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Research Article

A New Hybrid Iterative Algorithm for Fixed-Point Problems, Variational Inequality Problems, and Mixed Equilibrium Problems

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We introduce a new hybrid iterative algorithm for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings, the set of solutions of the variational inequality of a monotone mapping, and the set of solutions of a mixed equilibrium problem. This study, proves a strong convergence theorem by the proposed hybrid iterative algorithm which solves fixed-point problems, variational inequality problems, and mixed equilibrium problems.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a mapping $f : C \to C$ is called contractive if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x-y\|$ for all $x, y \in C$. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. Denote the set of fixed points of T by $F(T)$.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function and $\Theta : C \times C \to \mathbf{R}$ be an equilibrium bifunction, that is, $\Theta(u,u) = 0$ for each $u \in C$. The mixed equilibrium problem (for short, MEP) is to find $x^* \in C$ such that

$$\text{MEP:}\quad \Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in C. \tag{1.1}$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem (for short, EP), which is to find $x^* \in C$ such that

$$\text{EP:}\quad \Theta(x^*, y) \geq 0 \quad \forall y \in C. \tag{1.2}$$

Denote the set of solutions of MEP by Ω. The mixed equilibrium problems include fixed-point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and the equilibrium problems as special cases (see, e.g., [1–5]). Some methods have been proposed to solve the MEP and EP (see, e.g., [5–14]). In 1997, Combettes and Hirstoaga [13] introduced an iterative method of finding the best approximation to the initial data and proved a strong convergence theorem. Subsequently, S. Takahashi and W. Takahashi [8] introduced another iterative scheme for finding a common element of the set of solutions of EP and the set of fixed-point points of a nonexpansive mapping. Yao et al. [12] considered an iterative scheme for finding a common element of the set of solutions of EP and the set of common fixed points of an infinite nonexpansive mappings. Very recently, Zeng and Yao [14] considered a new iterative scheme for finding a common element of the set of solutions of MEP and the set of common fixed points of finitely many nonexpansive mappings. Their results extend and improve many results in the literature.

Let A of C into H be a nonlinear mapping. It is well known that the variational inequality problem is to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0 \quad \forall v \in C. \tag{1.3}$$

The set of solutions of the variational inequality problem is denoted by $VI(C,A)$. A mapping $A : C \to H$ is called β-inverse-strongly monotone if there exists a positive real number β such that

$$\langle Au - Av, u-v\rangle \geq \beta\|Au - Av\|^2 \quad \forall u, v \in C. \tag{1.4}$$

Recently, some authors have proposed new iterative algorithms to approximate a common element of the set of fixed points of a nonxpansive mapping and the set of solutions of the variational inequality. For the details, see [15, 16] and the references therein.

Motivated by the recent works, in this paper we introduce a new hybrid iterative algorithm for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings, the set of solutions of the variational inequality of a monotone mapping, and the set of solutions of a mixed equilibrium problem. We prove a strong convergence theorem by the proposed hybrid iterative algorithm which solves fixed-point problems, variational inequality problems, and mixed equilibrium problems.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Then for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \forall y \in C. \tag{2.1}$$

Such a P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x-y, P_Cx - P_Cy\rangle \geq \|P_Cx - P_Cy\|^2 \quad \forall x, y \in H. \tag{2.2}$$

Moreover, P_C is characterized by the following properties:

$$\langle x - P_C x, y - P_C x \rangle \le 0,$$
$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2 \quad \forall x \in H,\ y \in C. \tag{2.3}$$

It is clear that

$$u \in VI(C, A) \Longleftrightarrow u = P_C(u - \lambda A u) \quad \forall \lambda > 0. \tag{2.4}$$

In this paper, for solving the mixed equilibrium problems for an equilibrium bifunction $\Theta : C \times C \to \mathbf{R}$, we assume that Θ satisfies the following conditions:

(H1) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \le 0$ for all $x, y \in C$;

(H2) for each fixed $y \in C$, $x \mapsto \Theta(x, y)$ is concave and upper semicontinuous;

(H3) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex.

A mapping $\eta : C \times C \to H$ is called Lipschitz continuous if there exists a constant $\lambda > 0$ such that

$$\|\eta(x, y)\| \le \lambda \|x - y\| \quad \forall x, y \in C. \tag{2.5}$$

A differentiable function $K : C \to \mathbf{R}$ on a convex set C is called:

(i) η-convex if

$$K(y) - K(x) \ge \langle K'(x), \eta(y, x) \rangle \quad \forall x, y \in C, \tag{2.6}$$

where K' is the Fréchet derivative of K at x;

(ii) η-strongly convex if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y, x) \rangle \ge \left(\frac{\sigma}{2}\right) \|x - y\|^2 \quad \forall x, y \in C. \tag{2.7}$$

Let C be a nonempty closed convex subset of a real Hilbert space H, $\varphi : C \to \mathbf{R}$ be a real-valued function, and $\Theta : C \times C \to \mathbf{R}$ be an equilibrium bifunction. Let r be a positive number. For a given point $x \in C$, the auxiliary problem for MEP consists of finding $y \in C$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), \eta(z, y) \rangle \ge 0 \quad \forall z \in C. \tag{2.8}$$

Let $S_r : C \to C$ be the mapping such that for each $x \in C$, $S_r(x)$ is the solution set of the auxiliary problem MEP, that is,

$$S_r(x) = \left\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), \eta(z, y) \rangle \ge 0\ \forall z \in C \right\} \quad \forall x \in C. \tag{2.9}$$

We first need the following important and interesting result.

Lemma 2.1 (see [14]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Assume that*

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*

(a) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$,*

(b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*

(c) *for each fixed $y \in C$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$K : C \to \mathbf{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology;*

(iii) *for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0. \tag{2.10}$$

Then there hold the following:

(i) *S_r is single-valued;*

(ii) *S_r is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$ and*

$$\langle K'(x_1) - K'(x_2), \eta(u_1, u_2)\rangle \geq \langle K'(u_1) - K'(u_2), \eta(u_1, u_2)\rangle \quad \forall (x_1, x_2) \in C \times C, \tag{2.11}$$

where $u_i = S_r(x_i)$ for $i = 1, 2$;

(iii) *$F(S_r) = \Omega$;*

(vi) *Ω is closed and convex.*

We also need the following lemmas for proving our main results.

Lemma 2.2 (see [17]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.3 (see [18]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that*

(1) *$\sum_{n=1}^{\infty}\gamma_n = \infty$;*

(2) *$\limsup_{n\to\infty}\delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty}|\delta_n| < \infty$.*

Then $\lim_{n\to\infty}a_n = 0$.

3. Iterative algorithm and strong convergence theorems

In this section, we first introduce a new iterative algorithm. Consequently, we will establish a strong convergence theorem for this iteration algorithm. To be more specific, let $T_1, T_2, \ldots$ be infinite mappings of C into itself and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 \le \xi_i \le 1$ for every $i \in \mathbf{N}$. For any $n \in \mathbf{N}$, define a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \xi_n T_n U_{n,n+1} + (1-\xi_n) I, \\ U_{n,n-1} &= \xi_{n-1} T_{n-1} U_{n,n} + (1-\xi_{n-1}) I, \\ &\vdots \\ U_{n,k} &= \xi_k T_k U_{n,k+1} + (1-\xi_k) I, \\ U_{n,k-1} &= \xi_{k-1} T_{k-1} U_{n,k} + (1-\xi_{k-1}) I, \\ &\vdots \\ U_{n,2} &= \xi_2 T_2 U_{n,3} + (1-\xi_2) I, \\ W_n = U_{n,1} &= \xi_1 T_1 U_{n,2} + (1-\xi_1) I. \end{aligned} \tag{3.1}$$

Such W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_2, T_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_2, \xi_1$. For the iterative algorithm for a finite family of nonexpansive mappings, we refer the reader to [19].

We have the following crucial Lemmas 3.1 and 3.2 concerning W_n which can be found in [20]. Now we only need the following similar version in Hilbert spaces.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \le b < 1$ for any $i \in \mathbf{N}$. Then for every $x \in C$ and $k \in \mathbf{N}$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists.*

Lemma 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \le b < 1$ for any $i \in N$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

The following remark [12] is important to prove our main results.

Remark 3.3. Using Lemma 3.1, one can define a mapping W of C into itself as $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$ for every $x \in C$. If $\{x_n\}$ is a bounded sequence in C, then we have

$$\lim_{n\to\infty} \|W x_n - W_n x_n\| = 0. \tag{3.2}$$

Throughout this paper, we will assume that $0 < \xi_i \le b < 1$ for every $i \in \mathbf{N}$.

Now we introduce the following iteration algorithm.

Algorithm 3.4. Let $r > 0$ be a constant. Let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional and let $\Theta : C \times C \to \mathbf{R}$ be an equilibrium bifunction. Let $A : C \to H$ be a β-inverse-strongly monotone mapping and W_n be the W-mapping defined by (3.1). Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and given $x_0 \in C$ arbitrarily. Suppose that the sequences $\{x_n\}$ and $\{y_n\}$ are generated iteratively by

$$\begin{gathered}\Theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle \geq 0 \quad \forall x \in C,\\ y_n = P_C(z_n - \lambda_n A z_n),\\ x_{n+1} = \alpha_n f(W_n x_n) + \beta_n x_n + \gamma_n W_n P_C(y_n - \lambda_n A y_n) \quad \forall n \geq 0,\end{gathered} \tag{3.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $[0, 2\beta]$.

Now we study the strong convergence of the hybrid iterative algorithm (3.3).

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3) and let $T_1, T_2, \ldots$ be an infinite family of nonexpansive mappings of C into itself. Let $A : C \to H$ be a β-inverse-strongly monotone mapping such that $\cap_{n=1}^{\infty} F(T_n) \cap VI(A, C) \cap \Omega \neq \varnothing$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1, n \geq 0$. Assume that*

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*

  (a) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$,*
  (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*
  (c) *for each fixed $y \in C$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$K : C \to \mathbf{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$;*

(iii) *for each $x \in C$; there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0; \tag{3.4}$$

(iv) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, $\lambda_n \in [a, b] \subset (0, 2\beta)$, *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Let f be a contraction of C into itself and given $x_0 \in C$ arbitrarily. Then the sequence $\{x_n\}$ generated by (3.3) converges strongly to $x^* = P_\Gamma f(x^*)$, where $\Gamma = \cap_{n=1}^{\infty} F(T_n) \cap VI(A, C) \cap \Omega$ provided that S_r is firmly nonexpansive.

Proof. We first note that f is a contraction with coefficient $\alpha \in (0,1)$. Then $\|P_\Gamma f(x) - P_\Gamma f(y)\| \leq \|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$. Therefore $P_\Gamma f$ is a contraction of C into itself which implies that there exists a unique element $x^* \in C$ such that $x^* = P_\Gamma f(x^*)$.

Next we divide the following proofs into several steps.

Step 1 ($\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded)**.** *Let $x^* \in \Gamma$. From the definition of S_r, we know that $z_n = S_r x_n$. It follows that*

$$\left\|z_n - x^*\right\| = \left\|S_r x_n - S_r x^*\right\| \leq \left\|x_n - x^*\right\|. \tag{3.5}$$

For all $x, y \in C$ *and* $\lambda_n \in [0, 2\beta]$, *we note that*

$$\begin{aligned}\|(I-\lambda_n A)x-(I-\lambda_n A)y\|^2 &= \|(x-y)-\lambda_n(Ax-Ay)\|^2 \\ &= \|x-y\|^2-2\lambda_n\langle Ax-Ay, x-y\rangle+\lambda_n^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2+\lambda_n(\lambda_n-2\beta)\|Ax-Ay\|^2,\end{aligned} \tag{3.6}$$

which implies that $I-\lambda_n A$ *is nonexpansive.*

Set $u_n = P_C(y_n-\lambda_n Ay_n)$ *for all* $n \geq 0$. *From* (2.4), *we have that* $x^* = P_C(x^*-\lambda_n Ax^*)$. *It follows from* (3.6) *that*

$$\begin{aligned}\|y_n-x^*\| &= \|P_C(z_n-\lambda_n Az_n)-P_C(x^*-\lambda_n Ax^*)\| \\ &\leq \|(z_n-\lambda_n Az_n)-(x^*-\lambda_n Ax^*)\| \leq \|z_n-x^*\| \leq \|x_n-x^*\|, \\ \|u_n-x^*\| &= \|P_C(y_n-\lambda_n Ay_n)-P_C(x^*-\lambda_n Ax^*)\| \\ &\leq \|(y_n-\lambda_n Ay_n)-(x^*-\lambda_n Ax^*)\| \leq \|y_n-x^*\| \leq \|x_n-x^*\|.\end{aligned} \tag{3.7}$$

Hence we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n(f(W_n x_n)-x^*)+\beta_n(x_n-x^*)+\gamma_n(W_n u_n-x^*)\| \\ &\leq \alpha_n\|f(W_n x_n)-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|W_n u_n-x^*\| \\ &\leq \alpha_n\|f(W_n x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|u_n-x^*\| \\ &\leq \alpha_n\beta\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|x_n-x^*\| \\ &= (1-\beta)\alpha_n\frac{\|f(x^*)-x^*\|}{1-\beta}+[1-(1-\beta)\alpha_n]\|x_n-x^*\| \\ &\leq \max\left\{\|x_n-x^*\|, \frac{\|f(x^*)-x^*\|}{1-\beta}\right\} \leq \max\left\{\|x_0-x^*\|, \frac{\|f(x^*)-x^*\|}{1-\beta}\right\}.\end{aligned} \tag{3.8}$$

Therefore $\{x_n\}$ *is bounded, so are* $\{y_n\}$ *and* $\{z_n\}$.

Step 2 ($\|x_{n+1}-x_n\| \to 0$)**.** *Setting* $x_{n+1} = \beta_n x_n+(1-\beta_n)V_n$ *for all* $n \geq 0$. *It follows that*

$$\begin{aligned}V_{n+1}-V_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}f(W_{n+1}x_{n+1})+\gamma_{n+1}W_{n+1}u_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(W_n x_n)+\gamma_n W_n u_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}f(W_{n+1}x_{n+1})}{1-\beta_{n+1}}-\frac{\alpha_n f(W_n x_n)}{1-\beta_n}+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(W_{n+1}u_{n+1}-W_{n+1}u_n) \\ &\quad+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)W_{n+1}u_n+\frac{\gamma_n}{1-\beta_n}(W_{n+1}u_n-W_n u_n),\end{aligned} \tag{3.9}$$

which implies that

$$\begin{aligned}\|V_{n+1}-V_n\| \le{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|f(W_{n+1}x_{n+1})\|+\|W_{n+1}u_n\|\right)\\&+\frac{\alpha_n}{1-\beta_n}\left(\|f(W_nx_n)\|+\|W_{n+1}u_n\|\right)\\&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|u_{n+1}-u_n\|+\frac{\gamma_n}{1-\beta_n}\|W_{n+1}u_n-W_nu_n\|.\end{aligned}\tag{3.10}$$

Now we estimate $\|u_{n+1}-u_n\|$ *and* $\|W_{n+1}u_n-W_nu_n\|$.

From (3.1), *since* T_i *and* $U_{n,i}$ *are nonexpansive, we have*

$$\begin{aligned}\|W_{n+1}u_n-W_nu_n\| &= \|\xi_1T_1U_{n+1,2}u_n-\xi_1T_1U_{n,2}u_n\|\\&\le \xi_1\|U_{n+1,2}u_n-U_{n,2}u_n\|\\&= \xi_1\|\xi_2T_2U_{n+1,3}u_n-\xi_2T_2U_{n,3}u_n\|\\&\le \xi_1\xi_2\|U_{n+1,3}u_n-U_{n,3}u_n\|\\&\le\cdots\\&\le \xi_1\xi_2\cdots\xi_n\|U_{n+1,n+1}u_n-U_{n,n+1}u_n\|\\&\le M\prod_{i=1}^{n}\xi_i,\end{aligned}\tag{3.11}$$

where M *is a constant such that* $\sup\{\|U_{n+1,n+1}u_n-U_{n,n+1}u_n\|, n\ge 0\}\le M$.

At the same time, we observe that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|P_C(z_{n+1}-\lambda_{n+1}Az_{n+1})-P_C(z_n-\lambda_nAz_n)\|\\&\le \|(z_{n+1}-\lambda_{n+1}Az_{n+1})-(z_n-\lambda_nAz_n)\|\\&= \|(z_{n+1}-\lambda_{n+1}Az_{n+1})-(z_n-\lambda_{n+1}Az_n)+(\lambda_n-\lambda_{n+1})Az_n\|\\&\le \|(z_{n+1}-\lambda_{n+1}Az_{n+1})-(z_n-\lambda_{n+1}Az_n)\|+|\lambda_n-\lambda_{n+1}|\|Az_n\|\\&\le \|z_{n+1}-z_n\|+|\lambda_n-\lambda_{n+1}|\|Az_n\|,\\ \|u_{n+1}-u_n\| &= \|P_C(y_{n+1}-\lambda_{n+1}Ay_{n+1})-P_C(y_n-\lambda_nAy_n)\|\\&\le \|(y_{n+1}-\lambda_{n+1}Ay_{n+1})-(y_n-\lambda_nAy_n)\|\\&= \|(y_{n+1}-\lambda_{n+1}Ay_{n+1})-(y_n-\lambda_{n+1}Ay_n)+(\lambda_n-\lambda_{n+1})Ay_n\|\\&\le \|y_{n+1}-y_n\|+|\lambda_n-\lambda_{n+1}|\|Ay_n\|\\&\le \|z_{n+1}-z_n\|+|\lambda_n-\lambda_{n+1}|(\|Ay_n\|+\|Az_n\|).\end{aligned}\tag{3.12}$$

Since $z_n = S_r x_n$ *and* $z_{n+1} = S_r x_{n+1}$, *from the nonexpansivity of* S_r, *we get*

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\|. \tag{3.13}$$

Substituting (3.11)–(3.13) *into* (3.10), *we have*

$$\begin{aligned}\|V_{n+1} - V_n\| - \|x_{n+1} - x_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|f(W_{n+1}x_{n+1})\| + \|W_{n+1}u_n\|\right) \\ &+ \frac{\alpha_n}{1-\beta_n}\left(\|f(W_n x_n)\| + \|W_{n+1}u_n\|\right) + M\prod_{i=1}^{n}\xi_i \\ &+ |\lambda_n - \lambda_{n+1}|\left(\|Ay_n\| + \|Az_n\|\right).\end{aligned} \tag{3.14}$$

Since $\alpha_n \to 0$, $\lambda_{n+1} - \lambda_n \to 0$, *and* $\xi_i \in [a, b]$, *we have*

$$\limsup_{n\to\infty}\left(\|V_{n+1} - V_n\| - \|x_{n+1} - x_n\|\right) \leq 0. \tag{3.15}$$

Hence by Lemma 2.2, we have

$$\lim_{n\to\infty}\|V_n - x_n\| = 0. \tag{3.16}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.17}$$

Step 3 ($\|u_n - Wu_n\| \to 0$). *Note that* $x_{n+1} - x_n = \alpha_n(f(W_n x_n) - x_n) + \gamma_n(W_n u_n - x_n)$. *Then we have*

$$\|x_n - W_n u_n\| \leq \frac{1}{\gamma_n}\left\{\|x_n - x_{n+1}\| + \alpha_n\|f(W_n x_n) - x_n\|\right\} \longrightarrow 0. \tag{3.18}$$

For $x^* \in \Gamma$, *noting that* S_r *is firmly nonexpansive, we have*

$$\begin{aligned}\|z_n - x^*\|^2 &= \|S_r x_n - S_r x^*\|^2 \\ &\leq \langle S_r x_n - S_r x^*, x_n - x^*\rangle \\ &= \langle z_n - x^*, x_n - x^*\rangle \\ &= \frac{1}{2}\left(\|z_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - z_n\|^2\right),\end{aligned} \tag{3.19}$$

and hence

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2. \tag{3.20}$$

So, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|f(W_n x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|W_n u_n - x^*\|^2 \\ &\leq \alpha_n\|f(W_n x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\ &\leq \alpha_n\|f(W_n x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|z_n - x^*\|^2 \\ &\leq \alpha_n\|f(W_n x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\left(\|x_n - x^*\|^2 - \|x_n - z_n\|^2\right) \\ &\leq \alpha_n\|f(W_n x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n\|x_n - z_n\|^2,\end{aligned} \tag{3.21}$$

that is,

$$\begin{aligned}\|x_n - z_n\|^2 &\le \frac{1}{\gamma_n}\{\alpha_n\|f(W_nx_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\} \\ &\le \frac{1}{\gamma_n}\{\alpha_n\|f(W_nx_n) - x^*\|^2 + \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|)\} \\ &\longrightarrow 0.\end{aligned} \tag{3.22}$$

From (3.6)*, we obtain that*

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|y_n - x^*\|^2 \\ &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n[\|(z_n - \lambda_nAz_n) - (x^* - \lambda_nAx^*)\|^2] \\ &\le \alpha_n\|f(W_nx_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\{\|z_n-x^*\|^2+\lambda_n(\lambda_n - 2\beta)\|Az_n - Ax^*\|^2\} \\ &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_na(b - 2\beta)\|Az_n - Ax^*\|^2.\end{aligned} \tag{3.23}$$

Then we have

$$-\gamma_na(b - 2\beta)\|Az_n - Ax^*\|^2 \le \alpha_n\|f(W_nx_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \longrightarrow 0, \tag{3.24}$$

which implies that

$$\lim_{n\to\infty}\|Az_n - Ax^*\| = 0. \tag{3.25}$$

We note that

$$\begin{aligned}\|y_n - x^*\|^2 &= \|P_C(z_n - \lambda_nAz_n) - P_C(x^* - \lambda_nAx^*)\|^2 \\ &\le \langle(z_n - \lambda_nAz_n) - (x^* - \lambda_nAx^*), y_n - x^*\rangle \\ &= \frac{1}{2}\Big\{\|(z_n - \lambda_nAz_n) - (x^* - \lambda_nAx^*)\|^2 + \|y_n - x^*\|^2 \\ &\qquad - \|(z_n - \lambda_nAz_n) - (x^* - \lambda_nAx^*) - (y_n - x^*)\|^2\Big\} \\ &\le \frac{1}{2}\Big\{\|z_n - x^*\|^2 + \|y_n - x^*\|^2 - \|(z_n - y_n) - \lambda_n(Az_n - Ax^*)\|^2\Big\} \\ &= \frac{1}{2}\Big\{\|z_n-x^*\|^2+\|y_n-x^*\|^2-\|z_n-y_n\|^2+2\lambda_n\langle Az_n-Ax^*, z_n-y_n\rangle-\lambda_n^2\|Az_n-Ax^*\|^2\Big\}.\end{aligned} \tag{3.26}$$

Then we derive

$$\begin{aligned}\|y_n - x^*\|^2 &\le \|z_n - x^*\|^2 - \|z_n - y_n\|^2 + 2\lambda_n\langle Az_n - Ax^*, z_n - y_n\rangle - \lambda_n^2\|Az_n - Ax^*\|^2 \\ &\le \|x_n - x^*\|^2 - \|z_n - y_n\|^2 + 2\lambda_n\langle Az_n - Ax^*, z_n - y_n\rangle.\end{aligned} \tag{3.27}$$

Hence

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\\ &\le \alpha_n\|f(W_nx_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\Big\{\|x_n-x^*\|^2-\|z_n-y_n\|^2+2\lambda_n\langle Az_n-Ax^*,z_n-y_n\rangle\Big\},\end{aligned}\tag{3.28}$$

which implies that

$$\|z_n-y_n\|\le\frac{1}{\gamma_n}\Big\{\alpha_n\|f(W_nx_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\gamma_n\lambda_n\|Az_n-Ax^*\|\|z_n-y_n\|\Big\}\longrightarrow 0.\tag{3.29}$$

Since $\|y_n-u_n\|=\|P_C(z_n-\lambda_nAz_n)-P_C(y_n-\lambda_nAy_n)\|\le\|z_n-y_n\|$, *we have*

$$\begin{aligned}\|Wu_n-u_n\| &\le \|Wu_n-W_nu_n\|+\|W_nu_n-u_n\|\\ &\le \|Wu_n-W_nu_n\|+\|W_nu_n-x_n\|+\|x_n-z_n\|+\|z_n-y_n\|+\|y_n-u_n\|\\ &\le \|Wu_n-W_nu_n\|+\|W_nu_n-x_n\|+\|x_n-z_n\|+2\|z_n-y_n\|.\end{aligned}\tag{3.30}$$

Combining the above inequality, (3.18)–(3.29)*, and Remark 3.3, we have*

$$\lim_{n\to\infty}\|Wu_n-u_n\|=0.\tag{3.31}$$

Step 4 ($\lim_{n\to\infty}\langle f(x^*)-x^*,x_n-x^*\rangle$, where $x^*=P_\Gamma f(x^*)$)**.** *To show the above inequality, we can choose a subsequence* $\{u_{n_i}\}$ *of* $\{u_n\}$ *such that*

$$\lim_{j\to\infty}\langle f(x^*)-x^*,u_{n_j}-x^*\rangle=\limsup_{n\to\infty}\langle f(x^*)-x^*,u_n-x^*\rangle.\tag{3.32}$$

Since $\{u_{n_j}\}$ *is bounded, there exists a subsequence* $\{u_{n_{ji}}\}$ *of* $\{u_{n_j}\}$ *which converges weakly to* w*. Without loss of generality, we can assume that* $u_{n_j}\rightharpoonup w$*. From* $\|Wu_n-u_n\|\to 0$*, we obtain* $Wu_{n_j}\rightharpoonup w$*.*

First, we show $w\in F(W)=\cap_{n=1}^{\infty}F(T_n)$*. Assume that* $\omega\notin F(W)$*. Since* $u_{n_j}\rightharpoonup w$ *and* $w\ne Ww$*, by Opial's condition, we have*

$$\begin{aligned}\liminf_{j\to\infty}\|u_{n_j}-w\| &< \liminf_{j\to\infty}\|u_{n_j}-Ww\|\\ &\le \liminf_{j\to\infty}\big(\|u_{n_j}-Wu_{n_j}\|+\|Wu_{n_j}-Ww\|\big)\\ &\le \liminf_{j\to\infty}\|u_{n_j}-w\|,\end{aligned}\tag{3.33}$$

which is a contradiction. Hence we get $w\in F(W)$*. By the same argument as that in the proof of [21, Theorem 3.1], we can prove that* $w\in VI(A,C)$*; and by the same argument as that in the proof of [14, Theorem 4.1], we also can prove that* $w\in\Omega$*. Hence* $w\in\Gamma$*.*

Since $x^*=P_\Gamma f(x^*)\in\Gamma$ *and* $\|u_n-x_n\|\to 0$*, we have*

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*)-x^*,x_n-x^*\rangle &= \lim_{j\to\infty}\langle f(x^*)-x^*,x_{n_j}-x^*\rangle\\ &= \lim_{j\to\infty}\langle f(x^*)-x^*,u_{n_j}-x^*\rangle=\langle f(x^*)-x^*,w-x^*\rangle\le 0.\end{aligned}\tag{3.34}$$

Step 5 ($x_n \to x^*$, where $x^* = P_\Gamma f(x^*)$)**.** *From* (3.3), *we have*

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|\beta_n(x_n - x^*) + \gamma_n(W_n u_n - x^*)\|^2 + 2\alpha_n\langle f(W_n x_n) - x^*, x_{n+1} - x^*\rangle \\
&\le [\beta_n\|x_n - x^*\| + \gamma_n\|u_n - x^*\|]^2 + 2\alpha_n\langle f(W_n x_n) - f(x^*), x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\|f(W_n x_n) - f(x^*)\|\|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha\alpha_n\|x_n - x^*\|\|x_{n+1} - x^*\| + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)^2\|x_n - x^*\|^2 + \alpha\alpha_n(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle,
\end{aligned} \tag{3.35}$$

that is,

$$\|x_{n+1} - x^*\|^2 \le \left[1 - \frac{2(1-\alpha)}{1-\alpha\alpha_n}\alpha_n\right]\|x_n - x^*\|^2 + \frac{2(1-\alpha)}{1-\alpha\alpha_n}\alpha_n\left\{\frac{\alpha_n}{2(1-\alpha)}\|x_n - x^*\|^2 + \frac{1}{1-\alpha}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle\right\}. \tag{3.36}$$

It is easy to see that $\sum_{n=0}^{\infty}(2(1-\alpha)/(1-\alpha\alpha_n))\alpha_n = \infty$ *and*

$$\limsup_{n\to\infty}\left\{\frac{\alpha_n}{2(1-\alpha)}\|x_n - x^*\|^2 + \frac{1}{1-\alpha}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle\right\} \le 0. \tag{3.37}$$

Applying Lemma 2.3 and (3.34) *to* (3.36), *we conclude that* $x_n \to x^*$ *as* $n \to \infty$. *This completes the proof.* □

Concerning S_r, we give the following remark.

Remark 3.6. For each $x_1, x_2 \in C$, we denote $u_1 = S_r(x_1)$ and $u_2 = S_r(x_2)$. Then for all $y \in C$, we have

$$r[\Theta(u_1, y) + \varphi(y) - \varphi(u_1)] + \langle K'(u_1) - K'(x_1), \eta(y, u_1)\rangle \ge 0, \tag{3.38}$$

$$r[\Theta(u_2, y) + \varphi(y) - \varphi(u_2)] + \langle K'(u_2) - K'(x_2), \eta(y, u_2)\rangle \ge 0. \tag{3.39}$$

Taking $y = u_2$ in (3.38) and $y = u_1$ in (3.39), and adding up these two inequalities, we obtain

$$\begin{aligned}
&r[\Theta(u_1, u_2) + \varphi(u_2) - \varphi(u_1)] + \langle K'(u_1) - K'(x_1), \eta(u_2, u_1)\rangle \\
&\quad + r[\Theta(u_2, u_1) + \varphi(u_1) - \varphi(u_2)] + \langle K'(u_2) - K'(x_2), \eta(u_1, u_2)\rangle \ge 0.
\end{aligned} \tag{3.40}$$

Note that $\eta(u_1, u_2) + \eta(u_2, u_1) = 0$ and $\Theta(u_1, u_2) + \Theta(u_2, u_1) \le 0$. Hence from (3.40), we deduce

$$\langle K'(x_1) - K'(u_1), \eta(u_1, u_2)\rangle + \langle K'(u_2) - K'(x_2), \eta(u_1, u_2)\rangle \ge 0, \tag{3.41}$$

which implies that

$$\langle K'(x_1) - K'(x_2), \eta(u_1, u_2)\rangle \geq \langle K'(u_1) - K'(u_2), \eta(u_1, u_2)\rangle. \tag{3.42}$$

Since $K' : C \to H$ is η-strongly monotone with constant $\mu > 0$, then from (3.42), we conclude that

$$\langle K'(x_1) - K'(x_2), \eta(u_1, u_2)\rangle \geq \mu \|u_1 - u_2\|^2. \tag{3.43}$$

Take $K(x) = \|x\|^2/2$, $\eta(y, x) = y - x$, and $\mu = 1$. Then from (3.43), we have

$$\langle x_1 - x_2, u_1 - u_2\rangle \geq \|u_1 - u_2\|^2. \tag{3.44}$$

This indicates that S_r is firmly nonexpansive.

Corollary 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Let $A : C \to H$ be a β-inverse-strongly monotone mapping such that $VI(A, C) \cap \Omega \neq \varnothing$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1, n \geq 0$. Assume that*

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*

  (a) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$,*

  (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*

  (c) *for each fixed $y \in C$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$K : C \to \mathbf{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$;*

(iii) *for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that, for any $y \in C \setminus D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0; \tag{3.45}$$

(iv) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, $\lambda_n \in [a,b] \subset (0,2\beta)$, *and* $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0$.

Let f *be a contraction of* C *into itself and given* $x_0 \in C$ *arbitrarily. Let the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *be generated iteratively by*

$$\begin{aligned}&\Theta(z_n,x)+\varphi(x)-\varphi(z_n)+\frac{1}{r}\langle K'(z_n)-K'(x_n),\eta(x,z_n)\rangle \ge 0 \quad \forall x\in C,\\ &y_n = P_C(z_n-\lambda_n A z_n),\\ &x_{n+1} = \alpha_n f(x_n)+\beta_n x_n+\gamma_n P_C(y_n-\lambda_n A y_n) \quad \forall n\ge 0.\end{aligned} \tag{3.46}$$

Then the sequence $\{x_n\}$ *generated by* (3.46) *converges strongly to* $x^* = P_\Gamma f(x^*)$, *where* $\Gamma = VI(A,C)\cap\Omega$ *provided that* S_r *is firmly nonexpansive.*

Proof. Take $T_n x = x$ for all $n = 1,2,\ldots$, and for all $x \in C$ in (3.1). Then $W_n x = x$ for all $x \in C$. The conclusion follows immediately from Theorem 3.5. This completes the proof. □

Corollary 3.8. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T_1, T_2, \ldots$ *be an infinite family of nonexpansive mappings of* C *into itself. Let* $A : C \to H$ *be a* β*-inverse-strongly monotone mapping such that* $\cap_{n=1}^{\infty}F(T_n)\cap VI(A,C) \ne \emptyset$. *Suppose* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are three sequences in* $(0,1)$ *with* $\alpha_n+\beta_n+\gamma_n = 1$, $n \ge 0$. *Assume that*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\lambda_n \in [a,b] \subset (0,2\beta)$ *and* $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0$.

Let f be a contraction of C into itself and given $x_0 \in C$ arbitrarily. Then the sequence $\{x_n\}$, generated iteratively by

$$\begin{aligned}&y_n = P_C(x_n-\lambda_n A x_n),\\ &x_{n+1} = \alpha_n f(W_n x_n)+\beta_n x_n+\gamma_n W_n P_C(y_n-\lambda_n A y_n) \quad \forall n\ge 0,\end{aligned} \tag{3.47}$$

converges strongly to $x^* = P_\Gamma f(x^*)$, where $\Gamma = \cap_{n=1}^{\infty}F(T_n)\cap VI(A,C)$.

Proof. Set $\varphi(x) = 0$ and $\Theta(x,y) = 0$ for all $x,y \in C$ and put $r = 1$. Take $K(x) = \|x\|^2/2$ and $\eta(y,x) = y-x$ for all $x,y \in C$. Then we have $z_n = P_C x_n = x_n$. Hence the conclusion follows. This completes the proof. □

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