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Review Article

Remarks of Equivalence among Picard, Mann, and Ishikawa Iterations in Normed Spaces

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We show that the convergence of Picard iteration is equivalent to the convergence of Mann iteration schemes for various Zamfirescu operators. Our result extends of Soltuz (2005).

1. Introduction

Let E be a real normed space, D a nonempty convex subset of E, and T a self-map of D, let $p_0, u_0, x_0 \in D$. The Picard iteration is defined by

$$p_{n+1} = T p_n, \quad n \geq 0. \tag{1.1}$$

The Mann iteration is defined by

$$u_{n+1} = (1-a_n)u_n + a_n T u_n, \quad n \geq 0. \tag{1.2}$$

The Ishikawa iteration is defined by

$$\begin{aligned} y_n &= (1-b_n)x_n + b_n T x_n, \quad n \geq 0, \\ x_{n+1} &= (1-a_n)x_n + a_n T y_n, \quad n \geq 0, \end{aligned} \tag{1.3}$$

where $\{a_n\}, \{b_n\}$ are sequences of positive numbers in $[0,1]$. Obviously, for $a_n = 1$, the Mann iteration (1.2) reduces to the Picard iteration, and for $b_n = 0$, the Ishikawa iteration (1.3) reduces to the Mann iteration (1.2).

Definition 1.1 [1, Definition 1]. Let $T : D \to D$ be a map for which there exist real numbers a, b, c satisfying $0 < a < 1$, $0 < b < 1/2$, $0 < c < 1/2$. Then T is called a Zamfirescu operator if, for each pair x, y in D, T satisfies at least one of the following conditions given in (1)–(3):

(1) $\|Tx - Ty\| \le a\|x - y\|$;

(2) $\|Tx - Ty\| \le b(\|x - Tx\| + \|y - Ty\|)$;

(3) $\|Tx - Ty\| \le c(\|x - Ty\| + \|y - Tx\|)$.

It is easy to show that every Zamfirescu operator T satisfies the inequality

$$\|Tx - Ty\| \le \delta\|x - y\| + 2\delta\|x - Tx\| \tag{1.4}$$

for all $x, y \in D$, where $\delta = \max\{a, b/(1-b), c/(1-c)\}$ with $0 < \delta < 1$ (See Şoltuz [1]). Recently, Şoltuz [1] had studied that the equivalence of convergence for Picard, Mann, and Ishikawa iterations, and proved the following results.

Theorem 1.2 [1, Theorem 1]. *Let X be a normed space, D a nonempty, convex, closed subset of X, and $T : D \to D$ an operator satisfying condition Z (Zamfirescu operator). If $u_0 = x_0 \in D$, let $\{u_n\}_{n=0}^{\infty}$ be defined by (1.2) for $u_0 \in D$, and let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.3) for $x_0 \in D$ with $\{a_n\}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty} a_n = \infty$. Then the following are equivalent:*

(i) *the Mann iteration (1.2) converges to the fixed point of T;*

(ii) *the Ishikawa iteration (1.3) converges to the fixed point of T.*

Theorem 1.3 [1, Theorem 2]. *Let X be a normed space, D a nonempty, convex, closed subset of X, and $T : D \to D$ an operator satisfying condition Z (Zamfirescu operator). If $u_0 = p_0 \in D$, let $\{p_n\}_{n=0}^{\infty}$ be defined by (1.1) for $p_0 \in D$, and let $\{u_n\}_{n=0}^{\infty}$ be defined by (1.2) for $u_0 \in D$ with $\{a_n\}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty} a_n = \infty$ and $a_n \to 0$ as $n \to \infty$. Then*

(i) *if the Mann iteration (1.2) converges to x^* and $\lim_{n\to\infty}(\|u_{n+1} - u_n\|/a_n) = 0$, then the Picard iteration (1.1) converges to x^*,*

(ii) *if the Picard iteration (1.1) converges to x^* and $\lim_{n\to\infty}(\|p_{n+1} - p_n\|/a_n) = 0$, then the Mann iteration (1.2) converges to x^*.*

However, in the above-mentioned theorem, it is unnecessary that, for two conditions, $\lim_{n\to\infty}(\|u_{n+1} - u_n\|/a_n) = 0$ and $\lim_{n\to\infty}(\|p_{n+1} - p_n\|/a_n) = 0$. The aim of this paper is to show that the convergence of Picard iteration schemes is equivalent to the convergence of the Mann iteration for Zamfirescu operators in normed spaces. The result improves ones announced by Şoltuz [1, Theorem 2]. We will use a special case of the following lemma.

Lemma 1.4 [2]. *Let $\{a_n\}$ and $\{\sigma_n\}$ be nonnegative real sequences satisfying the following inequality:*

$$a_{n+1} \le (1 - \lambda_n)a_n + \sigma_n, \tag{1.5}$$

where $\lambda_n \in (0,1)$, for all $n \ge n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sigma_n/\lambda_n \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

This lemma is apparently due to Vasilen, it is given as [2, Lemma 2.3.6, page 96]. It was rediscovered with a different proof by Weng [3].

2. Main results

THEOREM 2.1. *Let E be a normed space, D a nonempty closed convex subset of E, and $T: D \to D$ a Zamfirescu operator. Suppose that T has a fixed point $q \in D$. Let $\{p_n\}_{n=0}^{\infty}$ be defined by (1.1) for $p_0 \in D$, and let $\{u_n\}_{n=0}^{\infty}$ be defined by (1.2) for $u_0 \in D$ with $\{a_n\}$ in $[0,1]$ and satisfying $\sum_{n=0}^{\infty} a_n = \infty$. Then the following are equivalent:*

(i) *the Picard iteration (1.1) converges to the fixed point of T;*

(ii) *the Mann iteration (1.2) converges to the fixed point of T.*

Proof. Let q be a fixed point of T. We will prove (ii)⇒(i). Suppose that $\|u_n - q\| \to 0$ as $n \to \infty$. Applying (1.1) and (1.2), we have

$$\begin{aligned} ||u_{n+1} - p_{n+1}|| &\le (1-a_n)||u_n - Tp_n|| + a_n||Tu_n - Tp_n|| \\ &\le (1-a_n)||u_n - Tu_n|| + ||Tu_n - Tp_n||. \end{aligned} \tag{2.1}$$

Using (1.4) with $x = u_n$, $y = p_n$, we have

$$||Tu_n - Tp_n|| \le \delta||u_n - p_n|| + 2\delta||u_n - Tu_n||. \tag{2.2}$$

Therefore, from (2.1), we get

$$\begin{aligned} ||u_{n+1} - p_{n+1}|| &\le \delta\|u_n - p_n\| + (1 - a_n + 2\delta)||u_n - Tu_n|| \\ &\le \delta||u_n - p_n|| + (1 - a_n + 2\delta)(||u_n - q|| + ||Tu_n - Tq||) \\ &\le \delta||u_n - p_n|| + (1 - a_n + 2\delta)(1+\delta)||u_n - q||, \end{aligned} \tag{2.3}$$

denoted by $A_n = \|u_n - p_n\|$, $\delta = 1 - \lambda$, and $B_n = (1 - a_n + 2\delta)(1+\delta)\|u_n - q\|$. By Lemma 1.4, we obtain $A_n = \|u_n - p_n\| \to 0$ as $n \to \infty$. Hence by $\|p_n - q\| \le \|u_n - p_n\| + \|u_n - q\|$, we get $\|p_n - q\| \to 0$ as $n \to \infty$.

Next, we will prove (i)⇒(ii), that is, if the Picard iteration converges, then the Mann iteration does too. Now by using Picard iteration (1.1) and Mann iteration (1.2), we have

$$\begin{aligned} ||u_{n+1} - p_{n+1}|| &\le (1-a_n)||u_n - Tp_n|| + a_n||Tu_n - Tp_n|| \\ &\le (1-a_n)||u_n - p_n|| + (1-a_n)||p_n - Tp_n|| + a_n||Tu_n - Tp_n|| \\ &\le (1-a_n)||u_n - p_n|| + (1-a_n)(||p_n - q|| + ||Tp_n - Tq||) + a_n||Tu_n - Tp_n||. \end{aligned} \tag{2.4}$$

On using (1.4) with $x = p_n$, $y = u_n$, we get

$$\begin{aligned} ||Tu_n - Tp_n|| &\le a_n\delta||u_n - p_n|| + 2a_n\delta||p_n - Tp_n|| \\ &\le a_n\delta||u_n - p_n|| + 2a_n\delta(||p_n - q|| + ||Tp_n - Tq||). \end{aligned} \tag{2.5}$$

Again, using (1.4) with $x = q$, $y = p_n$, we get

$$||Tp_n - Tq|| \le \delta||p_n - q||. \tag{2.6}$$

Hence by (2.4)–(2.6), we obtain

$$\begin{aligned}
||u_{n+1} - p_{n+1}|| &\le (1-(1-\delta)a_n)||u_n - p_n|| + (1 - a_n + 2a_n\delta)(1+\delta)||p_n - q|| \\
&\le (1-(1-\delta)a_n)||u_n - q|| + (1 - a_n + 2a_n\delta)(2+\delta)||p_n - q|| \\
&\le (1-\lambda a_n)(1-\lambda a_{n-1})||u_{n-1} - q|| + (1 - a_n + 2a_n\delta)(2+\delta)||p_n - q|| \\
&\le (1-\lambda a_n)(1-\lambda a_{n-1})\cdots(1-\lambda a_0)||u_0 - q|| \\
&\quad + (1 - a_n + 2a_n\delta)(2+\delta)||p_n - q|| \\
&\le \exp\left(-\lambda\sum_{i=0}^{n} a_i\right)||u_0 - q|| + (1 - a_n + 2a_n\delta)(2+\delta)||p_n - q||,
\end{aligned} \tag{2.7}$$

where $1 - \delta = \lambda$. Since $\sum_{n=0}^{\infty} a_n = \infty$ and $\|p_n - q\| \to 0$ as $n \to \infty$, hence $\|u_n - p_n\| \to 0$ as $n \to \infty$. And thus, $\|u_n - q\| \le \|u_n - p_n\| + \|p_n - q\| \to 0$ as $n \to \infty$. This completes the proof. □

Remark 2.2. Theorem 2.1 improves [1, Theorem 2] in the following sense.

(1) Both hypotheses $\lim_{n\to\infty}(\|u_{n+1} - u_n\|/a_n) = 0$ and $\lim_{n\to\infty}(\|p_{n+1} - p_n\|/a_n) = 0$ have been removed, and the conclusion remains valid.

(2) The assumption that $u_0 = p_0$ in [1] is superfluous.

Theorem 2.3. *Let E be a normed space, D a nonempty closed convex subset of E, and $T : D \to D$ a Zamfirescu operator. Suppose that T has a fixed point $q \in D$. Let $\{p_n\}_{n=0}^{\infty}$ be defined by (1.1) for $p_0 \in D$, and let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.3) for $x_0 \in D$ with $\{a_n\}$ and $\{b_n\}$ in* $[0,1]$ *and satisfying $\sum_{n=0}^{\infty} a_n = \infty$. Then the following are equivalent:*

(i) *the Picard iteration (1.1) converges to the fixed point of T;*

(ii) *the Ishikawa iteration (1.3) converges to the fixed point of T.*

Remark 2.4. As previously suggested, Theorem 2.3 reproduces exactly [1, Theorem 1]. Therefore we have the following conclusion: Picard iteration converges to the fixed point of T ⇔ Mann iteration converges to the fixed point of T ⇔ Ishikawa iteration converges to the fixed point of T.

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