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Research Article

Hybrid Iterative Methods for Convex Feasibility Problems and Fixed Point Problems of Relatively Nonexpansive Mappings in Banach Spaces

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The convex feasibility problem (CFP) of finding a point in the nonempty intersection $\bigcap_{i=1}^{N} C_i$ is considered, where $N \geqslant 1$ is an integer and the C_i's are assumed to be convex closed subsets of a Banach space E. By using hybrid iterative methods, we prove theorems on the strong convergence to a common fixed point for a finite family of relatively nonexpansive mappings. Then, we apply our results for solving convex feasibility problems in Banach spaces.

1. Introduction

We are concerned with the *convex feasibility problem* (CFP)

$$\text{finding an } x \in \bigcap_{i=1}^{N} C_i, \tag{1.1}$$

where $N \geqslant 1$ is an integer, and $C_1, \ldots, C_N$ are intersecting closed convex subsets of a Banach space E. This problem is a frequently appearing problem in diverse areas of mathematical and physical sciences. There is a considerable investigation on (CFP) in the framework of Hilbert spaces which captures applications in various disciplines such as image restoration [1–4], computer tomography [5], and radiation theraphy treatment planning [6]. In computer tomography with limited data, in which an unknown image has to be reconstructed from a priori knowledge and from measured results, each piece of information gives a constraint which in turn, gives rise to a convex set C_i to which the unknown image should belong (see [7]). The advantage of a Hilbert space H is that the (nearest point) projection P_K onto a closed convex subset K of H is nonexpansive (i.e., $\|P_K x - P_K y\| \leqslant \|x - y\|$, $x, y \in H$).

So projection methods have dominated in the iterative approaches to (CFP) in Hilbert spaces; see [6, 8–11] and the references therein. In 1993, Kitahara and Takahashi [12] deal with the convex feasibility problem by convex combinations of sunny nonexpansive retractions in uniformly convex Banach spaces (see also Takahashi and Tamura [13], O'Hara et al. [14], and Chang et al. [15] for the previous results on this subject). It is known that if C is a nonempty closed convex subset of a smooth, reflexive, and strictly convex Banach space E, then the *generalized projection* Π_C (see, Alber [16] or Kamimura and Takahashi [17]) from E onto C is relatively nonexpansive, whereas the metric projection P_C from E onto C is not generally nonexpansive. Our purpose in the present paper is to obtain an analogous result for a finite family of relatively nonexpansive mappings in Banach spaces. This notion was originally introduced by Butnariu et al. [18]. Recently, Matsushita and Takahashi [19] reformulated the definition of the notion and obtained weak and strong convergence theorems to approximate a fixed point of a single relatively nonexpansive mapping. Motivated by Nakajo and Takahashi [20], Matsushita and Takahashi [21] studied the strong convergence of the sequence $\{x_n\}$ generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geqslant 0\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{1.2}$$

where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$, T is a relatively nonexpansive mapping from C into itself, and $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.

Very recently, Plubtieng and Ungchittrakool [22] studied the strong convergence to a common fixed point of two relatively nonexpansive mappings of the sequence $\{x_n\}$ generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ z_n &= J^{-1}(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geqslant 0\}, \\ x_{n+1} &= P_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{1.3}$$

where J is the duality mapping on E, and $P_F(\cdot)$ is the generalized projection from C onto $F := F(S) \cap F(T)$.

We note that the block iterative method is a method which often used by many authors to solve the convex feasibility problem (CFP) (see, [23, 24], etc.). In 2008, Plubtieng and Ungchittrakool [25] established strong convergence theorems of block iterative methods for a finite family of relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. In this paper, we introduce the following iterative process by using the shrinking method proposed, whose studied by Takahashi et al. [26], which is different from the method in [25]. Let C be a closed convex subset of E and for each $i = 1, 2, \ldots, N$, let $T_i : C \to C$ be a relatively nonexpansive mapping such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Define $\{x_n\}$ in the two following ways:

$$
\begin{aligned}
x_0 &\in E, \quad C_1 = C, \quad x_1 = \Pi_{C_1} x_0, \\
y_n &= J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J z_n), \\
z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \sum_{i=1}^{N} \beta_n^{(i+1)} J T_i x_n\right), \\
C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\
x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned}
\tag{1.4}
$$

and

$$
\begin{aligned}
x_0 &\in C, \\
y_n &= J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J z_n), \\
z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \sum_{i=1}^{N} \beta_n^{(i+1)} J T_i x_n\right), \\
H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\
W_n &= \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \geqslant 0\}, \\
x_{n+1} &= \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned}
\tag{1.5}
$$

where $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0, 1]$, $\sum_{i=1}^{N+1} \beta_n^{(i)} = 1$ satisfy some appropriate conditions.

We will prove that both iterations (1.4) and (1.5) converge strongly to a common fixed point of $\bigcap_{i=1}^{N} F(T_i)$. Using this results, we also discuss the convex feasibility problem in Banach spaces. Moreover, we apply our results to the problem of finding a common zero of a finite family of maximal monotone operators and equilibrium problems.

Throughout the paper, we will use the notations:

(i) $\to$ for strong convergence and $\rightharpoonup$ for weak convergence;

(ii) $\omega_w(x_n) = \{x : \exists x_{n_r} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalized duality mapping J from E to E^* is defined by

$$
Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}
\tag{2.1}
$$

for $x \in E$.

A Banach space E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided that

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well known that ℓ^p and L^p $(1 < p < \infty)$ are uniformly convex and uniformly smooth; see Cioranescu [27] or Diestel [28]. We know that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [27, 29, 30]. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [27, 30] for more details. Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{2.3}$$

for all $x, y \in E$. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y,x) \leqslant (\|y\| + \|x\|)^2$,

(2) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle$,

(3) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$,

for all $x, y, z \in E$. Let E be a strictly convex, smooth, and reflexive Banach space, and let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [16]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.4}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping $V(x,\cdot) : E^* \to \mathbb{R}$ is a continuous and convex function from E^* into $\mathbb{R}$.

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, for any $x \in E$, there exists a point $x_0 \in C$ such that $\phi(x_0, x) = \min_{y\in C}\phi(y, x)$. The mapping $\Pi_C : E \to C$ defined by $\Pi_C x = x_0$ is called the *generalized projection* [16, 17, 31]. The following are well-known results. For example, see [16, 17, 31].

This section collects some definitions and lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Lemma 2.1 (see [27, 30, 32]). *If E is a strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$.*

Proof. It is sufficient to show that if $\phi(y, x) = 0$ then $x = y$. From (1), we have $\|x\| = \|y\|$. This implies $\langle y, Jx\rangle = \|y\|^2 = \|Jx\|^2$. From the definition of J, we have $Jx = Jy$. Since J is one-to-one, we have $x = y$. □

Lemma 2.2 (Kamimura and Takahashi [17]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let T be a mapping from C into itself, and let $F(T)$ be the set of all fixed points of T. Then a point $p \in C$ is said to be an *asymptotic fixed point* of T (see Reich [33]) if there exists a sequence $\{x_n\}$ in C converging weakly to p and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$ and we say that T is a *relatively nonexpansive mapping* if the following conditions are satisfied:

(R1) $F(T)$ is nonempty;

(R2) $\phi(u, Tx) \leqslant \phi(u, x)$ for all $u \in F(T)$ and $x \in C$;

(R3) $\widehat{F}(T) = F(T)$.

Lemma 2.3 (Alber [16], Alber and Reich [31], Kamimura and Takahashi [17]). *Let C be a nonempty closed convex subset of a smooth Banach space E, let $x \in E$, and let $x_0 \in C$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for all $y \in C$.*

Lemma 2.4 (Alber [16], Alber and Reich [31], Kamimura and Takahashi [17]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x)$ for all $y \in C$.*

Lemma 2.5. *Let E be a uniformly convex Banach space and let $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\left\|\sum_{i=1}^{N} \omega^{(i)} x_i\right\|^2 \leqslant \sum_{i=1}^{N} \omega^{(i)} \|x_i\|^2 - \omega^{(j)}\omega^{(k)} g(\|x_j - x_k\|), \quad \text{for any } j, k \in \{1, 2, \dots, N\}, \tag{2.5}$$

where $\{x_i\}_{i=1}^N \subset B_r(0)$ and $\{\omega^{(i)}\}_{i=1}^N \subset [0,1]$ with $\sum_{i=1}^N \omega^{(i)} = 1$.

Proof. It sufficient to show that

$$\left\|\sum_{i=1}^{N} \omega^{(i)} x_i\right\|^2 \leqslant \sum_{i=1}^{N} \omega^{(i)} \|x_i\|^2 - \omega^{(1)}\omega^{(2)} g(\|x_1 - x_2\|). \tag{2.6}$$

It is obvious that (2.6) holds for $N = 1, 2$ (see [34] for more details). Next, we assume that (2.6) is true for $N-1$. It remains to show that (2.6) holds for N. We observe that

$$\begin{aligned}
\left\|\sum_{i=1}^{N}\omega^{(i)}x_i\right\|^2 &= \left\|\omega^{(N)}x_N + \left(1-\omega^{(N)}\right)\left(\sum_{i=1}^{N-1}\frac{\omega^{(i)}}{1-\omega^{(N)}}x_i\right)\right\|^2 \\
&\leqslant \omega^{(N)}\left\|x_N\right\|^2 + \left(1-\omega^{(N)}\right)\left\|\sum_{i=1}^{N-1}\frac{\omega^{(i)}}{1-\omega^{(N)}}x_i\right\|^2 \\
&\leqslant \omega^{(N)}\left\|x_N\right\|^2 + \left(1-\omega^{(N)}\right)\left(\sum_{i=1}^{N-1}\frac{\omega^{(i)}}{1-\omega^{(N)}}\|x_i\|^2 - \frac{\omega^{(1)}\omega^{(2)}}{\left(1-\omega^{(N)}\right)^2}g\left(\left\|x_1-x_2\right\|\right)\right) \\
&= \sum_{i=1}^{N}\omega^{(i)}\left\|x_i\right\|^2 - \frac{\omega^{(1)}\omega^{(2)}}{\left(1-\omega^{(N)}\right)}g\left(\left\|x_1-x_2\right\|\right) \\
&\leqslant \sum_{i=1}^{N}\omega^{(i)}\left\|x_i\right\|^2 - \omega^{(1)}\omega^{(2)}g\left(\left\|x_1-x_2\right\|\right).
\end{aligned} \tag{2.7}$$

This completes the proof. □

Lemma 2.6. *Let C be a closed convex subset of a smooth Banach space E and let $x, y \in E$. Then the set $K := \{v \in C : \phi(v, y) \leqslant \phi(v, x)\}$ is closed and convex.*

Proof. As a matter of fact, the defining inequality in K is equivalent to the inequality

$$\left\langle v, 2(Jx - Jy)\right\rangle \leqslant \|x\|^2 - \|y\|^2. \tag{2.8}$$

This inequality is affine in v and hence the set K is closed and convex. □

3. Main result

In this section, we prove strong convergence theorems for finding a common fixed point of a finite family of relatively nonexpansive mappings in Banach spaces by using the hybrid method in mathematical programming.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N}$ be a finite family of relatively nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^{N} F(T_i)$ is nonempty and let $x_0 \in E$. For $C_1 = C$ and $x_1 = \Pi_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{aligned}
y_n &= J^{-1}\left(\alpha_n Jx_n + \left(1-\alpha_n\right)Jz_n\right), \\
z_n &= J^{-1}\left(\beta_n^{(1)}Jx_n + \sum_{i=1}^{N}\beta_n^{(i+1)}JT_i x_n\right), \\
C_{n+1} &= \left\{z \in C_n : \phi\left(z, y_n\right) \leqslant \phi\left(z, x_n\right)\right\}, \\
x_{n+1} &= \Pi_{C_{n+1}}x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0,1]$ *satisfy the following conditions:*

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 \leqslant \beta_n^{(i)} \leqslant 1$ *for all* $i = 1, 2, \ldots, N+1$, $\sum_{i=1}^{N+1} \beta_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$. *If either*

(a) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(i+1)} > 0$ *for all* $i = 1, 2, \ldots, N$ *or*

(b) $\lim_{n\to\infty} \beta_n^{(1)} = 0$ *and* $\liminf_{n\to\infty} \beta_n^{(k+1)} \beta_n^{(l+1)} > 0$ *for all* $i \neq j$, $k, l = 1, 2, \ldots, N$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* Π_F *is the generalized projection from* E *onto* F.

Proof. We first show by induction that $F \subset C_n$ for all $n \in \mathbb{N}$. $F \subset C_1$ is obvious. Suppose that $F \subset C_k$ for some $k \in \mathbb{N}$. Then, we have, for $u \in F \subset C_k$,

$$\begin{aligned}
\phi(u, y_k) &= \phi\left(u, J^{-1}\left(\alpha_k J x_k + (1-\alpha_k) J z_k\right)\right) = V\left(u, \alpha_k J x_k + (1-\alpha_k) J z_k\right) \\
&\leqslant \alpha_k V(u, J x_k) + (1-\alpha_k) V(u, J z_k) = \alpha_k \phi(u, x_k) + (1-\alpha_k) \phi(u, z_k), \\
\phi(u, z_k) &= V\left(u, \beta_k^{(1)} J x_k + \sum_{i=1}^{N} \beta_k^{(i+1)} J T_i x_k\right) \\
&\leqslant \beta_k^{(1)} V(u, J x_k) + \sum_{i=1}^{N} \beta_k^{(i+1)} V(u, J T_i x_k) \\
&\leqslant \phi(u, x_k).
\end{aligned} \tag{3.2}$$

It follow that

$$\phi(u, y_k) \leqslant \phi(u, x_k) \tag{3.3}$$

and hence $u \in C_{k+1}$. This implies that $F \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. Obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For $z \in C_k$, we note by Lemma 2.6 that C_{k+1} is closed and convex. Then for any $n \in \mathbb{N}$, C_n is closed and convex. This implies that $\{x_n\}$ is well-defined. From $x_n = \Pi_{C_n} x_0$, we have

$$\phi(x_n, x_0) \leqslant \phi(u, x_0) - \phi(u, x_n) \leqslant \phi(u, x_0) \quad \forall u \in C_n. \tag{3.4}$$

In particular, let $u \in F$, we have

$$\phi(x_n, x_0) \leqslant \phi(u, x_0) \quad \forall n \in \mathbb{N}. \tag{3.5}$$

Therefore $\phi(x_n, x_0)$ is bounded and hence $\{x_n\}$ is bounded by (1). From $x_n = \Pi_{C_n} x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) = \min_{y \in C_n} \phi(y, x_0) \leqslant \phi(x_{n+1}, x_0) \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. So there exists the limit of $\phi(x_n, x_0)$. By Lemma 2.4, we have

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x_0) \leqslant \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) = \phi(x_{n+1}, x_0) - \phi(x_n, x_0). \tag{3.7}$$

for each $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0$. Since $x_{n+1} \in C_{n+1}$ it follows from the definition of C_{n+1} that

$$\phi(x_{n+1}, y_n) \leqslant \phi(x_{n+1}, x_n) \quad \forall n \in \mathbb{N}. \tag{3.8}$$

Letting $n \to \infty$, we have $\lim_{n\to\infty}\phi(x_{n+1}, y_n) = 0$. By Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{3.10}$$

Since $\|Jx_{n+1} - Jy_n\| = \|Jx_{n+1} - \alpha_n Jx_n - (1-\alpha_n)Jz_n\| \geqslant (1-\alpha_n)\|Jx_{n+1} - Jz_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|$ for each $n \in \mathbb{N} \cup \{0\}$, we get that

$$\begin{aligned} \|Jx_{n+1} - Jz_n\| &\leqslant \frac{1}{1-\alpha_n}\left(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|\right) \\ &\leqslant \frac{1}{1-\alpha_n}\left(\|Jx_{n+1} - Jy_n\| + \|Jx_n - Jx_{n+1}\|\right). \end{aligned} \tag{3.11}$$

From (3.10) and $\limsup_{n\to\infty}\alpha_n < 1$, we have $\lim_{n\to\infty}\|Jx_{n+1} - Jz_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = \lim_{n\to\infty} \|J^{-1}(Jx_{n+1}) - J^{-1}(Jz_n)\| = 0. \tag{3.12}$$

From $\|x_n - z_n\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|$, we have $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

Next, we show that $\|x_n - T_i x_n\| \to 0$ for all $i = 1, 2, \ldots, N$. Since $\{x_n\}$ is bounded and $\phi(p, T_i x_n) \leqslant \phi(p, x_n)$ for all $i = 1, 2, \ldots, N$, where $p \in F$. We also obtain that $\{Jx_n\}$ and $\{JT_i x_n\}$ are bounded for all $i = 1, 2, \ldots, N$. Then there exists $r > 0$ such that $\{Jx_n\}, \{JT_i x_n\} \subset B_r(0)$ for all $i = 1, 2, \ldots, N$. Therefore Lemma 2.5 is applicable. Assume that (a) holds, we observe that

$$\begin{aligned}
\phi(p, z_n) &= \|p\|^2 - 2\left\langle p, \beta_n^{(1)} Jx_n + \sum_{i=1}^{N} \beta_n^{(i+1)} JT_i x_n \right\rangle + \left\| \beta_n^{(1)} Jx_n + \sum_{i=1}^{N} \beta_n^{(i+1)} JT_i x_n \right\|^2 \\
&\leqslant \|p\|^2 - 2\beta_n^{(1)} \langle p, Jx_n \rangle + \sum_{i=1}^{N} \beta_n^{(i+1)} \langle p, JT_i x_n \rangle + \beta_n^{(1)} \|x_n\|^2 + \sum_{i=1}^{N} \beta_n^{(i+1)} \|T_i x_n\|^2 \\
&\quad - \beta_n^{(1)} \beta_n^{(i+1)} g(\|Jx_n - JT_i x_n\|) \\
&= \beta_n^{(1)} (\|p\|^2 - 2\langle p, Jx_n \rangle + \|x_n\|^2) + \sum_{i=1}^{N} \beta_n^{(i+1)} (\|p\|^2 + 2\langle p, JT_i x_n \rangle + \|T_i x_n\|^2) \\
&\quad - \beta_n^{(1)} \beta_n^{(i+1)} g(\|Jx_n - JT_i x_n\|) \\
&= \beta_n^{(1)} \phi(p, x_n) + \sum_{i=1}^{N} \beta_n^{(i+1)} \phi(p, T_i x_n) - \beta_n^{(1)} \beta_n^{(i+1)} g(\|Jx_n - JT_i x_n\|) \\
&\leqslant \phi(p, x_n) - \beta_n^{(1)} \beta_n^{(i+1)} g(\|Jx_n - JT_i x_n\|)
\end{aligned} \tag{3.13}$$

and hence

$$\begin{aligned}
\beta_n^{(1)} \beta_n^{(i+1)} g(\|Jx_n - JT_i x_n\|) &\leqslant \phi(p, x_n) - \phi(p, z_n) \\
&= 2\langle p, z_n - x_n \rangle + (\|x_n\| + \|z_n\|)(\|x_n\| - \|z_n\|) \\
&\leqslant 2\|p\| \|z_n - x_n\| + (\|x_n\| + \|z_n\|)(\|x_n - z_n\|) \\
&\longrightarrow 0,
\end{aligned} \tag{3.14}$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous strictly increasing convex function with $g(0) = 0$ in Lemma 2.5. By (a), we have $\lim_{n\to\infty} g(\|Jx_n - JT_i x_n\|) = 0$ and then $\lim_{n\to\infty} \|Jx_n - JT_i x_n\| = 0$ for all $i = 1, 2, \ldots, N$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = \lim_{n\to\infty} \|J^{-1}(Jx_n) - J^{-1}(JT_i x_n)\| = 0, \tag{3.15}$$

for all $i = 1, 2, \dots, N$. If (b) holds, we get

$$
\begin{aligned}
\phi(p, z_n) &= \|p\|^2 - 2\left\langle p, \beta_n^{(1)} J x_n + \sum_{i=1}^{N} \beta_n^{(i+1)} J T_i x_n \right\rangle + \left\| \beta_n^{(1)} J x_n + \sum_{i=1}^{N} \beta_n^{(i+1)} J T_i x_n \right\|^2 \\
&\leqslant \|p\|^2 - 2\beta_n^{(1)} \langle p, J x_n\rangle + \sum_{i=1}^{N} \beta_n^{(i+1)} \langle p, J T_i x_n\rangle + \beta_n^{(1)} \|x_n\|^2 + \sum_{i=1}^{N} \beta_n^{(i+1)} \|T_i x_n\|^2 \\
&\quad - \beta_n^{(k+1)} \beta_n^{(l+1)} g(\|J T_k x_n - J T_l x_n\|) \\
&= \beta_n^{(1)} (\|p\|^2 - 2\langle p, J x_n\rangle + \|x_n\|^2) + \sum_{i=1}^{N} \beta_n^{(i+1)} (\|p\|^2 + 2\langle p, J T_i x_n\rangle + \|T_i x_n\|^2) \\
&\quad - \beta_n^{(k+1)} \beta_n^{(l+1)} g(\|J T_k x_n - J T_l x_n\|) \\
&= \beta_n^{(1)} \phi(p, x_n) + \sum_{i=1}^{N} \beta_n^{(i+1)} \phi(p, T_i x_n) - \beta_n^{(k+1)} \beta_n^{(l+1)} g(\|J T_k x_n - J T_l x_n\|) \\
&\leqslant \phi(p, x_n) - \beta_n^{(k+1)} \beta_n^{(l+1)} g(\|J T_k x_n - J T_l x_n\|)
\end{aligned}
\tag{3.16}
$$

and hence

$$
\begin{aligned}
\beta_n^{(k+1)} \beta_n^{(l+1)} g(\|J T_k x_n - J T_l x_n\|) &\leqslant \phi(p, x_n) - \phi(p, z_n) \\
&= 2\langle p, z_n - x_n\rangle + (\|x_n\| + \|z_n\|)(\|x_n\| - \|z_n\|) \\
&\leqslant 2\|p\| \|z_n - x_n\| + (\|x_n\| + \|z_n\|)(\|x_n - z_n\|) \\
&\longrightarrow 0.
\end{aligned}
\tag{3.17}
$$

Then by the same argument above, we have $\lim_{n\to\infty} \|T_k x_n - T_l x_n\| = 0$ for all $k, l = 1, 2, \dots, N$. Next, we observe that

$$
\begin{aligned}
\phi(T_k x_n, z_n) &= V\left(T_k x_n, \beta_n^{(1)} J x_n + \sum_{i=1}^{N} \beta_n^{(i+1)} J T_i x_n \right) \\
&\leqslant \beta_n^{(1)} V(T_k x_n, J x_n) + \sum_{i=1}^{N} \beta_n^{(i+1)} V(T_k x_n, J T_i x_n) \\
&= \beta_n^{(1)} \phi(T_k x_n, x_n) + \sum_{i=1}^{N} \beta_n^{(i+1)} \phi(T_k x_n, T_i x_n) \\
&\longrightarrow 0.
\end{aligned}
\tag{3.18}
$$

(as $\beta_n^{(1)} \to 0$). By Lemma 2.2, we have $\lim_{n\to\infty}\|T_k x_n - z_n\| = 0$ for all $k = 1,2,\dots,N$, and hence

$$\|T_i x_n - x_n\| \leqslant \|T_i x_n - z_n\| + \|z_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.19}$$

for all $i = 1,2,\dots,N$. Then $\omega_w(x_n) \subset \bigcap_{i=1}^N \widehat{F}(T_i) = \bigcap_{i=1}^N F(T_i) = F$.

Finally, we show that $x_n \to \Pi_F x_0$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v \in \omega_w(x_n) \subset F$. Put $w := \Pi_F x_0 \in F \subset C_{n_k}$, we observe that

$$\phi(x_{n_k}, x_0) = \phi(\Pi_{C_{n_k}} x_0, x_0) = \min_{y\in C_{n_k}} \phi(y, x_0) \leqslant \phi(w, x_0) = \min_{z\in F} \phi(z, x_0) \leqslant \phi(v, x_0). \tag{3.20}$$

Since $\phi(\cdot, x_0)$ is weakly lower semicontinuous, we obtain

$$\phi(v, x_0) \leqslant \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \leqslant \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \leqslant \phi(w, x_0) \leqslant \phi(v, x_0). \tag{3.21}$$

This implies that $v = w$ and $\lim_{k\to\infty}\|x_{n_k}\| = \|w\|$ and then the Kadec-Klee property of E yields $x_{n_k} \to w$. Since $\{x_{n_k}\}$ is an arbitrary, $x_n \to w$. This completes the proof. □

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{\Omega_i\}_{i=1}^N$ be a finite family of nonempty closed convex subset of C such that $\Omega := \bigcap_{i=1}^N \Omega_i$ is nonempty and let $x_0 \in E$. For $C_1 = C$ and $x_1 = \Pi_{C_1} x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{aligned} y_n &= J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) J z_n\right), \\ z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \sum_{i=1}^N \beta_n^{(i+1)} J \Pi_{\Omega_i} x_n\right), \\ C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n = 0,1,2,\dots, \end{aligned} \tag{3.22}$$

where $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0,1]$ satisfy the following conditions:

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 \leqslant \beta_n^{(i)} \leqslant 1$ *for all* $i = 1,2,\dots,N+1$, $\sum_{i=1}^{N+1} \beta_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$. *If either*

(a) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(i+1)} > 0$ *for all* $i = 1,2,\dots,N$ *or*

(b) $\lim_{n\to\infty} \beta_n^{(1)} = 0$ *and* $\liminf_{n\to\infty} \beta_n^{(k+1)} \beta_n^{(l+1)} > 0$ *for all* $i \neq j$, $k, l = 1,2,\dots,N$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection from E onto Ω.

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be a finite family of relatively nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^N F(T_i)$ is nonempty. Let a sequence $\{x_n\}$ defined by*

$$\begin{aligned} &x_0 \in C, \\ &y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ &z_n = J^{-1}\left(\beta_n^{(1)} J x_n + \sum_{i=1}^N \beta_n^{(i+1)} J T_i x_n\right), \\ &H_n = \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ &W_n = \{z \in C : \langle x_n - z, J x_0 - J x_n\rangle \geqslant 0\}, \\ &x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{3.23}$$

where $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0,1]$ satisfy the following conditions:

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 \leqslant \beta_n^{(i)} \leqslant 1$ *for all* $i = 1, 2, \ldots, N+1$, $\sum_{i=1}^{N+1} \beta_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$. *If either*

(a) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(i+1)} > 0$ *for all* $i = 1, 2, \ldots, N$ *or*

(b) $\lim_{n\to\infty} \beta_n^{(1)} = 0$ *and* $\liminf_{n\to\infty} \beta_n^{(k+1)} \beta_n^{(l+1)} > 0$ *for all* $i \neq j$, $k, l = 1, 2, \ldots, N$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from E onto F.

Proof. From the definition of H_n and W_n, it is obvious H_n and W_n are closed and convex for each $n \in \mathbb{N} \cup \{0\}$. Next, we show that $F \subset H_n \cap W_n$ for each $n \in \mathbb{N} \cup \{0\}$. Let $u \in F$ and let $n \in \mathbb{N} \cup \{0\}$. Then, as in the proof of Theorem 3.1, we have

$$\phi(u, z_n) \leqslant \phi(u, x_n) \tag{3.24}$$

for all $n \in \mathbb{N} \cup \{0\}$, and then $\phi(u, y_n) \leqslant \phi(u, x_n)$. Thus, we have $u \in H_n$. Therefore we obtain $F \subset H_n$ for each $n \in \mathbb{N} \cup \{0\}$. We note by [21, Proposion 2.4] that each $F(T_i)$ is closed and convex and so is F. Using the same argument presented in the proof of [21, Theorem 3.1; page 261-262], we have $F \subset H_n \cap W_n$ for each $n \in \mathbb{N} \cup \{0\}$, $\{x_n\}$ is well defined and bounded, and

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.25}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|J x_{n+1} - J y_n\| = \lim_{n\to\infty} \|J x_{n+1} - J x_n\| = 0. \tag{3.26}$$

As in the proof of Theorem 3.1, we also have that

$$\begin{aligned}\|Jx_{n+1}-Jz_n\| &\leqslant \frac{1}{1-\alpha_n}\left(\|Jx_{n+1}-Jy_n\|+\alpha_n\|Jx_n-Jx_{n+1}\|\right)\\ &\leqslant \frac{1}{1-\alpha_n}\left(\|Jx_{n+1}-Jy_n\|+\|Jx_n-Jx_{n+1}\|\right).\end{aligned}\tag{3.27}$$

From (3.26) and $\limsup_{n\to\infty}\alpha_n < 1$, we have $\lim_{n\to\infty}\|Jx_{n+1}-Jz_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-z_n\| = \lim_{n\to\infty}\|J^{-1}(Jx_{n+1})-J^{-1}(Jz_n)\| = 0.\tag{3.28}$$

From $\|x_n-z_n\| \leqslant \|x_n-x_{n+1}\|+\|x_{n+1}-z_n\|$ we have $\lim_{n\to\infty}\|x_n-z_n\| = 0$. By the same argument as in the proof of Theorem 3.1, we have $\{x_n\}$ converges strongly to $\Pi_F x_0$. □

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{\Omega_i\}_{i=1}^N$ be a finite family of nonempty closed convex subset of C such that $\Omega := \bigcap_{i=1}^N \Omega_i$ is nonempty. Let a sequence $\{x_n\}$ defined by*

$$\begin{aligned}x_0 &\in C,\\ y_n &= J^{-1}\left(\alpha_n Jx_n+(1-\alpha_n)Jz_n\right),\\ z_n &= J^{-1}\left(\beta_n^{(1)}Jx_n+\sum_{i=1}^N \beta_n^{(i+1)}J\Pi_{\Omega_i}x_n\right),\\ H_n &= \{z\in C:\phi(z,y_n)\leqslant\phi(z,x_n)\},\\ W_n &= \{z\in C:\langle x_n-z, Jx_0-Jx_n\rangle\geqslant 0\},\\ x_{n+1} &= \Pi_{H_n\cap W_n}x_0,\quad n=0,1,2,\ldots,\end{aligned}\tag{3.29}$$

where $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0,1]$ satisfy the following conditions:

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N}\cup\{0\}$ *and* $\limsup_{n\to\infty}\alpha_n < 1$,

(ii) $0 \leqslant \beta_n^{(i)} \leqslant 1$ *for all* $i = 1,2,\ldots,N+1$, $\sum_{i=1}^{N+1}\beta_n^{(i)} = 1$ *for all* $n \in \mathbb{N}\cup\{0\}$. *If either*

(a) $\liminf_{n\to\infty}\beta_n^{(1)}\beta_n^{(i+1)} > 0$ *for all* $i = 1,2,\ldots,N$ *or*

(b) $\lim_{n\to\infty}\beta_n^{(1)} = 0$ *and* $\liminf_{n\to\infty}\beta_n^{(k+1)}\beta_n^{(l+1)} > 0$ *for all* $i \neq j$, $k,l = 1,2,\ldots,N$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection from E onto Ω.

If $N = 2$, $T_1 = T$ and $T_2 = S$, then Theorem 3.3 reduces to the following corollary.

Corollary 3.5 (Plubtieng and Ungchittrakool [22, Theorem 3.1]). *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let S and T*

be two relatively nonexpansive mappings from C into itself with $F := F(S) \cap F(T)$ is nonempty. Let a sequence $\{x_n\}$ be defined by

$$\begin{aligned}
x_0 &= x \in C,\\
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n),\\
z_n &= J^{-1}(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n),\\
H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\},\\
W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geqslant 0\},\\
x_{n+1} &= P_{H_n \cap W_n} x, \quad n = 0, 1, 2, \dots,
\end{aligned} \tag{3.30}$$

with the following restrictions:

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 \leqslant \beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(3)} \leqslant 1$, $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$, $\lim_{n\to\infty} \beta_n^{(1)} = 0$ *and* $\liminf_{n\to\infty} \beta_n^{(2)} \beta_n^{(3)} > 0$.

Then the sequence $\{x_n\}$ converges strongly to $P_F x$, where P_F is the generalized projection from C onto F.

4. Applications

4.1. Maximal monotone operators

Let A be a multivalued operator from E to E^* with domain $D(A) = \{z \in E : Az \neq \varnothing\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$. An operator A is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geqslant 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}(0)$ is closed and convex. Let E be a reflexive, strictly convex and smooth Banach space, and let A be a monotone operator from E to E^*, we known from Rockafellar [35] that A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$. Let $J_r : E \to D(A)$ defined by $J_r = (J + rA)^{-1} J$ and such a J_r is called the resolvent of A. We know that J_r is a relatively nonexpansive; see [21] and $A^{-1}(0) = F(J_r)$ for all $r > 0$; see [30, 32] for more details.

Theorem 4.1. *Let E be a uniformly convex and uniformly smooth Banach space. Let $A_i \subset E \times E^*$ be a maximal monotone operator for each $i = 1, 2, \dots, N$ such that $\Lambda := \bigcap_{i=1}^N A_i^{-1}(0)$ is nonempty and let $x_0 \in E$. For $C_1 = E$, define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned}
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n),\\
z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \sum_{i=1}^{N} \beta_n^{(i+1)} J J_{r_i}^{A_i} x_n\right),\\
C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leqslant \phi(z, x_n)\},\\
x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n = 0, 1, 2, \dots,
\end{aligned} \tag{4.1}$$

where $J_{r_i}^{A_i}$ *is the resolvent of* A_i *with* $r_i > 0$ *for each* $i = 1, 2, \ldots, N$, *and* $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0, 1]$ *satisfy the following conditions:*

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 \leqslant \beta_n^{(i)} \leqslant 1$ *for all* $i = 1, 2, \ldots, N+1$, $\sum_{i=1}^{N+1} \beta_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$. *If either*

(a) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(i+1)} > 0$ *for all* $i = 1, 2, \ldots, N$ *or*

(b) $\lim_{n\to\infty} \beta_n^{(1)} = 0$ *and* $\liminf_{n\to\infty} \beta_n^{(k+1)} \beta_n^{(l+1)} > 0$ *for all* $i \neq j$, $k, l = 1, 2, \ldots, N$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_\Lambda x_0$, *where* Π_Λ *is the generalized projection from* E *onto* Λ.

Theorem 4.2. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* $A_i \subset E \times E^*$ *be a maximal monotone operator for each* $i = 1, 2, \ldots, N$ *such that* $\Lambda := \bigcap_{i=1}^{N} A_i^{-1}(0)$ *is nonempty. Let a sequence* $\{x_n\}$ *defined by*

$$
\begin{aligned}
x_0 &\in E, \\
y_n &= J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) J z_n\right), \\
z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \sum_{i=1}^{N} \beta_n^{(i+1)} J J_{r_i}^{A_i} x_n\right), \\
H_n &= \left\{z \in E : \phi(z, y_n) \leqslant \phi(z, x_n)\right\}, \\
W_n &= \left\{z \in E : \langle x_n - z, J x_0 - J x_n \rangle \geqslant 0\right\}, \\
x_{n+1} &= \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned}
\tag{4.2}
$$

where $J_{r_i}^{A_i}$ *is the resolvent of* A_i *with* $r_i > 0$ *for each* $i = 1, 2, \ldots, N$, *and* $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0, 1]$ *satisfy the following conditions:*

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 \leqslant \beta_n^{(i)} \leqslant 1$ *for all* $i = 1, 2, \ldots, N+1$, $\sum_{i=1}^{N+1} \beta_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$. *If either*

(a) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(i+1)} > 0$ *for all* $i = 1, 2, \ldots, N$ *or*

(b) $\lim_{n\to\infty} \beta_n^{(1)} = 0$ *and* $\liminf_{n\to\infty} \beta_n^{(k+1)} \beta_n^{(l+1)} > 0$ *for all* $i \neq j$, $k, l = 1, 2, \ldots, N$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_\Lambda x_0$, *where* Π_Λ *is the generalized projection from* E *onto* Λ.

4.2. Equilibrium problems

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leqslant 0$ for all $x, y \in C$;

(A3) f is upper-hemicontinuous, that is, for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f\left(tz + (1-t)x, y\right) \leqslant f(x, y); \tag{4.3}$$

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following result is in Blum and Oettli [36].

Lemma 4.3 (Blum and Oettli [36]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geqslant 0 \quad \forall y \in C. \tag{4.4}$$

The following result is in Takahashi and Zembayashi [37].

Lemma 4.4 (Takahashi and Zembayashi [37]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E and let $f : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geqslant 0, \quad \forall y \in C \right\} \tag{4.5}$$

for all $x \in E$. Then, the following hold;

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive-type mapping [38], that is, for any $x, y \in E$,*

$$\left\langle T_r x - T_r y, JT_r x - JT_r y\right\rangle \leqslant \left\langle T_r x - T_r y, Jx - Jy\right\rangle; \tag{4.6}$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex.*

Lemma 4.5 (Takahashi and Zembayashi [37]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E. let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r > 0$. Then for $x \in E$ and $q \in F(T_r)$,*

$$\phi\left(q, T_r x\right) + \phi\left(T_r x, x\right) \leqslant \phi(q, x). \tag{4.7}$$

Theorem 4.6. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $f_{(i)}$ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4)* *for each* $i = 1,2,\ldots,N$, *and* $\Theta := \bigcap_{i=1}^{N} EP(f_{(i)}) \neq \emptyset$, *and let* $x_0 \in E$. *For* $C_1 = C$ *and* $x_1 = \Pi_{C_1} x_0$, *define a sequence* $\{x_n\}$ *of* C *as follows:*

$$\begin{aligned}
&u_n^{(i)} \in C \quad \text{such that } f_{(i)}\left(u_n^{(i)}, y\right) \\
&\qquad + \frac{1}{r^{(i)}}\left\langle y - u_n^{(i)}, Ju_n^{(i)} - Jx_n\right\rangle \geqslant 0 \quad \forall y \in C, \text{ for each } i = 1,2,\ldots,N, \\
&y_n = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n)Jz_n\right), \\
&z_n = J^{-1}\left(\beta_n^{(1)} Jx_n + \sum_{i=1}^{N} \beta_n^{(i+1)} Ju_n^{(i)}\right), \\
&C_{n+1} = \left\{z \in C_n : \phi(z, y_n) \leqslant \phi(z, x_n)\right\}, \\
&x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n = 0,1,2,\ldots,
\end{aligned} \tag{4.8}$$

where $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0,1]$ *satisfy the following conditions:*

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 \leqslant \beta_n^{(i)} \leqslant 1$ *for all* $i = 1,2,\ldots,N+1$, $\sum_{i=1}^{N+1} \beta_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$. *If either*

(a) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(i+1)} > 0$ *for all* $i = 1,2,\ldots,N$ *or*

(b) $\lim_{n\to\infty} \beta_n^{(1)} = 0$ *and* $\liminf_{n\to\infty} \beta_n^{(k+1)} \beta_n^{(l+1)} > 0$ *for all* $i \neq j$, $k, l = 1,2,\ldots,N$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_\Theta x_0$, *where* Π_Θ *is the generalized projection from* E *onto* Θ.

Theorem 4.7. *Let* E *be a uniformly convex and uniformly smooth Banach space, and let* C *be a nonempty closed convex subset of* E. *Let* $f_{(i)}$ *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying (A1)–(A4) for each* $i = 1,2,\ldots,N$, *and* $\Theta := \bigcap_{i=1}^{N} EP(f_{(i)}) \neq \emptyset$. *Let a sequence* $\{x_n\}$ *defined by*

$$\begin{aligned}
&x_0 \in E, \\
&u_n^{(i)} \in C \quad \text{such that } f_{(i)}\left(u_n^{(i)}, y\right) \\
&\qquad + \frac{1}{r^{(i)}}\left\langle y - u_n^{(i)}, Ju_n^{(i)} - Jx_n\right\rangle \geqslant 0 \quad \forall y \in C, \text{ for each } i = 1,2,\ldots,N, \\
&y_n = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n)Jz_n\right), \\
&z_n = J^{-1}\left(\beta_n^{(1)} Jx_n + \sum_{i=1}^{N} \beta_n^{(i+1)} Ju_n^{(i)}\right), \\
&H_n = \left\{z \in E : \phi(z, y_n) \leqslant \phi(z, x_n)\right\}, \\
&W_n = \left\{z \in E : \left\langle x_n - z, Jx_0 - Jx_n\right\rangle \geqslant 0\right\}, \\
&x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0,1,2,\ldots,
\end{aligned} \tag{4.9}$$

where $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0,1]$ *and* $r^{(i)} > 0$ *for all* $i = 1, 2, \ldots, N$, *satisfy the following conditions:*

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 \leqslant \beta_n^{(i)} \leqslant 1$ *for all* $i = 1, 2, \ldots, N+1$, $\sum_{i=1}^{N+1} \beta_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$. *If either*

(a) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(i+1)} > 0$ *for all* $i = 1, 2, \ldots, N$ *or*

(b) $\lim_{n\to\infty} \beta_n^{(1)} = 0$ *and* $\liminf_{n\to\infty} \beta_n^{(k+1)} \beta_n^{(l+1)} > 0$ *for all* $i \neq j$, $k, l = 1, 2, \ldots, N$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_\Theta x_0$, *where* Π_Θ *is the generalized projection from* E *onto* Θ.

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