_0^t T(u)x_n du \right\| = 0. \tag{1.11}$$

Lemma 1.5. *Let C be a nonexpansive bounded closed convex subset of H and let $S = \{T(s) : 0 \le s < \infty\}$ be an asymptotically nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the properties* (i) $x_n \rightharpoonup z$; (ii) $\limsup_{s\to\infty} \limsup_{n\to\infty} \|T(s)x_n - x_n\| = 0$, *then $z \in F(S)$.*

Proof. This lemma is the continuous version of [12, Lemma 2.3]. The proof given in [12] is easily extended to the continuous case. □

2. Main results

In this section we propose a modification of the Halpern iteration method to have strong convergence for asymptotically nonexpansive mappings and asymptotically nonexpansive semigroup in Hilbert space.

Theorem 2.1. *Let C be a bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 1$, and M is an appropriate constant such that $M \ge \|x_0 - v\|^2$, for all $v \in C$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{gathered} x_0 \in C \quad \textit{chosen arbitrarily}, \\ z_n = \beta_n x_n + (1 - \beta_n) T^n x_n, \\ y_n = \alpha_n x_0 + (1 - \alpha_n) T^n z_n, \\ C_n = \left\{ v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + \|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v \rangle + \alpha_n M \right\}, \\ Q_n = \{ v \in C : \langle x_0 - x_n, x_n - v \rangle \ge 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{gathered} \tag{2.1}$$

Then $\{x_n\}$ converges to $P_{F(T)} x_0$, provided $k_n^2(1 - \alpha_n) - 1 \le 0$.

Proof. From [2] we know that T has a fixed point in C. That is, $F(T) \neq \emptyset$. It is obviously that C_n is closed and Q_n is closed and convex for each $n \geq 0$. Next observe that C is convex. For $v_1, v_2 \in C_n$ and $t \in (0,1)$, putting $v = tv_1 + (1-t)v_2$. It is sufficient to show that $v \in C_n$. Indeed, the defining inequality in C_n is equivalent to the inequality

$$2\langle z_n - y_n,\, v\rangle \leq \|z_n\|^2 - \|y_n\|^2 + \alpha_n M. \tag{2.2}$$

Therefore, we have

$$\begin{aligned}
2\langle z_n - y_n, v\rangle &= 2\langle z_n - y_n, tv_1 + (1-t)v_2\rangle \\
&= 2t\langle z_n - y_n, v_1\rangle + 2(1-t)\langle z_n - y_n, v_2\rangle \\
&\leq \|z_n\|^2 - \|y_n\|^2 + \alpha_n M,
\end{aligned} \tag{2.3}$$

which implies that C is convex. Next, we show that $F(T) \subset C_n$ for all n. Indeed, for each $p \in F(T)$,

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n x_0 - p + (1-\alpha_n)(T^n z_n - p)\|^2 \\
&\leq \alpha_n \|x_0 - p\|^2 + (1-\alpha_n)k_n^2 \|z_n - p\|^2 \\
&\leq \|x_n - p\|^2 - \|x_n - p\|^2 + \alpha_n \|x_0 - p\|^2 + (1-\alpha_n)k_n^2 \|z_n - p\|^2 \\
&\leq \|x_n - p\|^2 + \left(\|z_n - p\|^2 - \|x_n - p\|^2\right) + \alpha_n \|x_0 - p\|^2 \\
&\leq \|x_n - p\|^2 + \left(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, p\rangle\right) + \alpha_n M.
\end{aligned} \tag{2.4}$$

Therefore, $p \in C_n$ for each $n \geq 1$, which implies that $F(T) \subset C_n$. Next we show that

$$F(T) \subset Q_n \quad \forall n \geq 0. \tag{2.5}$$

We prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 1.2 we have

$$\langle x_0 - x_{n+1}, x_{n+1} - z\rangle \geq 0 \quad \forall z \in C_n \cap Q_n. \tag{2.6}$$

As $F(T) \subset C_n \cap Q_n$ by the induction assumptions, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$. Hence (2.5) holds for all $n \geq 0$. In order to prove $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, from the definition of Q_n we have $x_n = P_{Q_n} x_0$ which together with the fact that $x_{n+1} \in C_n \cap Q_n \subset Q_n$ implies that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|. \tag{2.7}$$

This shows that the sequence $\{x_n - x_0\}$ is nondecreasing. Since C is bounded. We obtain that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Notice again that $x_n = P_{Q_n} x_0$ and $x_{n+1} \in Q_n$ which give

$\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0$. Therefore, we have

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{2.8}$$

It follows that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2.9}$$

On the other hand, It follows from $x_{n+1} \in C_n$ that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, x_{n+1} \rangle + \alpha_n M. \tag{2.10}$$

It follows from (2.1) and $\lim_{n\to\infty} \beta_n = 1$ that

$$\|z_n - x_n\| = (1 - \beta_n)\|x_n - T^n x_n\| \longrightarrow 0. \tag{2.11}$$

Next, we consider

$$\begin{aligned} &\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, x_{n+1} \rangle \\ &\quad = \|z_n\|^2 + \|x_n\|^2 - 2\langle z_n, x_n \rangle + 2\langle x_n - z_n, x_{n+1} \rangle - 2\|x_n\|^2 + 2\langle z_n, x_n \rangle \\ &\quad = \|z_n - x_n\|^2 + 2\langle x_n - z_n, x_{n+1} \rangle - 2\|x_n\|^2 + 2\langle z_n, x_n \rangle \\ &\quad = \|z_n - x_n\|^2 + 2\langle z_n, x_n - x_{n+1} \rangle - 2\|x_n\|^2 + 2\langle x_n, x_{n+1} \rangle. \end{aligned} \tag{2.12}$$

Therefore, it follows from (2.9) and (2.11) that

$$\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, x_{n+1} \rangle \longrightarrow 0. \tag{2.13}$$

Furthermore, from (2.9), (2.13), and $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{2.14}$$

On the other hand, we consider

$$\begin{aligned} \|y_n - T^n x_n\| &\leq \|y_n - T^n z_n\| + \|T^n z_n - T^n x_n\| \\ &\leq \alpha_n \|x_0 - T^n z_n\| + k_n \|z_n - x_n\| \\ &= \alpha_n \|x_0 - T^n z_n\| + k_n (1 - \beta_n)\|x_n - T^n x_n\|. \end{aligned} \tag{2.15}$$

Therefore, it follows that

$$\begin{aligned} \|x_n - T^n x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - T^n x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \alpha_n \|x_0 - T^n z_n\| \\ &\quad + k_n(1-\beta_n)\|x_n - T^n x_n\|. \end{aligned} \tag{2.16}$$

That is,

$$(1 - k_n(1-\beta_n))\|x_n - T^n x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \alpha_n \|x_0 - T^n z_n\|. \tag{2.17}$$

It follows from $\lim_{n\to\infty} \beta_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$, (2.9), and (2.14) that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| \longrightarrow 0. \tag{2.18}$$

Putting $\overline{k} = \sup\{k_n : n \ge 1\} < \infty$, we obtain

$$\begin{aligned} \|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le \overline{k}\|x_n - T^n x_n\| + (1+\overline{k})\|x_n - x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - x_{n+1}\|, \end{aligned} \tag{2.19}$$

which implies that

$$\|Tx_n - x_n\| \longrightarrow 0. \tag{2.20}$$

Assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \widetilde{x}$. By Lemma 1.3 we have $\widetilde{x} \in F(T)$. Next we show that $\widetilde{x} = P_{F(T)}x_0$ and the convergence is strong. Put $\overline{x} = P_{F(T)}x_0$ and consider the sequence $\{x_0 - x_{n_i}\}$. Then we have $x_0 - x_{n_i} \rightharpoonup x_0 - \widetilde{x}$ and by the weak lower semicontinuity of the norm and by the fact that $\|x_0 - x_{n+1}\| \le \|x_0 - \overline{x}\|$ for all $n \ge 0$ which is implied by the fact that $x_{n+1} = P_{C_n \cap Q_n} x_0$, we have

$$\|x_0 - \overline{x}\| \le \|x_0 - \widetilde{x}\| \le \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \le \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \le \|x_0 - \overline{x}\|. \tag{2.21}$$

This gives

$$\|x_0 - \overline{x}\| = \|x_0 - \widetilde{x}\|, \qquad \|x_0 - x_{n_i}\| \longrightarrow \|x_0 - \overline{x}\|. \tag{2.22}$$

It follows that $x_0 - x_{n_i} \to x_0 - \overline{x}$; hence, $x_{n_i} \to \overline{x}$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \to \overline{x}$. The proof is completed. □

Theorem 2.2. *Let C be a nonempty bounded closed convex subset of H and let $S = \{T(s) : 0 \le s < \infty\}$ be an asymptotically nonexpansive semigroup on C. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\beta_n = 1$. $\{t_n\}$ is a positive real divergent sequence and M is an appropriate constant such that $M \ge \|x_0 - v\|$ for all $v \in C$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily}, \\
z_n = \beta_n x_n + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\
y_n = \alpha_n x_0 + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds, \\
C_n = \{v \in C : \|y_n - v\|^2 \le \alpha_n\|x_n - v\|^2 + \|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v\rangle + \alpha_n M\}, \\
Q_n = \{v \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\
x_{n+1} = P_{C_n\cap Q_n}x_0.
\end{gathered} \tag{2.23}$$

Then $\{x_n\}$ converges to $P_{F(S)}x_0$, provided $((1/t_n)\int_0^{t_n} L_s dt)^2(1-\alpha_n) - 1 \le 0$.

Proof. We only conclude the difference. First we show $F(S) \subset C_n$. It follows from C is bounded, we obtain that $F(S) \ne \varnothing$ (see [12]). Taking $p \in F(S)$, we have

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n\|x_0 - p\|^2 + (1-\alpha_n)\left\|\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds - p\right\|^2 \\
&\le \alpha_n\|x_0 - p\|^2 + (1-\alpha_n)\left(\frac{1}{t_n}\int_0^{t_n} \|T(s)z_n - p\| ds\right)^2 \\
&\le \alpha_n\|x_0 - p\|^2 + (1-\alpha_n)\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)^2 \|z_n - p\|^2 \\
&= \|x_n - p\|^2 + \left(\|z_n - p\|^2 - \|x_n - p\|^2\right) + \alpha_n\|x_0 - p\|^2 \\
&\le \|x_n - p\|^2 + \left(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, p\rangle\right) + \alpha_n M.
\end{aligned} \tag{2.24}$$

It follows that $F(S) \subset C_n$ for each $n \ge 0$. From the proof of Theorem 2.1 we have the sequence $\{x_n\}$ is well defined and $F(S) \subset C_n \cap Q_n$ for each $n \ge 0$. Similarly to the argument of Theorem 2.1 and noticing $q = P_{F(S)}x_0$, we have $\|x_{n+1} - x_0\| \le \|q - x_0\|$ for each $n \ge 0$ and $\|x_{n+1} - x_n\| \to 0$. Next, we assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to q. It follows that

$$\begin{aligned}
\|T(s)x_n - x_n\| &\le \left\|T(s)x_n - T(s)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right)\right\| \\
&\quad + \left\|T(s)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right) - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\| \\
&\quad + \left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n\right\| \\
&\le 2\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n\right\| \\
&\quad + \left\|T(s)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right) - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\|,
\end{aligned} \tag{2.25}$$

for each $n \ge 0$. It follows from (2.23) that

$$\left\|y_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\| \le \alpha_n \left\|x_0 - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\|. \tag{2.26}$$

Therefore, we obtain

$$\left\|y_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\| \longrightarrow 0. \tag{2.27}$$

Next, we consider the first term on the right-hand side of (2.25)

$$\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n\right\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \left\|y_n - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\|. \tag{2.28}$$

Since $x_{n+1} \in C_n$, we have

$$\|y_n - x_{n+1}\|^2 \le \alpha_n \|x_n - x_{n+1}\|^2 + \|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, x_{n+1}\rangle + \alpha_n M. \tag{2.29}$$

Similar to the proof of Theorem 2.1, we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0, \tag{2.30}$$

and hence

$$\begin{aligned}
&\left\|y_n - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\| \\
&\quad \le \left\|y_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\| + \left\|\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\| \\
&\quad \le \left\|y_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\| + \frac{1}{t_n}\int_0^{t_n} \|T(s)z_n - T(s)x_n\| ds \\
&\quad \le \left\|y_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\| + \left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)\|z_n - x_n\|^2.
\end{aligned} \tag{2.31}$$

Since $\lim_{n\to\infty}\beta_n = 1$, we have

$$||z_n - x_n|| = (1-\beta_n)\left|\left|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right|\right| \longrightarrow 0. \tag{2.32}$$

It follows from (2.27) and (2.32) that

$$\left|\left|y_n - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right|\right| \longrightarrow 0. \tag{2.33}$$

It follows from (2.30) and (2.33) that

$$\left|\left|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right|\right| \longrightarrow 0. \tag{2.34}$$

On the other hand, by using Lemma 1.4 we obtain

$$\limsup_{s\to\infty}\limsup_{n\to\infty}\left|\left|T(s)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right) - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right|\right| = 0. \tag{2.35}$$

It follows from (2.34) and (2.35) that

$$\limsup_{s\to\infty}\limsup_{n\to\infty}||T(s)x_n - x_n|| = 0. \tag{2.36}$$

Assume that a $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $\{x_{n_i}\} \rightharpoonup q \in C$, then $q \in F(S)$ (by Lemma 1.5). Next we show that $q = \Pi_{F(S)}x_0$ and the convergence is strong. Put $q' = \Pi_{F(S)}x_0$, from $x_{n+1} = \Pi_{C_n\cap Q_n}x_0$ and $q' \in F(S) \subset C_n \cap Q_n$, we have $\|x_{n+1} - x_0\| \le \|q' - x_0\|$. On the other hand, from weakly lower semicontinuity of the norm, we obtain

$$\begin{aligned} ||q' - x_0|| &\le ||x_0 - q|| \le \liminf_{i\to\infty}||x_0 - x_{n_i}|| \\ &\le \limsup_{i\to\infty}||x_0 - x_{n_i}|| \\ &\le ||q - x_0||. \end{aligned} \tag{2.37}$$

It follows from definition of $\Pi_{F(S)}x_0$ that we obtain $q = \Pi_{F(S)}x_0$ and hence

$$||q' - x_0|| = ||q - x_0||. \tag{2.38}$$

It follows that $x_{n_i} \to q'$. Since $\{x_{n_i}\}$ is an arbitrarily weakly convergent sequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to one point of $\Pi_{F(S)}x_0$. This completes the proof. □

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