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Research Article
Diametrically Contractive Multivalued Mappings

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Diametrically contractive mappings on a complete metric space are introduced by V. I. Istratescu. We extend and generalize this idea to multivalued mappings. An easy example shows that our fixed point theorem is more applicable than a former one obtained by H. K. Xu. A convergence theorem of Picard iteratives is also provided for multivalued mappings on hyperconvex spaces, thereby extending a Proinov's result.

1. Introduction

Let (X,d) be a complete metric space. A mapping $T : X \to X$ is a *contraction* if for some $\alpha \in (0,1)$,

$$d(Tx,Ty) \leq \alpha d(x,y) \quad \forall x,y \in X. \tag{1.1}$$

The mapping T is said to be *contractive* if

$$d(Tx,Ty) < d(x,y) \quad \forall x,y \in X,\ x \neq y. \tag{1.2}$$

By the well-known Banach's contraction principle, every contraction has a unique fixed point x_0 and the Picard iteration $\{T^n x\}$ converges to x_0 for every $x \in X$. Examples in [1, 2] show that a contractive mapping may fail to have a fixed point. However, a question of the existence of a fixed point is of interest. In fact, it has been left open the following question.

Question 1.1 [3]. Let M be a weakly compact subset of a Banach space and let $T : M \to M$ be contractive. Does T have a fixed point?

Istratescu [4] introduced a proper subclass of the class of the contractive mappings, whose elements are called the diametrically contractive mappings. Xu [2] proved, in the framework of Banach spaces, the following theorem.

THEOREM 1.2 [2, Theorem 2.3]. *Let M be a weakly compact subset of a Banach space X and let $T : M \to M$ be a diametrically contractive mapping. Then T has a fixed point.*

The following problems raised in [2] had been answered in the negative way in [5].

Problem 1.3. Can we substitute "weakly compact" subset with "closed convex bounded" one in Theorem 1.2?

Problem 1.4. If T is diametrically contractive and x^* is the fixed point of T, do we have $T^n x \to x^*$ for all (or at least for some) $x \in M$?

In this paper, we weaken the condition in the definition of diametrically contractive mappings and obtain a corresponding fixed point theorem for nonself multivalued mappings. Moreover, we also apply a Proinov's fixed point theorem to a selection of a multivalued mapping with externally hyperconvex values and obtain its unique fixed point on a hyperconvex metric space.

2. Diametrically contractive mappings

In [4], Istratescu introduced a new class of mappings strictly lying between contractions and contractive mappings.

Definition 2.1. A mapping T on a complete metric space (X,d) is said to be *diametrically contractive* if $\delta(TA) < \delta(A)$ for all closed subsets A with $0 < \delta(A) < \infty$.

(Here $\delta(A) := \sup\{d(x,y) : x,y \in A\}$ is the diameter of $A \subset X$.)

In the following, we consider a multivalued version of mappings in Theorem 1.2. We also can weaken the condition required in Definition 2.1.

Let $\mathcal{F}(X)$ be the collection of nonempty closed subsets of X and let $\operatorname{Fix} T$ denote the set of fixed points of T. Recall that $TA = \bigcup_{a \in A} Ta$.

THEOREM 2.2. *Let M be a weakly compact subset of a Banach space X and let $T : M \to \mathcal{F}(X)$, $Tx \cap M \neq \emptyset$ for all $x \in M$ and $\delta(TA \cap A) < \delta(A)$ for all closed sets A with $\delta(A) > 0$. Then T has a unique fixed point.*

Proof. The uniqueness of the fixed point is obvious. To prove the existence we consider the family $\mathcal{U} := \{A \subset M : A$ is a nonempty weakly compact subset of M, $TA \cap M \subset A\}$. Clearly, $\mathcal{U} \neq \emptyset$. Partially order $\mathcal{U}$ by saying that $A_1 \preceq A_2$ if $A_1 \supset A_2$ for $A_1, A_2 \in \mathcal{U}$. Every chain $\mathcal{C}$ in $\mathcal{U}$ has a finite intersection property, thus it has a nonempty intersection, that is, $B := \bigcap_{A \in \mathcal{C}} A \neq \emptyset$. Since $TB \cap M \subset TA \cap M \subset A$ for all $A \in \mathcal{C}$, $TB \cap M \subset B$, that is, $B \in \mathcal{U}$, and it is an upper bound of $\mathcal{C}$. Thus $\mathcal{U}$ has a maximal element, say A. Fix $x \in A$. As $A \in \mathcal{U}$ we see that $Tx \cap M \subset TA \cap M \subset A$. That is to say $Tx \cap A \neq \emptyset$ for all $x \in A$.

Put $A_0 = \overline{TA \cap A}^w$. Therefore $A_0 = \overline{TA \cap A}^w \subset \overline{TA \cap M}^w \subset \overline{A}^w = A$ and so $A_0 \subset A$. Moreover, we have $TA_0 \cap M \subset TA \cap M \subset A$. Therefore $TA_0 \cap M \subset TA \cap A \subset \overline{TA \cap A}^w = A_0$. Thus $A_0 \in \mathcal{U}$ and by maximality of A, we have $A = A_0 = \overline{TA \cap A}^w$. By lower semicontinuity of the norm of X we see that $\delta(A) = \delta(\overline{TA \cap A}^w) = \delta(TA \cap A)$. Since T is diametrically contractive we must have $\delta(A) = 0$ and A consists of exactly one point, say ξ. By the condition $\emptyset \neq TA \cap M \subset A$ we see that $\xi \in T\xi$, and we have a fixed point. $\square$

The proof given above is a modification of the proof of Theorem 1.2. The following example shows that Theorem 2.2 is strictly stronger than Theorem 1.2.

Example 2.3. $M = [0,5]$, $T : M \to \mathbb{R}$ defined by $Tx = x + 1$ if $x \leq 3$, and $Tx = 4$ if $x > 3$. Now, if A is a closed subset of M, then there will be a, b in M such that $A \subset [a,b]$ and $\delta(A) = b - a$. If $[a,b] \subset [0,3]$, then $TA \subset [a+1,b+1]$ and $TA \cap A \subset [a+1,b]$. Thus $\delta(TA \cap A) \leq b - a - 1 < \delta(A)$. If $a \leq 3 \leq b$, then $TA \subset [a+1,4]$ and therefore $\delta(TA \cap A) \leq 3 - a < b - a = \delta(A)$. The case when $[a,b] \subset [3,5]$, T clearly satisfies $\delta(TA \cap A) = 0 < \delta(A)$. Thus T has a fixed point by Theorem 2.2. Note that 4 is the unique fixed point of T. We observe that T does not satisfy the condition in Theorem 1.2 because $\delta(T[0,1]) = 1 = \delta([0,1])$.

Example 2.4. Let $Tx = [0, x - \log(x+1)]$ for $x \in [0,100]$. If A is a bounded closed subset of $[0,100]$, then for some $a,b > 0$ we have $A \subset [a,b]$, and $\delta(A) = b - a$. Clearly $TA \subset \bigcup_{x \in A}[0, x - \log(x+1)] \subset [0, b - \log(b+1)]$, and so $TA \cap A \subset [a, b - \log(b+1)]$. Therefore $\delta(TA \cap A) < \delta(A)$. 0 is the unique fixed point of T.

Next we will replace the diameter $\delta(A)$ of a set A by $\alpha(A)$, where α is the Kuratowski measure of noncompactness:

$$\alpha(A) = \inf\{\epsilon > 0 : A \text{ can be covered by finitely many sets with diameters } \leq \epsilon\}. \quad (2.1)$$

Definition 2.5. Let M be a nonempty subset of a metric space (X,d). A mapping $T : M \to 2^X$ is said to be *a k-set contraction* if, for each $A \subset M$ with A bounded, TA is bounded and $\alpha(TA) \leq k\alpha(A)$. If $\alpha(TA) < \alpha(A)$ for all such A, then T is said to be *α-condensing*.

Suppose that M is a bounded subset of a metric space (X,d). Then:

(i) $\text{co}(M) = \bigcap\{B \subset X : B \text{ is a closed ball in } X \text{ such that } M \subset B\}$, and

(ii) M is said to be *subadmissible* [6], if for each $A \in \langle M \rangle$, $\text{co}(A) \subset M$, where $\langle M \rangle$ denotes the class of all nonempty finite subsets of M.

For a nonempty subset M of X and a topological space Y, if two set-valued mappings $T, F : M \to 2^Y$ satisfy the condition $T(\text{co}(A) \cap M) \subset F(A)$, for any $A \in \langle M \rangle$, then F is called a generalized KKM mapping with respect to T.

Let $T : M \to 2^Y$ be a set-valued mapping such that the family $\{\overline{Fx} : x \in M\}$ has the finite intersection property (where $\overline{Fx}$ denotes the closure of Fx) for each generalized KKM mapping $F : M \to 2^Y$ with respect to T, then we say that T has the KKM property. Denote

$$\text{KKM}(M,Y) = \{T : M \longrightarrow 2^Y : T \text{ has the KKM property}\}. \quad (2.2)$$

Theorem 2.6 [7, Theorem 1]. *Let (X,d) be a complete metric space and let M be a nonempty bounded nearly subadmissible subset of X. If $T \in \mathrm{KKM}(M,M)$ is a k-set contraction, $0<k<1$, and closed with $\overline{TM} \subset M$, then T has a fixed point in M.*

The next result shows that we can replace k-set contractions in Theorem 2.6 by α-condensing mappings.

Theorem 2.7. *Let (X,d) be a complete metric space and let M be a nonempty bounded nearly subadmissible subset of X. If $T \in \mathrm{KKM}(M,M)$ is α-condensing, and closed with $\overline{TM} \subset M$, then T has a fixed point in M.*

In the course of the proof, we will apply the technique in the proof of the following lemma.

Lemma 2.8 [8, Lemma 2.2]. *Let F be a selfmapping of an arbitrary set Y and let $f : Y \to \mathbb{R}_+$ be a nonnegative valued function defined on Y. Suppose that the following conditions hold:*

(i) *there exists a function $\varphi \in \Phi_1$ (i.e., $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying: for any $\epsilon > 0$, there exists $\delta > \epsilon$ such that $\epsilon < t < \delta$ implies $\varphi(t) \leq \epsilon$) such that $f(Fy) \leq \varphi(f(y))$ for all $y \in Y$;*

(ii) *$f(y) > 0$ implies $f(Fy) < f(y)$ and $f(y) = 0$ implies $f(Fy) = 0$.*

Then $\lim f(F^n y) = 0$ *for each $y \in Y$.*

Proof of Theorem 2.7. We follow the proof of Theorem 2.6. Let $y \in M$ be any point, and let $M_0 = M$. Define $M_1 = \mathrm{co}(T(M_0) \cup \{y\}) \cap M$, and $M_{n+1} = \mathrm{co}(T(M_n) \cup \{y\}) \cap M$, for each n. Then

$$\alpha(M_{n+1}) \leq \alpha(T(M_n)) < \alpha(M_n) \leq \cdots < \alpha(M), \tag{2.3}$$

for each n (see [7]).

If we can prove that

$$\lim \alpha(M_n) = 0, \tag{2.4}$$

then the rest of the proof will follow the same lines as of Theorem 2.6. To achieve (2.4), we will apply the proof of Lemma 2.8. For each $t \in R_+$, let $A_t = \{M_n : \alpha(M_n) \leq t\}$ and $B_t = \{\alpha(M_{n+1}) : M_n \in A_t\}$. From (2.3), it is seen that $B_t \neq \emptyset$ and bounded if $A_t \neq \emptyset$. Define $\varphi(t) = \sup B_t$ if $A_t \neq \emptyset$. Otherwise, put $\varphi(t) = 0$. We claim that $\varphi \in \Phi_1$. Consider the set A_ϵ for $\epsilon > 0$. If $\alpha(M_n) \leq \epsilon$ for some n, let n_0 be the smallest such n. If $n_0 = 0$, then by (2.3) it is seen that $\varphi(t) \leq \epsilon$ for all $t > \epsilon$. Otherwise, let $\delta = \alpha(M_{n_0-1})$. Thus $\delta > \epsilon$, and if $\epsilon < t < \delta$, then by (2.3) we have $\varphi(t) \leq \alpha(M_{n_0+1}) < \alpha(M_{n_0}) \leq \epsilon$. Therefore $\varphi \in \Phi_1$. We now prove (2.4).

Clearly, we have

$$\alpha(M_{n+1}) \leq \varphi(\alpha(M_n)), \quad \alpha(M_{n+1}) < \alpha(M_n) \quad \text{by (2.3) for each } n. \tag{2.5}$$

It follows from (2.3) that $\{\alpha(M_n)\}$ is strictly decreasing, hence it converges to some $\epsilon \geq 0$. Suppose $\epsilon > 0$. Since $\varphi \in \Phi_1$, we have for some $\delta > \epsilon$, $\varphi(t) \leq \epsilon$ for all $t \in (\epsilon, \delta)$. Choose n_0 so that $\epsilon < \alpha(M_{n_0}) < \delta$. Thus $\varphi(\alpha(M_{n_0})) \leq \epsilon$. But then (2.5) implies $\alpha(M_n) \leq \epsilon$ for all $n > n_0$ which contradicts to (2.3). Hence (2.4) follows. □

3. Picard iteratives for multivalued mappings on hyperconvex metric spaces

A metric space (X,d) is *hyperconvex* if for any family of points $\{x_\alpha\}$ in X and any family of positive numbers $\{r_\alpha\}$ satisfying $d(x_\alpha,x_\beta)\le r_\alpha+r_\beta$, we have $\bigcap_\alpha B(x_\alpha,r_\alpha)\neq\varnothing$ where $B(x,r)$ is the closed ball with center at x and radius r. A subset E of X is said to be *externally hyperconvex* if for any of those families $\{x_\alpha\}$, $\{r_\alpha\}$ with $d(x_\alpha,x_\beta)\le r_\alpha+r_\beta$ and $\operatorname{dist}(x_\alpha,E)\le r_\alpha$, we have $\bigcap_\alpha B(x_\alpha,r_\alpha)\cap E\neq\varnothing$. The class of all externally hyperconvex subsets of X will be denoted by $\mathscr{E}(X)$. Let H be the Hausdorff metric.

Let t be a single-valued selfmapping on a metric space (X,d). A fixed point of t is said to be *contractive* (cf. [9]) if all Picard iteratives of t converge to this fixed point. A selfmapping t on a metric space (X,d) is said to be *asymptotically regular* (cf. [10]) if $\lim d(t^n(x),t^{n+1}(x))=0$ for each x in X. Extend the concept naturally to multivalued mappings with the Hausdorff metric taken into action.

Theorem 3.1 [8, Theorem 4.1]. *Let t be a continuous and asymptotically regular selfmapping on a complete metric space satisfying the following conditions:*

(i) *there exists $\varphi\in\Phi_1$ such that $d(t(x),t(y))\le\varphi(D(x,y))$ for all $x,y\in X$;*

(ii) *$d(t(x),t(y))<D(x,y)$ for all $x,y\in X$ with $x\neq y$.*

Then t has a contractive fixed point. Here $D(x,y)=d(x,y)+r[d(x,t(x))+d(y,t(y))]$, $r\ge 0$.

Replacing D by d, we present a multivalued version of Theorem 3.1 on a special setting, namely, on the class of hyperconvex metric spaces.

Theorem 3.2. *Let (X,d) be a bounded hyperconvex metric space, and let $T:X\to\mathscr{E}(X)$ be asymptotically regular satisfying the following conditions:*

(i) *there exists $\varphi\in\Phi_1$ such that $\varphi(x)\le x$, $\varphi(x+y)\le\varphi(x)+\varphi(y)$, $\varphi(x)=0$ if and only if $x=0$, and $H(Tx,Ty)\le\varphi(d(x,y))$ for all x,y in X;*

(ii) *$H(Tx,Ty)<d(x,y)$ for all $x,y\in X$ with $x\neq y$.*

Then, if $\delta(T^n x)\to 0$ for each $x\in X$, T has a contractive fixed point. That is, there exists a unique point ξ in X such that, for each $x\in X$, $T^n x\to\{\xi\}=\operatorname{Fix}T$.

Proof. The uniqueness of the fixed point is evident. We are going to find a selection $t: X\to X$ so that $t(x)\in Tx$ for all $x\in X$ and t satisfies the conditions in Theorem 3.1. Thus, there is a point ξ in $\operatorname{Fix}t$ satisfying $t^n(x)\to\xi$ for all $x\in X$. To find a selection t, we apply Zorn's lemma to the family $\mathscr{F}=\{(A,t):\varnothing\neq A\subset X,\ t:A\to A$ asymptotically regular, $t(a)\in Ta$ for all $a\in A$, and t satisfies (i) and (ii) in Theorem 3.1$\}$. Partially order $\mathscr{F}$ by $(A_1,t_1)\preceq(A_2,t_2)$ if $A_1\subset A_2$ and $t_2|_{A_1}=t_1$.

Suppose $A=\varnothing$ or $(A,t)\in\mathscr{F}$ and $x_0\in X\setminus A$. We will define a countable set $\{x_0,x_1,x_2,\dots\}$, possibly finite, and an extension function t^* of t over $A\cup\{x_0,x_1,x_2,\dots\}$ so that $(A\cup\{x_0,x_1,x_2,\dots\},t^*)\in\mathscr{F}$. Let $x_0,x_1,\dots,x_n\in X\setminus A$ have been defined for some $n\ge 0$ so that, for $1\le k\le n$, $x_k\in Tx_{k-1}$, and when $n\ge 2$, $d(x_{i+1},x_{j+1})\le\varphi(d(x_i,x_j))$ and $d(x_{i+1},x_{j+1})<d(x_i,x_j)$ for $i<j$ in $\{1,\dots,n-1\}$. Moreover, $d(t(x),x_{i+1})\le\varphi(d(x,x_i))$ and $d(t(x),x_{i+1})<d(x,x_i)$ for $i\in\{1,\dots,n-1\}$ and all $x\in A$.

Put $r_k = \varphi(d(x_{k-1},x_n))$, and $r_{t(x)} = \varphi(d(x,x_n))$ for each $1 \le k \le n$, and for all $x \in A$. Thus, for $1 \le k \le n$, $x \in A$, and for $i < j$ in $\{1,\dots,n-1\}$,

(i) $\operatorname{dist}(x_k, Tx_n) \le H(Tx_{k-1}, Tx_n) \le \varphi(d(x_{k-1},x_n)) = r_k$,

(ii) $\operatorname{dist}(t(x), Tx_n) \le H(Tx, Tx_n) \le r_{t(x)}$,

(iii) $d(t(x),x_k) \le \varphi(d(x,x_{k-1})) \le \varphi(d(x,x_n)) + \varphi(d(x_{k-1},x_n)) = r_{t(x)} + r_k$, and

(iv) $d(x_i,x_j) \le \varphi(d(x_{i-1},x_{j-1})) \le \varphi(d(x_{i-1},x_n)) + \varphi(d(x_{j-1},x_n)) = r_i + r_j$.

Finally, for $x, y \in A$, $d(t(x),t(y)) \le \varphi(d(x,y)) \le \varphi(d(x,x_n)) + \varphi(d(y,x_n)) = r_{t(x)} + r_{t(y)}$. Therefore there exists a point $x_{n+1} \in \bigcap_{x\in A} B(t(x), r_{t(x)}) \bigcap_{k=1}^{n} B(x_k, r_k) \cap Tx_n$. The point x_{n+1} has the following property: for $k \le n$, $d(x_k, x_{n+1}) \le r_k = \varphi(d(x_{k-1},x_n))$, and for each $x \in A$, $d(t(x),x_{n+1}) \le r_{t(x)} = \varphi(d(x,x_n))$. Clearly, $d(x_k,x_{n+1}) < d(x_{k-1},x_n)$ and $d(t(x),x_{n+1}) < d(x,x_n)$.

If $x_{n+1} \in A$, the process terminates. Otherwise, we obtain a subset $\{x_0,x_1,\dots,x_n,\dots\}$ of $X \setminus A$ satisfying the conditions (i) and (ii) in Theorem 3.1 where we extend t to t^* by defining $t^*(x_n) = x_{n+1}$ for $n \ge 0$. Thus $(A \cup \{x_0,x_1,\dots\}, t^*) \in \mathcal{F}$.

In summary, the above argument shows that, if $A = \emptyset$, then $(\{x_0,x_1,\dots\},t^*) \in \mathcal{F}$, that is, $\mathcal{F} \neq \emptyset$. On the other hand if (A,t) is a maximal element in $\mathcal{F}$ (by Zorn's lemma), we must have $A = X$, that is, (X,t) belongs to $\mathcal{F}$ for some t. Apply Theorem 3.1, to conclude that there exists a fixed point ξ of t such that $t^n(x) \to \xi$ for each $x \in X$. Consequently, $T^n x \to \{\xi\}$ for each $x \in X$ and $\operatorname{Fix} T = \{\xi\}$. □

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