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Research Article

A Fixed Point Approach to the Stability of Quadratic Functional Equation with Involution

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Cădariu and Radu applied the fixed point method to the investigation of Cauchy and Jensen functional equations. In this paper, we will adopt the idea of Cădariu and Radu to prove the Hyers-Ulam-Rassias stability of the quadratic functional equation with involution.

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1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. Indeed, he proved that each solution of the inequality $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. Rassias [3] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \varepsilon \left(\|x\|^p + \|y\|^p \right), \tag{1.1}$$

and generalized the result of Hyers. Since then, the stability problems for several functional equations have been extensively investigated.

The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. The terminology can also be applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [4–9].

Let E_1 and E_2 be real vector spaces. If an additive function $\sigma : E_1 \to E_1$ satisfies $\sigma(\sigma(x)) = x$ for all $x \in E_1$, then σ is called an involution of E_1 (see [10, 11]). For a given involution $\sigma : E_1 \to E_1$, the functional equation

$$f(x+y) + f\big(x+\sigma(y)\big) = 2f(x) + 2f(y) \tag{1.2}$$

is called the quadratic functional equation with involution. According to [11, Corollary 8], a function $f : E_1 \to E_2$ is a solution of (1.2) if and only if there exists an additive function $A : E_1 \to E_2$ and a biadditive symmetric function $B : E_1 \times E_1 \to E_2$ such that $A(\sigma(x)) = A(x)$, $B(\sigma(x), y) = -B(x,y)$ and $f(x) = B(x,x) + A(x)$ for all $x \in E_1$.

Indeed, if we set $\sigma = I$ in (1.2), where $I : E_1 \to E_1$ denotes the identity function, then (1.2) reduces to the additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1.3}$$

On the other hand, if $\sigma = -I$ in (1.2), then (1.2) is transformed into the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.4}$$

Recently, Belaid et al. have proved the Hyers-Ulam-Rassias stability of the quadratic functional equation with involution (1.2) (see [10]).

In this paper, we will apply the fixed point method to prove the Hyers-Ulam-Rassias stability of the functional equation (1.2) for a large class of functions from a vector space into a complete β-normed space. We remark that Isac and Rassias [12] were the first to apply the Hyers-Ulam-Rassias stability approach for the proof of new fixed point theorems.

2. Preliminaries

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

(M_1) $d(x,y) = 0$, if and only if $x = y$;

(M_2) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(M_3) $d(x,z) \le d(x,y) + d(y,z)$, for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

We now introduce one of fundamental results of fixed point theory. For the proof, refer to [13]. For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers et al. [14].

Theorem 2.1. *Let (X,d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $0 < L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the followings are true:*

(a) *the sequence* $\{\Lambda^n x\}$ *converges to a fixed point* x^* *of* Λ;

(b) x^* *is the unique fixed point of* Λ *in*

$$X^* = \{y \in X : d(\Lambda^k x, y) < \infty\}; \tag{2.1}$$

(c) *if* $y \in X^*$, *then*

$$d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y). \tag{2.2}$$

Throughout this paper, we fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Suppose E is a vector space over $\mathbb{K}$. A function $\|\cdot\|_\beta : E \to [0, \infty)$ is called a β-norm if and only if it satisfies

(N_1) $\|x\|_\beta = 0$, if and only if $x = 0$;

(N_2) $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$, for all $\lambda \in \mathbb{K}$ and all $x \in E$;

(N_3) $\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$, for all $x, y \in E$.

Recently, Cădariu and Radu [15] applied the fixed point method to the investigation of the Cauchy additive functional equation (see [16, 17]). Using such a clever idea, they could present a short, simple proof for the Hyers-Ulam stability of Cauchy and Jensen functional equations.

3. Main results

In this section, by using an idea of Cădariu and Radu (see [15, 16]), we will prove the Hyers-Ulam-Rassias stability of the quadratic functional equation with involution (1.2).

Theorem 3.1. *Let* E_1 *be a vector space over* $\mathbb{K}$ *and let* E_2 *be a complete* β*-normed space over* $\mathbb{K}$, *where* β *is a fixed real number with* $0 < \beta \leq 1$. *Suppose a function* $\varphi : E_1 \times E_1 \to [0, \infty)$ *is given and there exists a constant* L, $0 < L < 1$, *such that*

$$\begin{gathered} \varphi(2x, 2y) \leq \frac{4^\beta}{2} L \varphi(x, y), \\ \varphi(x + \sigma(x), y + \sigma(y)) \leq \frac{4^\beta}{2} L \varphi(x, y) \end{gathered} \tag{3.1}$$

for all $x, y \in E_1$. *Furthermore, let* $f : E_1 \to E_2$ *be a function satisfying the inequality*

$$\|f(x+y) + f(x + \sigma(y)) - 2f(x) - 2f(y)\|_\beta \leq \varphi(x, y) \tag{3.2}$$

for all $x, y \in E_1$, *where* $\sigma : E_1 \to E_1$ *is an involution of* E_1. *Then there exists a unique solution* $T : E_1 \to E_2$ *of* (1.2) *such that*

$$\|f(x) - T(x)\|_\beta \leq \frac{1}{4^\beta} \frac{1}{1-L} \varphi(x, x) \tag{3.3}$$

for all $x \in E_1$.

Proof. First, let us define X to be the set of all functions $h : E_1 \to E_2$ and introduce a generalized metric on X as follows:

$$d(g,h) = \inf\{C \in [0,\infty] : \|g(x) - h(x)\|_\beta \le C\varphi(x,x)\ \forall x \in E_1\}. \tag{3.4}$$

Let $\{f_n\}$ be a Cauchy sequence in (X,d). According to the definition of Cauchy sequences, there exists, for any given $\varepsilon > 0$, a positive integer N_ε such that $d(f_m, f_n) \le \varepsilon$ for all $m, n \ge N_\varepsilon$. By considering the definition of the generalized metric d, we see that

$$\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N} \quad \forall m,n \ge N_\varepsilon \quad \forall x \in E_1 : \|f_m(x) - f_n(x)\|_\beta \le \varepsilon\varphi(x,x). \tag{3.5}$$

If x is any given point of E_1, (3.5) implies that $\{f_n(x)\}$ is a Cauchy sequence in E_2. Since E_2 is complete, $\{f_n(x)\}$ converges in E_2 for each $x \in E_1$. Hence, we can define a function $f : E_1 \to E_2$ by

$$f(x) = \lim_{n\to\infty} f_n(x) \tag{3.6}$$

for any $x \in E_1$.

If we let m increase to infinity, it follows from (3.5) that for any $\varepsilon > 0$, there exists a positive integer N_ε with $\|f_n(x) - f(x)\|_\beta \le \varepsilon\varphi(x,x)$ for all $n \ge N_\varepsilon$ and for all $x \in E_1$, that is, for any $\varepsilon > 0$, there exists a positive integer N_ε such that $d(f_n, f) \le \varepsilon$ for any $n \ge N_\varepsilon$. This fact leads us to a conclusion that $\{f_n\}$ converges in (X,d). Hence, (X,d) is a complete space (cf. the proof of [15, Theorem 2.5]).

We now define an operator $\Lambda : X \to X$ by

$$(\Lambda h)(x) = \frac{1}{4}\left[h(2x) + h(x + \sigma(x))\right] \tag{3.7}$$

for all $x \in E_1$.

First, we assert that Λ is strictly contractive on X. Given $g, h \in X$, let $C \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le C$, that is,

$$\|g(x) - h(x)\|_\beta \le C\varphi(x,x) \tag{3.8}$$

for all $x \in E_1$. If we replace y by x in (3.2), then we obtain

$$\|f(2x) + f(x + \sigma(x)) - 4f(x)\|_\beta \le \varphi(x,x) \tag{3.9}$$

for every $x \in E_1$. It follows from (3.1) and (3.8) that

$$\begin{aligned}
\|(\Lambda g)(x) - (\Lambda h)(x)\|_\beta &= \frac{1}{4^\beta}\|g(2x) + g(x+\sigma(x)) - h(2x) - h(x+\sigma(x))\|_\beta \\
&\le \frac{1}{4^\beta}\|g(2x) - h(2x)\|_\beta + \frac{1}{4^\beta}\|g(x+\sigma(x)) - h(x+\sigma(x))\|_\beta \\
&\le \frac{C}{4^\beta}\varphi(2x,2x) + \frac{C}{4^\beta}\varphi(x+\sigma(x), x+\sigma(x)) \le LC\varphi(x,x)
\end{aligned} \tag{3.10}$$

for all $x \in E_1$, that is, $d(\Lambda g, \Lambda h) \leq LC$. We hence conclude that $d(\Lambda g, \Lambda h) \leq Ld(g,h)$ for any $g, h \in X$. Therefore, Λ is strictly contractive because L is a constant with $0 < L < 1$.

Next, we assert that $d(\Lambda f, f) < \infty$. If we put $y = x$ in (3.2) and we divide both sides by 4^β, then we get

$$\|(\Lambda f)(x) - f(x)\|_\beta = \left\| \frac{1}{4}\left[f(2x) + f(x+\sigma(x))\right] - f(x) \right\|_\beta \leq \frac{1}{4^\beta}\varphi(x,x) \tag{3.11}$$

for any $x \in E_1$, that is,

$$d(\Lambda f, f) \leq \frac{1}{4^\beta} < \infty. \tag{3.12}$$

Now, it follows from Theorem 2.1(a) that there exists a function $T : E_1 \to E_2$ which is a fixed point of Λ, such that $d(\Lambda^n f, T) \to 0$ as $n \to \infty$.

By mathematical induction, we can easily show (and hence we can omit to show) that

$$(\Lambda^n f)(x) = \frac{1}{2^{2n}}\left[f(2^n x) + (2^n - 1) f(2^{n-1}x + 2^{n-1}\sigma(x))\right] \tag{3.13}$$

for each $n \in \mathbb{N}$.

Since $d(\Lambda^n f, T) \to 0$ as $n \to \infty$, there exists a sequence $\{C_n\}$ such that $C_n \to 0$ as $n \to \infty$ and $d(\Lambda^n f, T) \leq C_n$ for every $n \in \mathbb{N}$. Hence, it follows from the definition of d that

$$\|(\Lambda^n f)(x) - T(x)\|_\beta \leq C_n \varphi(x,x) \tag{3.14}$$

for all $x \in E_1$. Thus, for each (fixed) $x \in E_1$, we have

$$\lim_{n\to\infty} \|(\Lambda^n f)(x) - T(x)\|_\beta = 0. \tag{3.15}$$

Therefore

$$T(x) = \lim_{n\to\infty} \frac{1}{2^{2n}}\left[f(2^n x) + (2^n - 1) f(2^{n-1}x + 2^{n-1}\sigma(x))\right] \tag{3.16}$$

for all $x \in E_1$. It follows from (3.1), (3.2), and (3.16) that

$$\begin{aligned}
&\|T(x+y) + T(x+\sigma(y)) - 2T(x) - 2T(y)\|_\beta \\
&\quad = \lim_{n\to\infty} \frac{1}{2^{2\beta n}} \| f(2^n x + 2^n y) + (2^n - 1) f(2^{n-1}(x+y) + 2^{n-1}(\sigma(x) + \sigma(y))) \\
&\qquad\qquad + f(2^n x + 2^n \sigma(y)) + (2^n - 1) f(2^{n-1}(x + \sigma(y)) + 2^{n-1}(\sigma(x) + y)) \\
&\qquad\qquad - 2f(2^n x) - 2(2^n - 1) f(2^{n-1}(x + \sigma(x))) \\
&\qquad\qquad - 2f(2^n y) - 2(2^n - 1) f(2^{n-1}(y + \sigma(y))) \|_\beta,
\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{n\to\infty}\frac{1}{4^{\beta n}}\left\|f(2^n x+2^n y)+f(2^n x+2^n\sigma(y))-2f(2^n x)-2f(2^n y)\right\|_\beta \\
&\quad+\lim_{n\to\infty}\frac{(2^n-1)^\beta}{4^{\beta n}}\Big\|f\big(2^{n-1}(x+\sigma(x))+2^{n-1}(y+\sigma(y))\big)+f\big(2^{n-1}(x+\sigma(x))+2^{n-1}(y+\sigma(y))\big) \\
&\qquad\qquad -2f\big(2^{n-1}(x+\sigma(x))\big)-2f\big(2^{n-1}(y+\sigma(y))\big)\Big\|_\beta \\
&\le \lim_{n\to\infty}\frac{1}{4^{\beta n}}\varphi(2^n x,2^n y)+\lim_{n\to\infty}\frac{(2^n-1)^\beta}{4^{\beta n}}\varphi\big(2^{n-1}(x+\sigma(x)),2^{n-1}(y+\sigma(y))\big) \\
&\le \lim_{n\to\infty}\frac{1}{4^{\beta n}}\left(\frac{4^\beta}{2}L\right)^n\varphi(x,y)+\lim_{n\to\infty}\frac{(2^n-1)^\beta}{4^{\beta n}}\left(\frac{4^\beta}{2}L\right)^n\varphi(x,y)=0
\end{aligned}
\tag{3.17}
$$

for all $x, y \in E_1$, which implies that T is a solution of (1.2).

By Theorem 2.1(c) and by (3.12), we obtain

$$
d(f,T)\le\frac{1}{1-L}d(\Lambda f,f)\le\frac{1}{4^\beta(1-L)},
\tag{3.18}
$$

that is, (3.3) is true for all $x \in E_1$.

Assume that $T_1 : E_1 \to E_2$ is another solution of (1.2) satisfying (3.3). (We know that T_1 is a fixed point of Λ.) In view of (3.3) and the definition of d, we can conclude that (3.18) is true with T_1 in place of T. Due to Theorem 2.1(b), we get $T = T_1$. This proves the uniqueness of T. □

In a similar way, by applying Theorem 2.1, we can prove the following theorem.

Theorem 3.2. *Let E_1 be a vector space over $\mathbb{K}$ and let E_2 be a complete β-normed space over $\mathbb{K}$, where β is a fixed real number with $0 < \beta \le 1$. Assume that a function $\varphi : E_1 \times E_1 \to [0,\infty)$ is given and there exists a constant L, $0 < L < 1$, such that*

$$
\begin{gathered}
\varphi(x,y)\le\frac{L}{2\cdot 4^\beta}\varphi(2x,2y),\\
\varphi\big(x+\sigma(x),y+\sigma(y)\big)\le 2^\beta\varphi(2x,2y)
\end{gathered}
\tag{3.19}
$$

for all $x, y \in E_1$. Furthermore, let $f : E_1 \to E_2$ be a function satisfying (3.2) for all $x, y \in E_1$, where $\sigma : E_1 \to E_1$ is an involution of E_1. Then there exists a unique solution $T : E_1 \to E_2$ of (1.2) such that

$$
\|f(x)-T(x)\|_\beta\le\frac{1}{4^\beta}\frac{L}{1-L}\varphi(x,x)
\tag{3.20}
$$

for all $x \in E_1$.

Proof. We use the same definitions for X and d as in the proof of Theorem 3.1. Then, we can similarly prove that (X, d) is complete. Let us define an operator $\Lambda : X \to X$ by

$$
(\Lambda h)(x)=4\left[h\left(\frac{x}{2}\right)-\frac{1}{2}h\left(\frac{x}{4}+\frac{\sigma(x)}{4}\right)\right]
\tag{3.21}
$$

for all $x \in E_1$. By induction, we can prove that

$$(\Lambda^n f)(x) = 2^{2n}\left[f\left(\frac{x}{2^n}\right) + \left(\frac{1}{2^n} - 1\right)f\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right)\right] \tag{3.22}$$

for all $x \in E_1$ and for every $n \in \mathbb{N}$.

We apply the same argument as in the proof of Theorem 3.1 and prove that Λ is a strictly contractive operator. Given $g, h \in X$, let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le C$, that is, $\|g(x) - h(x)\|_\beta \le C\varphi(x, x)$ for all $x \in E_1$. It then follows from (3.19) and (3.21) that

$$\begin{aligned}
&\|(\Lambda g)(x) - (\Lambda h)(x)\|_\beta \\
&\quad = 4^\beta\left\|g\left(\frac{x}{2}\right) - \frac{1}{2}g\left(\frac{x}{4} + \frac{\sigma(x)}{4}\right) - h\left(\frac{x}{2}\right) + \frac{1}{2}h\left(\frac{x}{4} + \frac{\sigma(x)}{4}\right)\right\|_\beta \\
&\quad \le 4^\beta\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\|_\beta + 2^\beta\left\|g\left(\frac{x}{4} + \frac{\sigma(x)}{4}\right) - h\left(\frac{x}{4} + \frac{\sigma(x)}{4}\right)\right\|_\beta \\
&\quad \le 4^\beta C\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + 2^\beta C\varphi\left(\frac{x}{4} + \frac{\sigma(x)}{4}, \frac{x}{4} + \frac{\sigma(x)}{4}\right) \le LC\varphi(x, x)
\end{aligned} \tag{3.23}$$

for all $x \in E_1$, that is, $d(\Lambda g, \Lambda h) \le Ld(g, h)$.

If we replace $x/2$, respectively, $x/4 + \sigma(x)/4$, for x and y in (3.2), then we obtain

$$\left\|f(x) + f\left(\frac{x}{2} + \frac{\sigma(x)}{2}\right) - 4f\left(\frac{x}{2}\right)\right\|_\beta \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \tag{3.24}$$

respectively,

$$\left\|f\left(\frac{x}{2} + \frac{\sigma(x)}{2}\right) - 2f\left(\frac{x}{4} + \frac{\sigma(x)}{4}\right)\right\|_\beta \le \frac{1}{2^\beta}\varphi\left(\frac{x}{4} + \frac{\sigma(x)}{4}, \frac{x}{4} + \frac{\sigma(x)}{4}\right). \tag{3.25}$$

Therefore, it follows from (3.19), (3.21), (3.24), and (3.25) that

$$\begin{aligned}
&\|f(x) - (\Lambda f)(x)\|_\beta \\
&\quad = \left\|f(x) - 4\left[f\left(\frac{x}{2}\right) - \frac{1}{2}f\left(\frac{x}{4} + \frac{\sigma(x)}{4}\right)\right]\right\|_\beta \\
&\quad \le \left\|f(x) + f\left(\frac{x}{2} + \frac{\sigma(x)}{2}\right) - 4f\left(\frac{x}{2}\right)\right\|_\beta + \left\|2f\left(\frac{x}{4} + \frac{\sigma(x)}{4}\right) - f\left(\frac{x}{2} + \frac{\sigma(x)}{2}\right)\right\|_\beta \\
&\quad \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{1}{2^\beta}\varphi\left(\frac{x}{4} + \frac{\sigma(x)}{4}, \frac{x}{4} + \frac{\sigma(x)}{4}\right) \le \frac{1}{4^\beta}L\varphi(x, x)
\end{aligned} \tag{3.26}$$

for all $x \in E_1$. This means that

$$d(\Lambda f, f) \le \frac{1}{4^\beta}L. \tag{3.27}$$

According to Theorem 2.1(a) there exists a unique function $T : E_1 \to E_2$, which is a fixed point of Λ, such that

$$T(x) = \lim_{n\to\infty} 2^{2n}\left[f\left(\frac{x}{2^n}\right) + \left(\frac{1}{2^n} - 1\right) f\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right)\right] \tag{3.28}$$

for all $x \in E_1$. Analogously to the proof of Theorem 3.1, we can show that T is a solution of (1.2).

Using Theorem 2.1(c) and (3.27), we get

$$d(f,T) \le \frac{1}{4^\beta}\frac{L}{1-L}, \tag{3.29}$$

which implies the validity of (3.20). □

In the following corollaries, we will investigate some special cases of Theorems 3.1 and 3.2.

Corollary 3.3. *Fix a nonnegative number p less than 1 and choose a constant β with $(p+1)/2 < \beta \le 1$. Let E_1 be a normed space over $\mathbb{K}$ and let E_2 be a complete β-normed space over $\mathbb{K}$. If a function $f : E_1 \to E_2$ satisfies*

$$\left\| f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y) \right\|_\beta \le \varepsilon\left(\|x\|^p + \|y\|^p\right) \tag{3.30}$$

and $\|x + \sigma(x)\|^p \le 2^p\|x\|^p$ for all $x \in E_1$ and for some $\varepsilon > 0$, then there exists a unique solution $T : E_1 \to E_2$ of (1.2) *such that*

$$\left\| f(x) - T(x) \right\|_\beta \le \frac{2\varepsilon}{4^\beta - 2^{p+1}}\|x\|^p \tag{3.31}$$

for any $x \in E_1$.

Proof. If we set $\varphi(x,y) = \varepsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$ and if we set $L = 2^{p+1}/4^\beta$, then we have $0 < L < 1$ and

$$\varphi(2x,2y) = 2^p\varepsilon\left(\|x\|^p + \|y\|^p\right) = \frac{4^\beta}{2}L\varphi(x,y) \tag{3.32}$$

for all $x, y \in E_1$. Furthermore, we get

$$\varphi\left(x+\sigma(x), y+\sigma(y)\right) \le \frac{4^\beta}{2}L\varphi(x,y) \tag{3.33}$$

for any $x, y \in E_1$.

According to Theorem 3.1, there exists a unique solution $T : E_1 \to E_2$ of (1.2) such that (3.31) holds for every $x \in E_1$. □

Remark 3.4. It may be remarked that if we set $p = 0$ and $\beta = 1$ in Corollary 3.3, then it reduces to [10, Theorem 2.1].

If we set $\sigma(x) = -x$ in Corollary 3.3, then $\|x + \sigma(x)\|^p = \|0\|^p \leq 2^p\|x\|^p$ is true for all $x \in E_1$. In this case, (3.30) reduces to

$$\left\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \right\|_\beta \leq \varepsilon\left(\|x\|^p + \|y\|^p\right), \tag{3.34}$$

and the quadratic function T is defined by

$$T(x) = \lim_{n\to\infty} \frac{1}{2^{2n}} f\left(2^n x\right). \tag{3.35}$$

For the case when $\sigma(x) = -x$ and $\beta = 1$, Corollary 3.3 reduces to [10, Corollary 3.3].

If we let $\sigma(x) = x$ in Corollary 3.3, then $\|x + \sigma(x)\|^p = 2^p\|x\|^p$ holds for all $x \in E_1$, (3.30) reduces to

$$\left\| f(x+y) - f(x) - f(y) \right\|_\beta \leq \frac{\varepsilon}{2^\beta}\left(\|x\|^p + \|y\|^p\right), \tag{3.36}$$

and the additive function T is given by

$$T(x) = \lim_{n\to\infty} \frac{1}{2^{n}} f\left(2^n x\right). \tag{3.37}$$

If we set $\sigma(x) = x$ and $\beta = 1$, then the upper bound of (3.31) is smaller than that of [10, Corollary 3.2].

Corollary 3.5. *Fix a number p larger than 1 and choose a constant β with $0 < \beta < (p-1)/2$. Let E_1 be a normed space over $\mathbb{K}$ and let E_2 be a complete β-normed space over $\mathbb{K}$. If a function $f : E_1 \to E_2$ satisfies (3.30) and $\|x + \sigma(x)\|^p \leq 2^{p+\beta}\|x\|^p$ for all $x, y \in E_1$ and for some $\varepsilon > 0$, then there exists a unique solution $T : E_1 \to E_2$ of (1.2) such that*

$$\left\| f(x) - T(x) \right\|_\beta \leq \frac{2\varepsilon}{2^{p-1} - 4^\beta}\|x\|^p \tag{3.38}$$

for any $x \in E_1$.

Proof. If we set $\varphi(x, y) = \varepsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$ and if we set $L = 4^\beta/2^{p-1}$, then we have $0 < L < 1$ and

$$\varphi(x,y) = \varepsilon\left(\|x\|^p + \|y\|^p\right) = \frac{L}{2\cdot 4^\beta}\varphi(2x, 2y) \tag{3.39}$$

for all $x, y \in E_1$. Furthermore, we get

$$\varphi\left(x + \sigma(x), y + \sigma(y)\right) \leq 2^\beta \varphi(2x, 2y) \tag{3.40}$$

for any $x, y \in E_1$.

According to Theorem 3.2, there exists a unique solution $T : E_1 \to E_2$ of (1.2) such that (3.38) holds for every $x \in E_1$. □

Remark 3.6. If $\sigma(x) = -x$ in Corollary 3.5, then $\|x + \sigma(x)\|^p = \|0\|^p \le 2^{p+\beta}\|x\|^p$ is true for all $x \in E_1$. In this case, (3.30) reduces to

$$\left\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \right\|_\beta \le \varepsilon\left(\|x\|^p + \|y\|^p\right), \tag{3.41}$$

and the quadratic function T is defined by

$$T(x) = \lim_{n\to\infty} 2^{2n} f\left(\frac{x}{2^n}\right) \tag{3.42}$$

for all $x \in E_1$. If we let $\sigma(x) = -x$, $p > 3$ and $\beta = 1$ in Corollary 3.5, then the upper bound of (3.38) is smaller than that of [10, Corollary 4.3].

We cannot expect the Hyers-Ulam-Rassias stability for (3.41) when $p = 2$ and the range space E_2 of the relevant functions f is a Banach space (i.e., E_2 is a complete 1-normed space) (see [18]). However, if E_2 is a complete β-normed space over $\mathbb{K}$, where β is a fixed real number with $0 < \beta < 1/2$, then (3.41) is stable in the sense of Hyers, Ulam, and Rassias in spite of $p = 2$.

If we set $\sigma(x) = x$ in Corollary 3.5, then $\|x + \sigma(x)\|^p = 2^p\|x\|^p \le 2^{p+\beta}\|x\|^p$ for all $x \in E_1$, (3.30) reduces to

$$\left\| f(x+y) - f(x) - f(y) \right\|_\beta \le \frac{\varepsilon}{2^\beta}\left(\|x\|^p + \|y\|^p\right), \tag{3.43}$$

and the additive function T is given by

$$T(x) = \lim_{n\to\infty} 2^{n} f\left(\frac{x}{2^n}\right). \tag{3.44}$$

Unfortunately, if we set $\sigma(x) = x$, $p > 3$ and $\beta = 1$ in Corollary 3.5, then the upper bound of (3.38) is larger than that of [10, Corollary 4.2].

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