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Research Article

Some Fixed Point Theorem for Mapping on Complete G-Metric Spaces

Zead Mustafa, Hamed Obiedat, and Fadi Awawdeh

Department of Mathematics, The Hashemite University, P.O. Box 330127, Zarqa 13115, Jordan

Correspondence should be addressed to Zead Mustafa, zmagablh@hu.edu.jo

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Recommended by Brailey Sims

We prove some fixed point results for mapping satisfying sufficient conditions on complete G-metric space, also we showed that if the G-metric space (X, G) is symmetric, then the existence and uniqueness of these fixed point results follow from well-known theorems in usual metric space (X, d_G), where (X, d_G) is the usual metric space which defined from the G-metric space (X, G).

1. Introduction

During the sixties, the notion of 2-metric space introduced by Gähler (see [1, 2]) as a generalization of usual notion of metric space (X, d). But different authors proved that there is no relation between these two functions, for instance, Ha et al. in [3] show that 2-metric need not be continuous function, further there is no easy relationship between results obtained in the two settings.

In 1992, Bapure Dhage in his Ph.D. thesis introduce a new class of generalized metric space called D-metric spaces ([4, 5]).

In a subsequent series of papers, Dhage attempted to develop topological structures in such spaces (see [5–7]). He claimed that D-metrics provide a generalization of ordinary metric functions and went on to present several fixed point results.

But in 2003 in collaboration with Brailey Sims, we demonstrated in [8] that most of the claims concerning the fundamental topological structure of D-metric space are incorrect, so, we introduced more appropriate notion of generalized metric space as follows.

Definition 1.1 (see [9]). Let X be a nonempty set, and let $G : X \times X \times X \to \mathbf{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$, for all $x,y,z \in X$, with $z \neq y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$, (symmetry in all three variables);

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a *generalized metric*, or, more specifically, a *G-metric* on X, and the pair (X,G) is called a *G-metric space*.

Definition 1.2 (see [9]). Let (X,G) be a G-metric space, and let (x_n) be sequence of points of X, a point $x \in X$ is said to be the *limit* of the sequence (x_n), if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, and one says that the sequence (x_n) is *G-convergent* to x.

Thus, that if $x_n \to x$ in a G-metric space (X,G), then for any $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x,x_n,x_m) < \epsilon$, for all $n,m \geq N$.

Proposition 1.3 (see [9]). *Let (X,G) be a G-metric space, then the following are equivalent.*

(1) (x_n) *is G-convergent to* x.

(2) $G(x_n,x_n,x) \to 0$, *as* $n \to \infty$.

(3) $G(x_n,x,x) \to 0$, *as* $n \to \infty$.

(4) $G(x_m,x_n,x) \to 0$, *as* $m,n \to \infty$.

Definition 1.4 (see [9]). Let (X,G) be a G-metric space, a sequence (x_n) is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n,x_m,x_l) < \epsilon$, for all $n,m,l \geq N$; that is, if $G(x_n,x_m x_l) \to 0$ as $n,m,l \to \infty$.

Proposition 1.5 (see [8]). *If (X,G) is a G-metric space, then the following are equivalent.*

(1) *The sequence (x_n) is G-Cauchy.*

(2) *For every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n,x_m,x_m) < \epsilon$, for all $n,m \geq N$.*

Definition 1.6 (see [9]). Let (X,G) and (X',G') be two G-metric spaces, and let $f : (X,G) \to (X',G')$ be a function, then f is said to be *G-continuous at a point* $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x,y \in X$; and $G(a,x,y) < \delta$ implies $G'(f(a),f(x),f(y)) < \epsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Proposition 1.7 (see [9]). *Let (X,G), (X',G') be two G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.*

Definition 1.8 (see [9]). A G-metric space (X,G) is called symmetric G-metric space if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Proposition 1.9 (see [9]). *Let (X,G) be a G-metric space, then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Proposition 1.10 (see [8]). *Every G-metric space (X,G) will define a metric space (X,d_G) by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \quad \forall x,y \in X. \tag{1.1}$$

Note that if (X,G) is a symmetric G-metric space, then

$$d_G(x,y) = 2G(x,y,y), \quad \forall x,y \in X. \tag{1.2}$$

However, if (X, G) is not symmetric, then it holds by the G-metric properties that

$$\frac{3}{2}G(x,y,y) \leq d_G(x,y) \leq 3G(x,y,y), \quad \forall x,y \in X, \tag{1.3}$$

and that in general these inequalities cannot be improved.

Definition 1.11 (see [9]). A G-metric space (X, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.12 (see [9]). *A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

2. Main results

Here we start our work with the following theorem.

Theorem 2.1. *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a mapping satisfying one of the following conditions:*

$$G(T(x),T(y),T(z)) \leq \{aG(x,y,z)+bG(x,T(x),T(x))+cG(y,T(y),T(y))+dG(z,T(z),T(z))\} \tag{2.1}$$

or

$$G(T(x),T(y),T(z)) \leq \{aG(x,y,z) + bG(x,x,T(x)) + cG(y,y,T(y)) + dG(z,z,T(z))\} \tag{2.2}$$

for all $x, y, z \in X$ where $0 \leq a + b + c + d < 1$, then T has a unique fixed point (say u, i.e., $Tu = u$), and T is G-continuous at u.

Proof. Suppose that T satisfies condition (2.1), then for all $x, y \in X$, we have

$$\begin{aligned} G(Tx,Ty,Ty) &\leq aG(x,y,y) + bG(x,Tx,Tx) + (c+d)G(y,Ty,Ty), \\ G(Ty,Tx,Tx) &\leq aG(y,x,x) + bG(y,Ty,Ty) + (c+d)G(x,Tx,Tx). \end{aligned} \tag{2.3}$$

Suppose that (X, G) is symmetric, then by definition of metric (X, d_G) and (1.2), we get

$$d_G(Tx,Ty) \leq ad_G(x,y) + \frac{c+d+b}{2}d_G(x,Tx) + \frac{c+d+b}{2}d_G(y,Ty), \quad \forall x,y \in X. \tag{2.4}$$

In this line, since $0 < a + b + c + d < 1$, then the existence and uniqueness of the fixed point follows from well-known theorem in metric space (X, d_G) (see [10]).

However, if (X, G) is not symmetric then by definition of metric (X, d_G) and (1.3), we get

$$d_G(Tx,Ty) \leq ad_G(x,y) + \frac{2(c+d+b)}{3}d_G(x,Tx) + \frac{2(c+d+b)}{3}d_G(y,Ty), \tag{2.5}$$

for all $x, y \in X$, then the metric condition gives no information about this map since $0 < a + 2(c + d + b)/3 + 2(c + d + b)/3$ need not be less than 1. But this can be proved by G-metric.

Let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$. By (2.1), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq aG(x_{n-1}, x_n, x_n) + bG(x_{n-1}, x_n, x_n) + (c+d)G(x_n, x_{n+1}, x_{n+1}), \tag{2.6}$$

then

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a+b}{1-(c+d)} G(x_{n-1}, x_n, x_n). \tag{2.7}$$

Let $q = (a+b)/(1-(c+d))$, then $0 \leq q < 1$ since $0 \leq a+b+c+d < 1$.

So,

$$G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n). \tag{2.8}$$

Continuing in the same argument, we will get

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1). \tag{2.9}$$

Moreover, for all $n, m \in \mathbf{N};\ n < m$, we have by rectangle inequality that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq (q^n + q^{n+1} + \cdots + q^{m-1}) G(x_0, x_1, x_1) \\ &\leq \frac{q^n}{1-q} G(x_0, x_1, x_1), \end{aligned} \tag{2.10}$$

and so $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$. Thus (x_n) is G-Cauchy sequence. Due to the completeness of (X, G), there exists $u \in X$ such that (x_n) is G-converge to u.

Suppose that $T(u) \neq u$, then

$$G(x_n, T(u), T(u)) \leq aG(x_{n-1}, u, u) + bG(x_{n-1}, x_n, x_n) + (c+d)G(u, T(u), T(u)), \tag{2.11}$$

taking the limit as $n \to \infty$, and using the fact that the function G is continuous, then $G(u, T(u), T(u)) \leq (c+d)G(u, T(u), T(u))$. This contradiction implies that $u = T(u)$.

To prove uniqueness, suppose that $u \neq v$ such that $T(v) = v$, then

$$G(u, v, v) \leq aG(u, v, v) + bG(u, T(u), T(u)) + (c+d)G(v, T(v), T(v)) = aG(u, v, v), \tag{2.12}$$

which implies that $u = v$.

To show that T is G-continuous at u, let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n) = u$. we can deduce that

$$\begin{aligned} G(u, T(y_n), T(y_n)) &\leq aG(u, y_n, y_n) + bG(u, T(u), T(u)) + (c+d)G(y_n, T(y_n), T(y_n)) \\ &= aG(u, y_n, y_n) + (c+d)G(y_n, T(y_n), T(y_n)), \end{aligned} \tag{2.13}$$

and since $G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n))$, we have that $G(u, T(y_n), T(y_n)) \leq (a/(1-(c+d)))G(u, y_n, y_n) + ((c+d)/(1-(c+d)))G(y_n, u, u)$.

Taking the limit as $n \to \infty$, from which we see that $G(u, T(y_n), T(y_n)) \to 0$ and so, by Proposition 1.7, $T(y_n) \to u = Tu$. It is proved that T is G-continuous at u.

If T satisfies condition (2.2), then the argument is similar to that above. However, to show that the sequence (x_n) is G-Cauchy, we start with

$$G(x_n, x_n, x_{n+1}) \leq aG(x_{n-1}, x_{n-1}, x_n) + (b+c)G(x_{n-1}, x_{n-1}, x_n) + dG(x_n, x_n, x_{n+1}), \tag{2.14}$$

then

$$G(x_n, x_n, x_{n+1}) \leq \frac{a+b+c}{1-d} G(x_{n-1}, x_{n-1}, x_n). \tag{2.15}$$

Let $q = (a+b+c)/(1-d)$, then $0 \leq q < 1$ since $0 \leq a+b+c+d < 1$.

Continuing in the same way, we find that

$$G(x_n, x_n, x_{n+1}) \leq q^n G(x_0, x_0, x_1). \tag{2.16}$$

Then for all $n, m \in \mathbf{N}$; $n < m$, we have by repeated use of the rectangle inequality $G(x_n, x_n, x_m) \leq (q^n/(1-q))G(x_0, x_0, x_1)$. □

Corollary 2.2. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying one of the following conditions:*

$$\begin{aligned} &G(T^m(x), T^m(y), T^m(z)) \\ &\quad \leq \{aG(x,y,y) + bG(x, T^m(x), T^m(x)) + cG(y, T^m(y), T^m(y)) + dG(z, T^m(z), T^m(z))\} \end{aligned} \tag{2.17}$$

or

$$G(T^m(x), T^m(y), T^m(z)) \leq \{aG(x,y,y) + bG(x,x,T^m(x)) + cG(y,y,T^m(y)) + dG(z,z,T^m(z))\}, \tag{2.18}$$

for all $x, y, z \in X$, where $0 \leq a+b+c+d < 1$. Then T has a unique fixed point (say u), and T^m is G-continuous at u.

Proof. From the previous theorem, we see that T^m has a unique fixed point (say u), that is, $T^m(u) = u$. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$. □

Theorem 2.3. *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a mapping satisfying one of the following conditions:*

$$G(T(x), T(y), T(z)) \leq k \max \begin{Bmatrix} G(x, T(x), T(x)), \\ G(y, T(y), T(y)), \\ G(z, T(z), T(z)) \end{Bmatrix} \tag{2.19}$$

or

$$G(T(x), T(y), T(z)) \leq k \max \begin{Bmatrix} G(x, x, T(x)), \\ G(y, y, T(y)), \\ G(z, z, T(z)) \end{Bmatrix}, \tag{2.20}$$

for all $x, y, z \in X$, where $0 \leq k < 1$. Then T has a unique fixed point (say u), and T is G-continuous at u.

Proof. Suppose that T satisfies condition (2.19), then for all $x, y \in X$,

$$\begin{aligned} G(Tx,Ty,Ty) &\le k\max\{G(x,Tx,Tx),G(y,Ty,Ty)\},\\ G(Ty,Tx,Tx) &\le k\max\{G(y,Ty,Ty),G(x,Tx,Tx)\}. \end{aligned} \tag{2.21}$$

Suppose that (X, G) is symmetric, then by definition of the metric (X, d_G) and (1.2) we get

$$d_G(Tx,Ty) \le k\max\{d_G(x,Tx),d_G(y,Ty)\}, \quad \forall x,y \in X. \tag{2.22}$$

Since $k < 1$, then the existence and uniqueness of the fixed point follows from a theorem in metric space (X, d_G) (see [11]).

However, if (X, G) is not symmetric, then by definition of the metric (X, d_G) and (1.3), we get

$$d_G(Tx,Ty) \le \frac{4k}{3}\max\{d_G(x,Tx),d_G(y,Ty)\}, \quad \forall x,y \in X. \tag{2.23}$$

The metric condition gives no information about this map since $4k/3$ need not be less than 1, but we will proof it by G-metric.

Let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$. By (2.19), we can verify that

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) &\le k\max\{G(x_{n-1},x_n,x_n),G(x_n,x_{n+1},x_{n+1})\}\\ &= kG(x_{n-1},x_n,x_n) \quad (\text{since } 0 \le k < 1). \end{aligned} \tag{2.24}$$

Continuing in the same argument, we will find

$$G(x_n,x_{n+1},x_{n+1}) \le k^n G(x_0,x_1,x_1). \tag{2.25}$$

For all $n, m \in \mathbf{N};\ n < m$, we have by rectangle inequality that

$$\begin{aligned} G(x_n,x_m,x_m) &\le G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2})+G(x_{n+2},x_{n+3},x_{n+3})+\cdots+G(x_{m-1},x_m,x_m)\\ &\le \left(k^n+k^{n+1}+\cdots+k^{m-1}\right)G(x_0,x_1,x_1)\\ &\le \frac{k^n}{1-k}G(x_0,x_1,x_1). \end{aligned} \tag{2.26}$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, and thus (x_n) is G-Cauchy sequence. Due to the completeness of (X, G), there exists $u \in X$ such that $(x_n) \to u$.

Suppose that $T(u) \ne u$, then $G(x_{n+1}, T(u), T(u)) \le k\max\{G(x_{n+1}, x_{n+2}, x_{n+2}), G(u, T(u), T(u))\}$ and by taking the limit as $n \to \infty$, and using the fact that the function G is continuous, we get that $G(u, T(u), T(u)) \le kG(u, T(u), T(u))$. This contradiction implies that $u = T(u)$.

To prove uniqueness, suppose that $u \ne v$ such that $T(v) = v$, then $G(u, v, v) \le k\max\{G(v, v, v), G(u, u, u)\} = 0$ which implies that $u = v$.

To show that T is G-continuous at u, let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n) = u$, then

$$G(u,T(y_n),T(y_n)) \le k\max\{G(u,T(u),T(u)),G(y_n,T(y_n),T(y_n))\} = kG(y_n,T(y_n),T(y_n)). \tag{2.27}$$

But, $G(y_n, T(y_n), T(y_n)) \le G(y_n, u, u) + G(u, T(y_n), T(y_n))$, then $G(u, T(y_n), T(y_n)) \le (k/(1-k))G(y_n, u, u)$. Taking the limit as $n \to \infty$, from which we see that $G(u, T(y_n), T(y_n)) \to 0$, and so by Proposition 1.7, $T(y_n) \to u = Tu$. So, T is G-continuous at u □

Corollary 2.4. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying one of the following conditions for some $m \in N$:*

$$G(T^m(x), T^m(y), T^m(z)) \leq k \max \left\{ \begin{array}{c} G(x, T^m(x), T^m(x)), \\ G(y, T^m(y), T^m(y)), \\ G(z, T^m(z), T^m(z)) \end{array} \right\} \tag{2.28}$$

or

$$G(T^m(x), T^m(y), T^m(z)) \leq k \max \left\{ \begin{array}{c} G(x, x, T^m(x)), \\ G(y, y, T^m(y)), \\ G(z, z, T^m(z)) \end{array} \right\}, \tag{2.29}$$

for all $x, y, z \in X$, then T has a unique fixed point (say u) and T^m is G-continuous at u.

Proof. We use the same argument as in Corollary 2.2. □

Theorem 2.5. *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a mapping satisfying one of the following conditions:*

$$G(T(x), T(y), T(y)) \leq k \max\{G(x, T(y), T(y)), G(y, T(x), T(x)), G(y, T(y), T(y))\} \tag{2.30}$$

or

$$G(T(x), T(y), T(y)) \leq k \max\{G(x, x, T(y)), G(y, y, T(x)), G(y, y, T(y))\}, \tag{2.31}$$

for all $x, y \in X$, where $k \in [0, 1)$. Then T has a unique fixed point (say u), and T is G-continuous at u.

Proof. Suppose that T satisfies condition (2.30), then for all $x, y \in X$,

$$\begin{aligned} G(Tx, Ty, Ty) &\leq k \max\{G(x, Ty, Ty), G(y, Tx, Tx), G(y, Ty, Ty)\}, \\ G(Ty, Tx, Tx) &\leq k \max\{G(x, Ty, Ty), G(y, Tx, Tx), G(x, Tx, Tx)\}. \end{aligned} \tag{2.32}$$

Suppose that (X, G) is symmetric, then by definition of the metric (X, d_G) and (1.2), we have.

$$\begin{aligned} d_G(Tx, Ty) &\leq \frac{k}{2} \max \left\{ \begin{array}{c} d_G(x, Ty), \\ d_G(y, Tx), \\ d_G(y, Ty) \end{array} \right\} + \frac{k}{2} \max \left\{ \begin{array}{c} d_G(x, Ty), \\ d_G(y, Tx), \\ d_G(x, Tx) \end{array} \right\} \\ &\leq k \max\{d_G(x, Ty), d_G(y, Tx), d_G(x, Tx), d_G(y, Ty)\}, \quad \forall x, y \in X. \end{aligned} \tag{2.33}$$

Since $0 \leq k < 1$, then the existence and uniqueness of the fixed point follows from a theorem in metric space (X, d_G) (see [12]).

However, if (X, G) is not symmetric, then by definition of the metric (X, d_G) and (1.3), we have

$$d_G(Tx, Ty) \leq \frac{2k}{3} \max \left\{ \begin{array}{c} d_G(x, Ty), \\ d_G(y, Tx), \\ d_G(y, Ty) \end{array} \right\} + \frac{2k}{3} \max \left\{ \begin{array}{c} d_G(x, Ty), \\ d_G(y, Tx), \\ d_G(x, Tx) \end{array} \right\}, \tag{2.34}$$

for all $x, y \in X$, then the metric space (X, d_G) gives no information about this map since $4k/3$ need not be less than 1. But we will proof it by G-metric.

Let $x_0 \in X$ be arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then by (2.30) and using $k < 1$, we deduce that

$$G(x_n, x_{n+1}, x_{n+1}) \le k \max\{G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\} = kG(x_{n-1}, x_{n+1}, x_{n+1}). \tag{2.35}$$

So,

$$G(x_n, x_{n+1}, x_{n+1}) \le kG(x_{n-1}, x_{n+1}, x_{n+1}), \tag{2.36}$$

and using

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \le k \max\{G(x_{n-2}, x_{n+1}, x_{n+1}), G(x_n, x_{n-1}, x_{n-1}), G(x_n, x_{n+1}, x_{n+1})\}, \tag{2.37}$$

then,

$$G(x_n, x_{n+1}, x_{n+1}) \le k^2 \max\{G(x_{n-2}, x_{n+1}, x_{n+1}), G(x_n, x_{n-1}, x_{n-1})\}. \tag{2.38}$$

Continuing in this procedure, we will have

$$G(x_n, x_{n+1}, x_{n+1}) \le k^n \Gamma_n, \tag{2.39}$$

where $\Gamma_n = \max\{G(x_i, x_j, x_j);\ \text{for all } i, j \in \{0, 1, \ldots, n+1\}\}$.

For $n, m \in \mathbf{N};\ n < m$, let $\Gamma = \max\{\Gamma_i;\ \text{ for all } i = n, \ldots, m-1\}$.

Then, for all $n, m \in \mathbf{N};\ n < m$, we have by rectangle inequality

$$\begin{aligned} G(x_n, x_m, x_m) &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\le k^n \Gamma_n + k^{n+1} \Gamma_{n+1} + \cdots + k^{m-1} \Gamma_{m-1} \\ &\le (k^n + k^{n+1} + \cdots + k^{m-1}) \Gamma \\ &\le \frac{k^n}{1-k} \Gamma. \end{aligned} \tag{2.40}$$

This prove that $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, and thus (x_n) is G-Cauchy sequence. Since (X, G) is G-complete then there exists $u \in X$ such that (x_n) is G-converge to u.

Suppose that $T(u) \ne u$, then

$$G(x_n, T(u), T(u)) \le k \max\{G(x_{n-1}, T(u), T(u)), G(u, x_{n+1}, x_{n+1}), G(u, T(u), T(u))\}. \tag{2.41}$$

Taking the limit as $n \to \infty$, and using the fact that the function G is continuous, we get $G(u, T(u), T(u)) \le kG(u, T(u), T(u))$, this contradiction implies that $u = T(u)$.

To prove the uniqueness, suppose that $u \ne v$ such that $T(v) = v$. So, by (2.30), we have that

$$G(u, v, v) \le k \max\{G(u, v, v), G(v, u, u)\} \Longrightarrow G(u, v, v) \le kG(v, u, u). \tag{2.42}$$

Again we will find $G(v, u, u) \le kG(u, v, v)$, so

$$G(u, v, v) \le k^2 G(u, v, v); \tag{2.43}$$

since $k < 1$, this implies that $u = v$.

To show that T is G-continuous at u, let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n) = u$, then

$$G(u, T(y_n), T(y_n)) \leq k \max\{G(u, T(y_n), T(y_n)), G(y_n, T(u), T(u)), G(y_n, T(y_n), T(y_n))\}. \tag{2.44}$$

But, $G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n))$, so, $G(u, T(y_n), T(y_n)) \leq (k/(1 - k))G(y_n, u, u)$.

Taking the limit as $n \to \infty$, from which we see that $G(u, T(y_n), T(y_n)) \to 0$ and so, by Proposition 1.7, we have $T(y_n) \to u = Tu$ which implies that T is G-continuous at u. □

Corollary 2.6. *Let (X, G) be a complete G-metric space, and let $T : X \to X$, be a mapping satisfying one of the following conditions:*

$$G(T(x), T(y), T(z)) \leq k \max\left\{\begin{matrix} G(x, T(y), T(y)), G(x, T(z), T(z)), \\ G(y, T(x), T(x)), G(y, T(z), T(z)), \\ G(z, T(x), T(x)), G(z, T(y), T(y)) \end{matrix}\right\} \tag{2.45}$$

or

$$G(T(x), T(y), T(z)) \leq k \max\left\{\begin{matrix} G(x, x, T(y)), G(x, x, T(z)), \\ G(y, y, T(x)), G(y, y, T(z)), \\ G(z, z, T(x)), G(z, z, T(y)) \end{matrix}\right\}, \tag{2.46}$$

for all $x, y, z \in X$ where $k \in [0, 1)$, then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. If we let $z = y$ in conditions (2.45) and (2.46), then they become conditions (2.30) and (2.31), respectively, in Theorem 2.5; so the proof follows from Theorem 2.5. □

Corollary 2.7. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying one of the following conditions:*

$$G(T^m(x), T^m(y), T^m(z)) \leq k \max\left\{\begin{matrix} G(x, T^m(y), T^m(y)), G(x, T^m(z), T^m(z)), \\ G(y, T^m(x), T^m(x)), G(y, T^m(z), T^m(z)), \\ G(z, T^m(x), T^m(x)), G(z, T^m(y), T^m(y)) \end{matrix}\right\},$$

$$G(T^m(x), T^m(y), T^m(z)) \leq k \max\left\{\begin{matrix} G(x, x, T^m(y)), G(x, x, T^m(z)), \\ G(y, y, T^m(x)), G(y, y, T^m(z)), \\ G(z, z, T^m(x)), G(z, z, T^m(y)) \end{matrix}\right\},$$

$$G(T^m(x), T^m(y), T^m(y)) \leq k \max\{G(x, T^m(y), T^m(y)), G(y, T^m(x), T^m(x)), G(y, T^m(y), T^m(y))\}, \tag{2.47}$$

or,

$$G(T^m(x), T^m(y), T^m(y)) \leq k \max\{G(x, x, T^m(y)), G(y, y, T^m(x)), G(y, y, T^m(y))\}, \tag{2.48}$$

for all $x, y, z \in X$, for some $m \in \mathbf{N}$, where $k \in [0, 1)$, then T has a unique fixed point (say u), and T^m is G-continuous at u.

Proof. The proof follows from Theorem 2.5, Corollary 2.6, and from an argument similar to that used in Corollary 2.2. □

Theorem 2.8. *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a mapping satisfying one of the following conditions:*

$$G(T(x), T(y), T(y)) \leq k \max\{G(x, T(y), T(y)), G(y, T(x), T(x))\} \tag{2.49}$$

or

$$G(T(x), T(y), T(y)) \leq k \max\{G(x, x, T(y)), G(y, y, T(x))\}, \tag{2.50}$$

for all $x, y \in X$, where $k \in [0, 1)$, then T has a unique fixed point (say u), and T is G-continuous at u.

Proof. Since whenever the mapping satisfies condition (2.49), or (2.50), then it satisfies condition (2.45), or (2.46), respectively, in Theorem 2.5. Then the proof follows from Theorem 2.5. □

Theorem 2.9. *Let (X, G) be a complete G-metric space, and let $T : X \to X$, be a mapping satisfying one of these conditions*

$$G(T(x), T(y), T(y)) \leq a\{G(x, T(y), T(y)) + G(y, T(x), T(x))\} \tag{2.51}$$

or

$$G(T(x), T(y), T(y)) \leq a\{G(x, x, T(y)) + G(y, y, T(x))\}, \tag{2.52}$$

for all $x, y \in X$, where $a \in [0, 1/2)$, then T has a unique fixed point (say u), and T is G-continuous at u.

Proof. Suppose that T satisfies condition (2.51), then we have

$$\begin{aligned} G(Tx, Ty, Ty) &\leq a\{G(y, Tx, Tx) + G(x, Ty, Ty)\}, \\ G(Ty, Tx, Tx) &\leq a\{G(x, Ty, Ty) + G(y, Tx, Tx)\}, \end{aligned} \tag{2.53}$$

for all $x, y \in X$.

Suppose that (X, G) is symmetric, then by definition of the metric (X, d_G) and (1.2), we get

$$d_G(Tx, Ty) \leq a\{d_G(x, Ty) + d_G(y, Tx)\} \quad \forall x, y \in X. \tag{2.54}$$

Since $0 \leq 2a < 1$, then the existence and uniqueness of the fixed point follow from a theorem in metric space (X, d_G) (see [13]).

However, if (X, G) is not symmetric, then by definition of the metric (X, d_G) and (1.3), we have

$$d_G(Tx, Ty) \leq \frac{4a}{3} d_G(x, Ty) + \frac{4a}{3} d_G(y, Tx) \quad \forall x, y \in X. \tag{2.55}$$

So, the metric space (X, d_G) gives no information about this map since $8a/3$ need not be less than 1. But this can be proved by G-metric.

Let $x_0 \in X$ be arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then by (2.51), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq a\{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\} = aG(x_{n-1}, x_{n+1}, x_{n+1}). \tag{2.56}$$

But

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \leq aG(x_{n-1}, x_n, x_n) + aG(x_n, x_{n+1}, x_{n+1}), \tag{2.57}$$

thus we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a}{1-a} G(x_{n-1}, x_n, x_n). \tag{2.58}$$

Let $k = a/(1-a)$, hence $0 \leq k < 1$ then continue in this procedure, we will get that $G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1)$.

For all $n, m \in \mathbf{N}$; $n < m$, we have by rectangle inequality

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq (k^n + k^{n+1} + \cdots + k^{m-1}) G(x_0, x_1, x_1) \\ &\leq \frac{k^n}{1-k} G(x_0, x_1, x_1). \end{aligned} \tag{2.59}$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, and so, (x_n) is G-Cauchy sequence. By completeness of (X, G), there exists $u \in X$ such that (x_n) is G-converge to u.

Suppose that $T(u) \neq u$, then

$$G(x_n, T(u), T(u)) \leq a\{G(x_{n-1}, T(u), T(u)) + G(u, x_n, x_n)\}. \tag{2.60}$$

Taking the limit as $n \to \infty$, and using the fact that the function G is continuous, then $G(u, T(u), T(u)) \leq aG(u, T(u), T(u))$. This contradiction implies that $u = T(u)$.

To prove uniqueness, suppose that $u \neq v$ such that $T(v) = v$, then $G(u, v, v) \leq a\{G(u, v, v) + G(v, u, u)\}$, so

$$G(u, v, v) \leq \left(k = \frac{a}{1-a}\right) G(v, u, u) \tag{2.61}$$

again by the same argument, we can verify that $G(u, v, v) \leq k^2 G(u, v, v)$, which implies that $u = v$.

To show that T is G-continuous at u, let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n) = u$, then

$$G(u, T(y_n), T(y_n)) \leq a\{G(u, T(y_n), T(y_n)) + G(y_n, T(u), T(u))\}, \tag{2.62}$$

and so $G(u, T(y_n), T(y_n)) \leq (a/(1-a)) G(y_n, T(u), T(u)$.

Taking the limit as $n \to \infty$, from which we see that $G(u, T(y_n), T(y_n)) \to 0$. By Proposition 1.7, we have $T(y_n) \to u = Tu$ which implies that T is G-continuous at u. □

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