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Research Article

Strong Convergence Theorems of the CQ Method for Nonexpansive Semigroups

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Motivated by T. Suzuki, we show strong convergence theorems of the CQ method for nonexpansive semigroups in Hilbert spaces by hybrid method in the mathematical programming. The results presented extend and improve the corresponding results of Kazuhide Nakajo and Wataru Takahashi (2003).

1. Introduction and preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. We use $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ will symbolize strong convergence. we denote by $\mathbb{N}$ and $\mathbb{R}_+$ the sets of nonnegative integers and nonnegative real numbers, respectively. let C be a closed convex subset of a Hilbert space H, and Let $T: C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$). We use $\text{Fix}(T)$ to denote the set of fixed points of T; that is, $\text{Fix}(T) = \{x \in C : x = Tx\}$. We know that $\text{Fix}(T)$ is nonempty if C is bounded, for more details see [1].

In [2], Shioji and Takahashi introduce in a Hilbert space the implicit iteration

$$x_n = \alpha_n u + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad n \in \mathbb{N}, \tag{1.1}$$

Where $\{\alpha_n\}$ is a sequence in (0,1), $\{t_n\}$ is a sequence of positive real numbers divergent to ∞, for each $t \ge 0$ and $u \in C$. In 2003, Suzuki [3] is the first to introduce again in a Hilbert space the following implicit iteration process:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)(x_n), \quad n \ge 1, \tag{1.2}$$

for the nonexpansive semigroup case. In 2005, Xu [4] established a Banach space version of the sequence (1.2) of Suzuki [3], he proved that if E is a uniformly convex Banach space with a weakly continuous duality map (e.g., l^p for $1 < p < \infty$), if C is a closed convex subset of E, and if $\{T(t) : t \in \mathbb{R}_+\}$ is a nonexpansive semigroup on a closed convex subset C such that $\mathrm{Fix}(T) \neq \emptyset$, then under certain appropriate assumptions made and the sequences α_n and t_n of the parameters, he showed that the sequence x_n implicitly defined by (1.2) for all $n \geq 1$ converges strongly to a member of $F = \bigcap_{t\geq 0} \mathrm{Fix}(T(t))$.

Recently, Chen and He [5] extend and improve the corresponding results of Suzuki [3], if E is a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*, suppose C is a nonempty closed convex subset of E. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a nonexpansive semigroup on C such that $F(T) \neq \emptyset$, and $f : C \to C$ is a fixed contraction on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. Define a sequence $\{x_n\}$ in C by

$$x_n = \alpha_n f(x_n) + (1-\alpha_n) T(t_n)(x_n), \quad n \geq 1. \tag{1.3}$$

Then $\{x_n\}$ converges strongly to q, as $n \to \infty$. q is the element of F, such that q is the unique solution in F to the following variational inequality:

$$\langle (f-I)q, j(x-q)\rangle \leq 0 \quad \forall x \in F(T). \tag{1.4}$$

Some other results can be seen in [6–8].

Nakajo and Takahashi [9] introduced an iteration procedure for nonexpansive self-mappings T on C as follows:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n &= \{z \in C;\ \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C;\ \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0) \end{aligned} \tag{1.5}$$

for each $n \in \mathbb{N} \cup \{0\}$, where $\alpha_n \in [0,a]$ for some $a \in [0,1)$, and $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)} x_0$

Let $\{T(t) : t \in \mathbb{R}_+\}$ be a nonexpansive semigroup on a closed convex subset C of a Hilbert space H, that is,

(1) for each $t \in \mathbb{R}_+$, $T(t)$ is a nonexpansive mapping on C;
(2) $T(0)x = x$ for all $x \in C$;
(3) $T(s+t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}_+$;
(4) for each $x \in X$, the mapping $T(\cdot)x$ from $\mathbb{R}_+$ into C is continuous. We put $F = \bigcap_{t\geq 0} \mathrm{Fix}(T(t))$. We know that F is nonempty if C is bounded, see [10].

Let C be a nonempty closed convex subset of H and *let*$\{T(t) : t \in \mathbb{R}_+\}$ be a nonexpansive semigroup on a closed convex subset C of a Hilbert space H such that $F \neq \emptyset$, Nakajo and Takahashi [9] also introduced an iteration procedure for nonexpansive semigroup $\{T(t) : t \in \mathbb{R}_+\}$ on C as follows:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n &= \{z \in C;\ \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z \in C;\ \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0) \end{aligned} \tag{1.6}$$

for each $n \in \mathbb{N} \cup \{0\}$, where $\alpha_n \in [0,a]$ for some $a \in [0,1)$ and $\{t_n\}$ is a positive real number divergent sequence, and the sequence $\{x_n\}$ converges strongly to $P_F x_0$.

In 2006, Martinez-Yanes and Xu [11] employ Nakajo-Takahashi [9] idea and prove some strong convergence theorems for nonexpansive mappings and maximal monotone operators.

In this paper, we consider an iteration procedure for nonexpansive semigroups $\{T(t) : t \in \mathbb{R}_+\}$ on C as follows:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n x_n + (1-\alpha_n)T(t_n)x_n, \\ C_n &= \{z \in C; \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z \in C; \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0) \end{aligned} \tag{1.7}$$

for each $n \in \mathbb{N} \cup \{0\}$, where $\alpha_n \in [0,a]$ for some $a \in [0,1)$ and $t_n \ge 0$ $\lim_{n\to\infty} t_n = 0$. then the sequence $\{x_n\}$ converges strongly to $P_F x_0$.

In the sequel, we will need the following definitions and results.

Definition 1.1. A Banach space E is said to satisfy Opial's condition [12] if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, as $n \to \infty$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \text{ with } x \ne y. \tag{1.8}$$

It is well known that Hilbert space and $l^p(1 < l < \infty)$ space satisfy Opial's condition [13].

Lemma 1.2 [14]. *Let C be a nonempty closed convex subset of a Hilbert space H. Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if $\langle x - y, y - z\rangle \ge 0$, is satisfied for all $z \in C$.*

Lemma 1.3 [14, 15]. *Every Hilbert space H has Radon-Riesz property or Kadets-Klee property, that is, for a sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then there holds $x_n \to x$.*

2. Main results

Lemma 2.1. *Let C be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a nonexpansive semigroup on C such that $F \neq \emptyset$, and the sequence $\{x_n\}$ generated by (1.7), where $\alpha_n \in [0,a]$ for some $a \in [0,1)$, Then $\{x_n\}$ is well defined and $F \subset C_n \cap Q_n$ for every $n \in \mathbb{N} \cup \{0\}$.*

Proof. It is obvious that C_n is closed and Q_n is closed and convex for every $n \in \mathbb{N} \cup \{0\}$. It follows from that C_n is convex for every $n \in \mathbb{N} \cup \{0\}$ because $\|y_n - z\| \leq \|x_n - z\|$ is equivalent to

$$||y_n - x_n||^2 + 2\langle y_n - x_n, x_n - z\rangle \leq 0. \tag{2.1}$$

So, $C_n \cap Q_n$ is closed and convex for every $n \in \mathbb{N} \cup \{0\}$. Let $u \in F$. Then from

$$\begin{aligned} ||y_n - u|| &= ||\alpha_n x_n + (1-\alpha_n)T(t_n)x_n - u|| \\ &\leq \alpha_n ||x_n - u|| + (1-\alpha_n)||T(t_n)x_n - u|| \\ &\leq ||x_n - u||. \end{aligned} \tag{2.2}$$

we have $u \in C_n$ for each $n \in \mathbb{N} \cup \{0\}$. So, we have $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Next, we show by mathematical induction that $\{x_n\}$ is well defined and $F \subset C_n \cap Q_n$ for every $n \in \mathbb{N} \cup \{0\}$. For $n = 0$, we have $x_0 = x \in C$ and $Q_0 = C$, and hence $F \subset C_0 \cap Q_0$. Suppose that x_k is given and $F \subset C_k \cap Q_k$ for some $k \in \mathbb{N} \cup \{0\}$.There exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x_0)$. From $x_{k+1} = P_{C_k \cap Q_k}(x_0)$, it holds that

$$\langle x_{k+1} - z, x_0 - x_{k+1}\rangle \geq 0 \tag{2.3}$$

for each $z \in C_k \cap Q_k$. Since $F \subset C_k \cap Q_k$, we get $F \subset Q_{k+1}$, therefore we have $F \subset C_{k+1} \cap Q_{k+1}$.

The proof is completed. □

Lemma 2.2. *Let C be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a nonexpansive semigroup on C such that $F \neq \emptyset$, and the sequence $\{x_n\}$ generated by (1.7), where $\alpha_n \in [0,a]$ for some $a \in [0,1)$, Then* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof. At first, we show that F is a closed convex subset of C. Since $T(t) : C \to C$, $t > 0$ is nonexpansive, we claim that F is closed. In fact, if $p_n \subset F = \bigcap_{t\geq 0} \mathrm{Fix}(T(t))$, $n \geq 1$, such that $\lim_{n\to\infty} p_n = p$, then we have

$$T(t)p = \lim_{n\to\infty} T(t)p_n = \lim_{n\to\infty} p_n = p \quad \forall t \in \mathbb{R}_+. \tag{2.4}$$

Thus $p \in F$.

Next, we show that F is convex, we will use the following identity in Hilbert space:

$$||tx + (1-t)y||^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \tag{2.5}$$

which holds for all $x, y \in H$ and for all $t \in [0,1]$ indeed,

$$\begin{aligned} \|tx+(1-t)y\|^2 &= t^2\|x\|^2+(1-t)^2\|y\|^2+2t(1-t)\langle x,y\rangle \\ &= t\|x\|^2+(1-t)\|y\|^2+2t(1-t)\langle x,y\rangle \\ &\quad - t(1-t)\|x\|^2 - t(1-t)\|y\|^2 \\ &= t\|x\|^2+(1-t)\|y\|^2 - t(1-t)\big(\|x\|^2+\|y\|^2-2\langle x,y\rangle\big) \\ &= t\|x\|^2+(1-t)\|y\|^2 - t(1-t)\|x-y\|^2. \end{aligned} \tag{2.6}$$

Let $p_1, p_2 \in F$ and for all $t \in [0,1]$, $p = tp_1+(1-t)p_2$, then

$$p-p_1 = (1-t)(p_2-p_1), \qquad p-p_2 = (1-t)(p_1-p_2). \tag{2.7}$$

From (2.5) and (2.7), we have

$$\begin{aligned} \|p-T(t)p\|^2 &= \|t(p_1-T(t)p)+(1-t)(p_2-T(t)p)\|^2 \\ &= t\|p_1-T(t)p\|^2+(1-t)\|p_2-T(t)p\|^2 - t(1-t)\|p_1-p_2\|^2 \\ &\le t\|p_1-p\|^2+(1-t)\|p_2-p\|^2 - t(1-t)\|p_1-p_2\|^2 \\ &= t(1-t)^2\|p_1-p_2\|^2+t^2(1-t)\|p_1-p_2\|^2 - t(1-t)\|p_1-p_2\|^2 \\ &= t(1-t)(1-t+t-1)\|p_1-p_2\|^2 = 0. \end{aligned} \tag{2.8}$$

Thus $p = T(t)p$, for all $t > 0$, that is, $p \in F$.

Secondly, we show that $\{x_n\}$ is bounded. Since F is a nonempty closed convex subset of C, there exists a unique element $z_0 \in F$ such that $z_0 = P_F(x_0)$. From $x_{n+1} = P_{C_n\cap Q_n}(x_0)$, we have

$$\|x_{n+1}-x_0\| \le \|z-x_0\| \quad \forall z \in C_n \cap Q_n. \tag{2.9}$$

It follows from Lemma 2.1 that $F \subset C_n \cap Q_n$ for every $n \in \mathbb{N} \cup \{0\}$, together with $z_0 \in F(T)$, we have

$$\|x_{n+1}-x_0\| \le \|z_0-x_0\| \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{2.10}$$

This implies that $\{x_n\}$ is bounded, so $\{T(t)x_n\}$ is also bounded, and moreover so is $\{y_n\}$ since $\|y_n\| \le \alpha_n\|x_n\|+(1-\alpha_n)\|T(t)x_n\|$.

Thirdly, we show that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$. Since $Q_n = \{z \in C;\ \langle x_n - z, x_0 - x_n\rangle \ge 0\}$, $x_n = P_{Q_n}(x_0)$. As $x_{n+1} \in C_n \cap Q_n \subset Q_n$, we obtain

$$\|x_{n+1}-x_0\| \ge \|x_n-x_0\|, \quad \forall z \in C_n \cap Q_n. \tag{2.11}$$

Therefore the sequence $\{\|x_n - x_0\|\}$ is bounded and nondecreasing. So

$$\lim_{n\to\infty} ||x_n - x_0|| \text{ exists.} \tag{2.12}$$

On the other hand, from $x_{n+1} \in Q_n$, we get $\langle x_n - x_{n+1}, x_0 - x_n\rangle \geq 0$, and hence

$$\begin{aligned} ||x_n - x_{n+1}||^2 &= ||(x_n - x_0) - (x_{n+1} - x_0)||^2 \\ &= ||x_n - x_0||^2 - 2\langle x_n - x_0, x_{n+1} - x_0\rangle + ||x_{n+1} - x_0||^2 \\ &= ||x_n - x_0||^2 + ||x_{n+1} - x_0||^2 \\ &\quad - 2\langle x_n - x_0, x_{n+1} - x_n + x_n - x_0\rangle \\ &= ||x_{n+1} - x_0||^2 - ||x_n - x_0||^2 - 2\langle x_n - x_{n+1}, x_0 - x_n\rangle \\ &\leq ||x_{n+1} - x_0||^2 - ||x_n - x_0||^2 \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{2.13}$$

So

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{2.14}$$

This proof is completed. □

Theorem 2.3. *Let C be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a nonexpansive semigroup on C such that $F \neq \emptyset$, and the sequence $\{x_n\}$ generated by (1.7), where $\alpha_n \in [0,a]$ for some $a \in [0,1)$, and $t_n \geq 0$ $\lim_{n\to\infty} t_n = 0$. then the sequence $\{x_n\}$ converges strongly to $P_F x_0$.*

Proof. It follows from $x_{n+1} \in C_n$ that

$$\begin{aligned} ||T(t_n)x_n - x_n|| &= \frac{1}{1-\alpha_n}||y_n - x_n|| \\ &\leq \frac{1}{1-\alpha_n}(||y_n - x_{n+1}|| + ||x_{n+1} - x_n||) \\ &\leq \frac{2}{1-\alpha_n}||x_{n+1} - x_n|| \end{aligned} \tag{2.15}$$

for every $n \in \mathbb{N} \cup \{0\}$. By Lemma 2.2, we get $\|T(t_n)x_n - x_n\| \to 0$.

We claim that $\{x_n\}$ is relatively sequentially compact. Indeed, there exists a weakly convergence subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ by reflexivity of H and boundedness of the sequence $\{x_n\}$, now we suppose $x_{n_j} \rightharpoonup x \in C (j \to \infty)$. Now we show that $x \in F$. Put $x_j = x_{n_j}$, $\beta_j = \alpha_{n_j}$, and $s_j = t_{n_j}$ for $j \in \mathbb{N}$, let $s_j \geq 0$ be such that

$$s_j \longrightarrow 0, \quad \frac{||T(s_j)x_j - x_j||}{s_j} \longrightarrow 0, \quad j \longrightarrow \infty. \tag{2.16}$$

Fix $t > 0$, from

$$\begin{aligned}\|x_j - T(t)x\| &\le \sum_{k=0}^{[t/s_j]-1} \|T((k+1)s_j)x_j - T(ks_j)x_j\| \\ &\quad + \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)x_j - T\left(\left[\frac{t}{s_j}\right]s_j\right)x\right\| + \|T([t/s_j]s_j)x - T(t)x\| \\ &\le \left[\frac{t}{s_j}\right]\|T(s_j)x_j - x_j\| + \|x_j - x\| + \left\|T\left(t - \left[\frac{t}{s_j}\right]s_j\right)x - x\right\| \\ &\le t\frac{\|T(s_j)x_j - x_j\|}{s_j} + \|x_j - x\| + \max\{\|T(s)x - x\| : 0 \le s \le s_j\}.\end{aligned} \tag{2.17}$$

For all $j \in \mathbb{N} \cup \{0\}$, as every Hilbert space satisfies Opial's condition, then we have

$$\limsup_{j\to\infty} \|x_j - T(t)x\| \le \limsup_{j\to\infty} \|x_j - x\|. \tag{2.18}$$

This implies that $T(t)x = x$. Therefore,

$$x \in F. \tag{2.19}$$

If $z_0 = P_F(x_0)$, it follows from (2.10), (2.19), and the lower semicontinuity of the norm that

$$\|x_0 - z_0\| \le \|x_0 - x\| \le \liminf_{j\to\infty} \|x_0 - x_{n_j}\| \le \limsup_{j\to\infty} \|x_0 - x_{n_j}\| \le \|x_0 - z_0\|. \tag{2.20}$$

Thus, we obtain

$$\lim_{j\to\infty} \|x_{n_j} - x_0\| = \|x_0 - x\| = \|x_0 - z_0\|. \tag{2.21}$$

This implies that

$$x_{n_j} \longrightarrow x = z_0. \tag{2.22}$$

This shows that $\{x_n\}$ is relatively sequentially compact. Therefore, we have $x_n \to z_0$.

The proof is completed. □

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