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Research Article

Weak and Strong Convergence Theorems of an Implicit Iteration Process for a Countable Family of Nonexpansive Mappings

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Using the implicit iteration and the hybrid method in mathematical programming, we prove weak and strong convergence theorems for finding common fixed points of a countable family of nonexpansive mappings in a real Hilbert space. Our results include many convergence theorems by Xu and Ori (2001) and Zhang and Su (2007) as special cases. We also apply our method to find a common element to the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem. Finally, we propose an iteration to obtain convergence theorems for a continuous monotone mapping.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C. \tag{1.1}$$

We denote by $F(T)$ the set of all fixed points of T. If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [1]). We write $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x. There are many methods for approximating fixed points of a nonexpansive mapping. Xu and Ori [2] introduced the following implicit iteration process to approximate a common fixed point of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^{N}$: an initial point $x_0 \in C$,

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1x_1,\\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2x_2,\\
&\ \ \vdots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_Nx_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1x_{N+1}\\
&\ \ \vdots
\end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. The iteration above can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_nx_n, \quad n \geq 1, \tag{1.3}$$

where $T_n \equiv T_{n \bmod N}$, here the mod N function takes values in $\{1,2,\ldots,N\}$.They proved that this process converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$. Recently, to obtain a strong convergence theorem, Zhang and Su [3] modify iteration processes (1.3) by the implicit hybrid method for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$: an initial point $x_0 \in C$,

$$\begin{aligned}
&x_0 \in C \text{ is arbitrary},\\
&y_n = \alpha_n x_n + (1-\alpha_n)T_nz_n,\\
&z_n = \beta_n y_n + (1-\beta_n)T_ny_n,\\
&C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\},\\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\
&x_{n+1} = P_{C_n \cap Q_n}x_0, \quad n = 0,1,2,\ldots,
\end{aligned} \tag{1.4}$$

where $T_n \equiv T_{n \bmod N}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1]$ with $\alpha_n < 1$.

In this paper, we establish weak and strong convergence theorems for finding common fixed points of a countable family of nonexpansive mappings in a real Hilbert space. Our results include many convergence theorems by [2, Theorems 2] and [3, Theorems 2.4] as special cases. The new iteration introduced in this paper is applied to find a common element to the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem. We also propose an iteration to obtain convergence theorems for a continuous monotone mapping.

2. Preliminaries

Let H be a real Hilbert space. Then,

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. It is also known that H satisfies the following.

(1) Opial's condition [4], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.3}$$

holds for every $y \in H$ with $y \neq x$.

(2) The Kadec-Klee property [1], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together implies $\|x_n - x\| \to 0$.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists the nearest point $P_C x$ in C such that

$$\|x - P_C x\| \le \|x - y\| \quad \forall y \in C. \tag{2.4}$$

Such a mapping, P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Furthermore, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \text{iff } \langle x - z, z - y\rangle \ge 0 \; \forall y \in C. \tag{2.5}$$

Lemma 2.1 (see [5, Lemma 1]). *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative real numbers such that*

$$a_{n+1} \le a_n + b_n \quad \forall n \ge 1, \tag{2.6}$$

and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (see [6, Lemma 2.2]). *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$. Then, $\liminf_{n\to\infty} b_n = 0$.*

Lemma 2.3 (see [7, Lemma 3.2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H such that*

$$\|x_{n+1} - y\| \le \|x_n - y\| \quad \forall y \in C,\ n \in \mathbb{N}. \tag{2.7}$$

Then, the sequence $\{P_C(x_n)\}$ converges strongly to some $z \in C$.

To deal with a family of mappings, the following conditions are introduced. Let C be a subset of a Banach space, let $\{T_n\}$ and $\mathcal{T}$ be families of mappings of C with $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \varnothing$, where $F(\mathcal{T})$ is the set of all common fixed points of all mappings in $\mathcal{T}$.

(a) $\{T_n\}$ is said to satisfy the AKTT-condition [8] if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty. \tag{2.8}$$

(b) $\{T_n\}$ is said to satisfy the NST-condition (I) with $\mathcal{T}$ [9] if for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \text{ implies } \lim_{n\to\infty} \|z_n - T z_n\| = 0 \quad \forall T \in \mathcal{T}. \tag{2.9}$$

In particular, if $\mathcal{T} = \{T\}$, that is, $\mathcal{T}$ consists of one mapping T, then $\{T_n\}$ is said to satisfy the NST-condition (I) with T.

(c) $\{T_n\}$ is said to satisfy the NST-condition (II) [9] if for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty}\|z_{n+1}-T_nz_n\|=0 \text{ implies } \lim_{n\to\infty}\|z_n-T_mz_n\|=0 \quad \forall m\in\mathbb{N}. \tag{2.10}$$

Inspired by conditions above, we introduce the following one.

(d) $\{T_n\}$ is said to satisfy the NST*-condition with $\mathfrak{T}$ if for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty}\|z_n-T_nz_n\|=0, \qquad \lim_{n\to\infty}\|z_n-z_{n+1}\|=0 \tag{2.11}$$

imply that $\lim_{n\to\infty}\|z_n-Tz_n\|=0$ for all $T\in\mathfrak{T}$. In particular, if $\mathfrak{T}=\{T\}$, then we simply say that $\{T_n\}$ satisfies the NST*-condition with T.

Remark 2.4. (i) If $\{T_n\}$ satisfies the NST-condition (I) with $\mathfrak{T}$, then $\{T_n\}$ satisfies the NST*-condition with $\mathfrak{T}$.

(ii) If $\{T_n\}$ satisfies the NST-condition (II), then $\{T_n\}$ satisfies the NST*-condition with $\{T_n\}$.

Lemma 2.5 (see [8, Lemma 3.2]). *Let C be a nonempty closed subset of a Banach space, and let $\{T_n\}$ be a family of mappings of C into itself which satisfies the AKTT-condition, then there exists a mapping $T: C \to C$ such that*

$$Tx=\lim_{n\to\infty}T_nx \quad \forall x\in C, \tag{2.12}$$

and $\lim_{n\to\infty}\sup\{\|Tz-T_nz\| : z\in B\}=0$ *for each bounded subset B of C.*

Lemma 2.6. *Let C be a nonempty closed subset of a Banach space, and let $\{T_n\}$ be a family of mappings of C into itself which satisfies AKTT-condition and $\bigcap_{n=1}^{\infty}F(T_n)\neq\varnothing$. Let T be the mapping from C into itself defined by $Tz=\lim_{n\to\infty}T_nz$ for all $z\in C$ and suppose that $F(T)=\bigcap_{n=1}^{\infty}F(T_n)$. Then, $\{T_n\}$ satisfies the NST-condition (I) with T. This implies that $\{T_n\}$ satisfies the NST*-condition with T.*

Proof. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty}\|z_n-T_nz_n\|=0$. We apply Lemma 2.5 to get

$$\begin{aligned}\|z_n-Tz_n\| &\le \|z_n-T_nz_n\|+\|T_nz_n-Tz_n\| \\ &\le \|z_n-T_nz_n\|+\sup\{\|T_nz-Tz\| : z\in\{z_n\}\}\longrightarrow 0.\end{aligned} \tag{2.13}$$

Hence, we obtain that $\{T_n\}$ satisfies the NST-condition (I) with T. This completes the proof. □

Lemma 2.7. *Let C be a nonempty subset of a Banach space, and let $\{T_n\}_{n=1}^{N}$ be a finite family of nonexpansive mappings of C into itself with a common fixed point. Then, $\{T_n\}$ satisfies NST*-condition with $\mathfrak{T}=\{T_1,T_2,\dots,T_N\}$, where $T_n\equiv T_{n \bmod N}$.*

Proof. Let $\{z_n\}$ be a bounded sequence in C such that

$$\lim_{n\to\infty}\|z_n-T_nz_n\|=0, \qquad \lim_{n\to\infty}\|z_{n+1}-z_n\|=0. \tag{2.14}$$

Obviously, it is easy to see that $\lim_{n\to\infty}\|z_{n+i}-z_n\|=0$ for each $i=1,2,\ldots,N$. Consequently,

$$\begin{aligned}\|z_n-T_{n+i}z_n\| &\le \|z_n-z_{n+i}\|+\|z_{n+i}-T_{n+i}z_{n+i}\|+\|T_{n+i}z_{n+i}-T_{n+i}z_n\|\\ &\le 2\|z_n-z_{n+i}\|+\|z_{n+i}-T_{n+i}z_{n+i}\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.\end{aligned}\tag{2.15}$$

This implies that $\lim_{n\to\infty}\|z_n-T_mz_n\|=0$ for each $m=1,2,\ldots,N$. This completes the proof. □

Remark 2.8. There are families of mappings $\{T_n\}$ and $\mathcal{T}$ such that

(1) $\{T_n\}$ satisfies the NST*-condition with $\mathcal{T}$;

(2) $\{T_n\}$ fails the NST-condition (I) with $\mathcal{T}$ and the NST-condition (II).

The following example shows that the NST*-condition with $\mathcal{T}$ is strictly weaker than NST-condition (I) with $\mathcal{T}$ and the NST-condition (II).

Example 2.9. Let $H:=\mathbb{R}^2$ and $C:=[0,1]\times[0,1]$. Define $T_1,T_2:C\to C$ as follows:

$$T_1(x,y)=(x,1-y),\qquad T_2(x,y)=(1-x,y)\tag{2.16}$$

for all $(x,y)\in C$. Hence, T_1 and T_2 are nonexpansive mappings with

$$F(T_1)\cap F(T_2)=\left([0,1]\times\left\{\frac{1}{2}\right\}\right)\cap\left(\left\{\frac{1}{2}\right\}\times[0,1]\right)=\left\{\left(\frac{1}{2},\frac{1}{2}\right)\right\}\neq\varnothing.\tag{2.17}$$

Let $T_n=T_{n(\bmod 2)}$. By Lemma 2.7, we have $\{T_n\}$ satisfies NST*-condition with $\{T_1,T_2\}$.

(a) $\{T_n\}$ fails the NST-condition (I) with $\mathcal{T}=\{T_1,T_2\}$. In fact, let $z_{2n-1}=(1,1/2)$ and $z_{2n}=(1/2,1)$ for all $n\in\mathbb{N}$. Then, $z_{2n-1}\in F(T_{2n-1})=F(T_1)$ and $z_{2n}\in F(T_{2n})=F(T_2)$. In particular, $\|z_n-T_nz_n\|\equiv 0$. Clearly,

$$\|z_n-T_1z_n\|\nrightarrow 0,\qquad \|z_n-T_2z_n\|\nrightarrow 0.\tag{2.18}$$

Hence, $\{T_n\}$ fails the NST-condition (I) with $\{T_1,T_2\}$.

(b) $\{T_n\}$ fails the NST-condition (II). To this end, let $z_{4n-3}=(1/4,1/4)$, $z_{4n-2}=(1/4,3/4)$, $z_{4n-1}=(3/4,3/4)$, and $z_{4n}=(3/4,1/4)$ for all $n\in\mathbb{N}$. Then, $\|z_{n+1}-T_nz_n\|\equiv 0$. But,

$$\|z_n-T_1z_n\|\nrightarrow 0,\qquad \|z_n-T_2z_n\|\nrightarrow 0.\tag{2.19}$$

Hence, $\{T_n\}$ fails the NST-condition (II).

Lemma 2.10 (see [10]). *Let C be a nonempty closed convex subset of a strictly convex Banach space, S and T be two nonexpansive mappings of C into itself with a common fixed point, and $0<\beta<1$. Let U be a mapping defined by*

$$U=T(\beta I+(1-\beta)S),\tag{2.20}$$

where I is the identity mapping. Then, U is a nonexpansive mapping from C into itself and $F(U)=F(T)\cap F(S)$.

Lemma 2.11. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_n\}$ and $\mathcal{T}$ be two families of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$, and suppose that $\{T_n\}$ satisfies the NST*-condition with $\mathcal{T}$. Let $\{U_n\}$ be a family of nonexpansive mappings from C into itself defined by*

$$U_n = T_n(\beta_n I + (1-\beta_n)T_n) \tag{2.21}$$

for all $n \in \mathbb{N}$, where I is the identity mapping, and $\{\beta_n\}$ is a sequence in $[a,1]$ for some $a \in (0,1]$. Then, $\{U_n\}$ satisfies the NST-condition with $\mathcal{T}$.*

Proof. By Lemma 2.10, we have $F(U_n) = F(T_n)$ for all $n \in \mathbb{N}$ and so,

$$\bigcap_{n=1}^{\infty} F(U_n) = F(\mathcal{T}) \neq \emptyset. \tag{2.22}$$

Let $\{z_n\}$ be a bounded sequence in C such that

$$\lim_{n\to\infty} \|z_n - U_n z_n\| = 0, \qquad \lim_{n\to\infty} \|z_{n+1} - z_n\| = 0. \tag{2.23}$$

Since

$$\begin{aligned} \|z_n - T_n z_n\| &\le \|z_n - U_n z_n\| + \|T_n(\beta_n z_n + (1-\beta_n)T_n z_n) - T_n z_n\| \\ &\le \|z_n - U_n z_n\| + (1-\beta_n)\|z_n - T_n z_n\| \\ &\le \|z_n - U_n z_n\| + (1-a)\|z_n - T_n z_n\|, \end{aligned} \tag{2.24}$$

it follows that

$$\|z_n - T_n z_n\| \le \frac{1}{a}\|z_n - U_n z_n\| \longrightarrow 0. \tag{2.25}$$

Since $\{T_n\}$ satisfies the NST*-condition with $\mathcal{T}$, we have

$$\lim_{n\to\infty} \|z_n - T z_n\| = 0 \quad \forall T \in \mathcal{T}. \tag{2.26}$$

Hence, we obtain that $\{U_n\}$ satisfies the NST*-condition with $\mathcal{T}$. This completes the proof. □

3. Weak convergence theorems

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a family of nonexpansive mappings from C into itself with a common fixed point. Let $\{x_n\}$ be a sequence in C defined by $x_0 \in C$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0,1)$. Then,

(i) $\lim_{n\to\infty}\|x_n - p\|$ *exists for each* $p \in \bigcap_{n=1}^{\infty} F(T_n)$*;*

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)\|x_{n-1} - T_n x_n\|^2 < \infty$.

Proof. Observe that if C is a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ is a nonexpansive mapping, then for every $u \in C$, $\alpha \in (0,1]$, the mapping $S = S_{(\alpha,T)} : C \to C$ defined by

$$Sx = \alpha u + (1-\alpha)Tx \quad (x \in C) \tag{3.2}$$

is a $(1-\alpha)$-contraction, that is, for all $x, y \in C$,

$$\|Sx - Sy\| = (1-\alpha)\|Tx - Ty\| \leq (1-\alpha)\|x - y\|. \tag{3.3}$$

Consequently, S has a unique fixed point $x^* \in C$. Thus, there exists a unique $x^* \in C$, that is,

$$x^* = \alpha u + (1-\alpha)Tx^*. \tag{3.4}$$

This implies that the implicit iteration scheme (3.1) is well defined. To see (i), we let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. It follows from (2.2) that

$$\begin{aligned} \|x_n - p\|^2 &= \|\alpha_n(x_{n-1} - p) + (1-\alpha_n)(T_n x_n - p)\|^2 \\ &= \alpha_n\|x_{n-1} - p\|^2 + (1-\alpha_n)\|T_n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1} - T_n x_n\|^2 \\ &\leq \alpha_n\|x_{n-1} - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1} - T_n x_n\|^2. \end{aligned} \tag{3.5}$$

Since $\alpha_n > 0$, we have

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\|^2 - (1-\alpha_n)\|x_{n-1} - T_n x_n\|^2. \tag{3.6}$$

In particular,

$$\|x_n - p\| \leq \|x_{n-1} - p\|. \tag{3.7}$$

So, $\lim_{n\to\infty}\|x_n - p\|$ exists. Furthermore, from (3.6), we have

$$(1-\alpha_n)\|x_{n-1} - T_n x_n\|^2 \leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2. \tag{3.8}$$

Summing from 1 to m and tending to infinity for m, we have (ii). This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ and $\mathcal{T}$ be two families of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$, and suppose that $\{T_n\}$ satisfies the NST*-condition with $\mathcal{T}$. Then, the sequence $\{x_n\}$ in C defined by* (3.1)*, where $\{\alpha_n\}$ is a sequence in $(0,b]$ for some $b \in (0,1)$, converges weakly to $w \in F(\mathcal{T})$. Moreover, $\lim_{n\to\infty} P_{F(\mathcal{T})} x_n = w$.*

Proof. It follows from Lemma 3.1(i) that $\{x_n\}$ is bounded. By Lemma 3.1(ii) and $\alpha_n \leq b$, we have

$$\sum_{n=1}^{\infty}\|x_{n-1} - T_n x_n\|^2 < \infty. \tag{3.9}$$

It follows that $\lim_{n\to\infty}\|x_{n-1} - T_n x_n\| = 0$. From (3.1), we immediately have

$$\lim_{n\to\infty}\left\|x_n - T_n x_n\right\| = \lim_{n\to\infty}\alpha_n\left\|x_{n-1} - T_n x_n\right\| = 0, \tag{3.10}$$

and so,

$$\lim_{n\to\infty}\left\|x_n - x_{n-1}\right\| = 0. \tag{3.11}$$

Since $\{T_n\}$ satisfies the NST*-condition with $\mathcal{T}$, we have

$$\lim_{n\to\infty}\left\|x_n - Tx_n\right\| = 0 \quad \forall T \in \mathcal{T}. \tag{3.12}$$

We now extract a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. So, by the demiclosedness principle, $w \in F(\mathcal{T})$. To prove that $x_n \rightharpoonup w$, suppose that there exists another subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $x_{m_j} \rightharpoonup w' \neq w$. So, we have $w' \in F(\mathcal{T})$. It follows from Lemma 3.1(i) and Opial's condition that

$$\begin{aligned}\lim_{n\to\infty}\left\|x_n - w\right\| &= \lim_{i\to\infty}\left\|x_{n_i} - w\right\| < \lim_{i\to\infty}\left\|x_{n_i} - w'\right\| \\ &= \lim_{j\to\infty}\left\|x_{m_j} - w'\right\| < \lim_{j\to\infty}\left\|x_{m_j} - w\right\| \\ &= \lim_{n\to\infty}\left\|x_n - w\right\|,\end{aligned} \tag{3.13}$$

arriving at a contradiction. Hence, $x_n \rightharpoonup w \in F(\mathcal{T})$. Finally, we prove that $\lim_{n\to\infty} z_n = w$, where $z_n = P_{F(\mathcal{T})}x_n$ for each $n \in \mathbb{N}$. By (3.7) and Lemma 2.3, there is $w_0 \in F(\mathcal{T})$ such that $z_n \to w_0$. From $z_n = P_{F(\mathcal{T})}x_n$ and $w \in F(\mathcal{T})$, we have

$$\left\langle x_n - z_n, z_n - w\right\rangle \geq 0 \quad \forall n \in \mathbb{N}. \tag{3.14}$$

It follows from $z_n \to w_0$ and $x_n \rightharpoonup w$ that

$$\left\langle w - w_0, w_0 - w\right\rangle \geq 0, \tag{3.15}$$

and then $w_0 = w$. This completes the proof. □

Using Theorem 3.2 and Lemma 2.7, we have the following result.

Corollary 3.3 (see [2, Theorem 2]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $\{T_n\}_{n=1}^{N}$ be a finite family of nonexpansive mappings of C into itself with a common fixed point. Then, the sequence $\{x_n\}$ in C defined by* (1.3)*, where $\{\alpha_n\}$ is a sequence in* $(0, b]$ *for some* $b \in (0,1)$*, converges weakly to $w = \lim_{n\to\infty} P_{\bigcap_{n=1}^{N} F(T_n)} x_n$.*

In the presence of the stronger condition than NST*-condition with $\mathcal{T}$, we are able to weaken the restriction on $\{\alpha_n\}$.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself which satisfies the AKTT-condition and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let T be the mapping from C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$, and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then, the sequence in C defined by* (3.1)*, where $\{\alpha_n\}$ is a sequence in* $(0,1)$ *with $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, converges weakly to $w = \lim_{n\to\infty} P_{F(T)} x_n$.*

Proof. By Lemmas 2.2 and 3.1(ii) and $\sum_{n=1}^{\infty}(1-\alpha_n)=\infty$, we have

$$\liminf_{n\to\infty}\|x_{n-1}-T_nx_n\|=0, \tag{3.16}$$

and hence,

$$\liminf_{n\to\infty}\|x_n-T_nx_n\|=\liminf_{n\to\infty}\alpha_n\|x_{n-1}-T_nx_n\|=0. \tag{3.17}$$

Next, we prove that the limit $\lim_{n\to\infty}\|x_n-T_nx_n\|$ exists. Since $\{x_n\}$ is bounded, it follows from AKTT-condition that

$$\sum_{n=1}^{\infty}\sup\{\|T_nz-T_{n-1}z\|:z\in\{x_n\}\}<\infty. \tag{3.18}$$

Notice that

$$\begin{aligned}\|x_n-x_{n-1}\|&=(1-\alpha_n)\|x_{n-1}-T_nx_n\|\\&\le(1-\alpha_n)(\|x_{n-1}-T_{n-1}x_{n-1}\|+\|T_{n-1}x_{n-1}-T_{n-1}x_n\|+\|T_{n-1}x_n-T_nx_n\|)\\&\le(1-\alpha_n)\|x_{n-1}-T_{n-1}x_{n-1}\|+(1-\alpha_n)\|x_{n-1}-x_n\|\\&\quad+(1-\alpha_n)\sup\{\|T_nz-T_{n-1}z\|:z\in\{x_n\}\},\end{aligned} \tag{3.19}$$

so we have

$$\alpha_n\|x_n-x_{n-1}\|\le(1-\alpha_n)\|x_{n-1}-T_{n-1}x_{n-1}\|+(1-\alpha_n)\sup\{\|T_nz-T_{n-1}z\|:z\in\{x_n\}\}. \tag{3.20}$$

It follows that

$$\begin{aligned}\|x_n-T_nx_n\|&=\frac{\alpha_n}{1-\alpha_n}\|x_n-x_{n-1}\|\\&\le\|x_{n-1}-T_{n-1}x_{n-1}\|+\sup\{\|T_nz-T_{n-1}z\|:z\in\{x_n\}\}.\end{aligned} \tag{3.21}$$

By Lemma 2.1 and (3.18), we have $\lim_{n\to\infty}\|x_n-T_nx_n\|$ exists. Thus, we have

$$\lim_{n\to\infty}\|x_n-T_nx_n\|=0. \tag{3.22}$$

From the definition of T, we have T is nonexpansive. By Lemma 2.6, we have $\{T_n\}$ satisfies the NST*-condition with T. As in the proof of Theorem 3.2, $\{x_n\}$ converges weakly to $w=\lim_{n\to\infty}P_{F(T)}x_n$. □

Remark 3.5. Since the NST*-condition is implied by the AKTT-condition, Theorem 3.4 still holds under the same condition of $\{\alpha_n\}$ as in Theorem 3.2.

As in [8, Theorem 4.1], we can generate a family $\{T_n\}$ of nonexpansive mappings satisfying the AKTT-condition by using convex combination of a general family $\{S_k\}$ of nonexpansive mappings with a common fixed point.

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{\infty}(1-\alpha_n)=\infty$. Let $\{\beta_n^k\}$ be a family of positive real numbers with indices $n,\ k\in\mathbb{N}$ with $k\le n$ such that*

(i) $\sum_{k=1}^{n}\beta_n^k = 1$ *for every* $n\in\mathbb{N}$;

(ii) $\lim_{n\to\infty}\beta_n^k > 0$ *for every* $k\in\mathbb{N}$;

(iii) $\sum_{n=1}^{\infty}\sum_{k=1}^{n}|\beta_{n+1}^k-\beta_n^k|<\infty$.

Let $\{S_k\}$ be a family of nonexpansive mappings from C into itself with a common fixed point. Then, the sequence $\{x_n\}$ in C defined by (3.1), where $T_n \equiv \sum_{k=1}^{n}\beta_n^k S_k$, converges weakly to $w=\lim_{n\to\infty}P_{\bigcap_{k=1}^{\infty}F(S_k)}x_n$.

4. Strong convergence theorems

We next use the hybrid method from mathematical programming to obtain several strong convergence theorems.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ and $\mathcal{T}$ be two families of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty}F(T_n) = F(\mathcal{T})\neq\emptyset$, and suppose that $\{T_n\}$ satisfies the NST*-condition with $\mathcal{T}$. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$
\begin{aligned}
x_0 &\in C \text{ is arbitrary},\\
y_n &= \alpha_n x_n + (1-\alpha_n)T_n y_n,\\
C_n &= \{z\in C : \|y_n - z\| \le \|x_n - z\|\},\\
Q_n &= \{z\in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
x_{n+1} &= P_{C_n\cap Q_n}x_0, \quad n=0,1,2,\dots,
\end{aligned}
\tag{4.1}
$$

where $\{\alpha_n\}$ is a sequence in $(0,b]$ for some $b\in(0,1)$. Then, $\{x_n\}$ converges strongly to $P_{F(\mathcal{T})}x_0$.

Proof. We first prove that C_n and Q_n are closed and convex for each $n\in\mathbb{N}\cup\{0\}$. From the definitions of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n\in\mathbb{N}\cup\{0\}$. We prove that C_n is convex. Since $\|y_n - z\|\le\|x_n - z\|$ is equivalent to

$$
\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \le 0,
\tag{4.2}
$$

(by (2.1)) it follows that C_n is convex. Next, we show that

$$
F(\mathcal{T})\subset C_n \quad \forall n\in\mathbb{N}\cup\{0\}.
\tag{4.3}
$$

Let $p\in F(\mathcal{T})$ and $n\in\mathbb{N}\cup\{0\}$. Since

$$
\begin{aligned}
\|y_n - p\| &= \|\alpha_n x_n + (1-\alpha_n)T_n y_n - p\|\\
&\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|T_n y_n - p\|\\
&\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\|,
\end{aligned}
\tag{4.4}
$$

it follows that

$$\|y_n - p\| \leq \|x_n - p\|, \tag{4.5}$$

and hence, $p \in C_n$. Therefore, we obtain (4.3). Now, we show that

$$F(\mathcal{T}) \subset Q_n \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{4.6}$$

We prove this by induction. For $n = 0$, we have $F(\mathcal{T}) \subset C = Q_0$. Suppose that $F(\mathcal{T}) \subset Q_n$. Then, $\emptyset \neq F(\mathcal{T}) \subset C_n \cap Q_n$ and there exists a unique element $x_{n+1} \in C_n \cap Q_n$ such that $x_{n+1} = P_{C_n \cap Q_n} x_0$. Then,

$$\langle x_{n+1} - z, x_0 - x_{n+1} \rangle \geq 0 \tag{4.7}$$

for each $z \in C_n \cap Q_n$. In particular,

$$\langle x_{n+1} - p, x_0 - x_{n+1} \rangle \geq 0 \tag{4.8}$$

for each $p \in F(\mathcal{T})$. It follows that $F(\mathcal{T}) \subset Q_{n+1}$, and hence (4.6) holds. Therefore,

$$F(\mathcal{T}) \subset C_n \cap Q_n \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{4.9}$$

This implies that $\{x_n\}$ is well defined. It follows from the definition of Q_n that $x_n = P_{Q_n} x_0$, that is,

$$\|x_n - x_0\| \leq \|z - x_0\| \quad \forall z \in Q_n \text{ and all } n \in \mathbb{N} \cup \{0\}. \tag{4.10}$$

In particular,

$$\|x_n - x_0\| \leq \|z - x_0\| \quad \forall z \in F(\mathcal{T}) \text{ and all } n \in \mathbb{N} \cup \{0\}. \tag{4.11}$$

On the other hand, from $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{4.12}$$

Therefore, $\{\|x_n - x_0\|\}$ is nondecreasing and bounded. So, $\lim_{n \to \infty} \|x_n - x_0\|$ exists. This implies that $\{x_n\}$ is bounded. Since $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\langle x_n - x_{n+1}, x_0 - x_n \rangle \geq 0. \tag{4.13}$$

It follows from (2.1) that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \end{aligned} \tag{4.14}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{4.15}$$

Since $x_{n+1} \in C_n$, we have

$$\begin{aligned} \|y_n - x_n\| &\le \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \\ &\le 2\|x_n - x_{n+1}\| \longrightarrow 0. \end{aligned} \tag{4.16}$$

It follows from $\alpha_n \le b < 1$ that

$$\begin{aligned} \|x_n - T_n x_n\| &\le \|x_n - T_n y_n\| + \|T_n y_n - T_n x_n\| \\ &\le \|x_n - T_n y_n\| + \|y_n - x_n\| \\ &= \frac{1}{1-\alpha_n}\|y_n - x_n\| + \|y_n - x_n\| \\ &\le \frac{1}{1-b}\|y_n - x_n\| + \|y_n - x_n\| \longrightarrow 0. \end{aligned} \tag{4.17}$$

Since $\{T_n\}$ satisfies the NST*-condition with $\mathcal{T}$, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \quad \forall T \in \mathcal{T}. \tag{4.18}$$

Finally, we show that $x_n \to w$, where $w = P_{F(\mathcal{T})}x_0$. Since $\{x_n\}$ is bounded, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w'$. Since $I - T$ is demiclosed and by using (4.18), we have $w' \in F(\mathcal{T})$. By (4.11), we have

$$\|x_n - x_0\| \le \|w - x_0\|. \tag{4.19}$$

It follows from $w = P_{F(\mathcal{T})}x_0$ and the lower semicontinuity of the norm that

$$\|w - x_0\| \le \|w' - x_0\| \le \liminf_{k\to\infty} \|x_{n_k} - x_0\| \le \limsup_{k\to\infty} \|x_{n_k} - x_0\| \le \|w - x_0\|. \tag{4.20}$$

Thus, we obtain that $\lim_{k\to\infty}\|x_{n_k} - x_0\| = \|w' - x_0\| = \|w - x_0\|$. Using the Kadec-Klee property of H, we obtain that $\lim_{k\to\infty} x_{n_k} = w' = w$. Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that the whole sequence $\{x_n\}$ converges strongly to $P_{F(\mathcal{T})}x_0$. □

Using Theorem 4.1 and Lemmas 2.7 and 2.11, we have the following result.

Corollary 4.2 (see [3, Theorem 2.4]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $\{T_n\}_{n=1}^N$ be a finite family of nonexpansive mappings of C into itself with a common fixed point. Then, the sequence $\{x_n\}$ in C defined by* (1.4)*, where $\{\alpha_n\}$ is a sequence in $(0, a]$ for some $a \in (0,1)$, and $\{\beta_n\}$ is a sequence in $[b, 1]$ for some $b \in (0,1]$, converges strongly to $P_{\bigcap_{n=1}^N F(T_n)}x_0$.*

5. Applications

5.1. Equilibrium problems

Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \ge 0 \quad \forall y \in C. \tag{5.1}$$

The set of solutions of (5.1) is denoted by $\mathrm{EP}(f)$. Numerous problems in physics, optimization, and economics are reduced to find a solution of (5.1). Some methods have been proposed to solve the equilibrium problem [11–17]. In 2005, Combettes and Hirstoaga [12] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(f)$ is nonempty, and they also proved a strong convergence theorem.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions (see [11]).

(A1) $f(x,x)=0$ for all $x\in C$;

(A2) f is monotone, that is, $f(x,y)+f(y,x)\le 0$ for any $x,y\in C$;

(A3) f is upper-hemicontinuous, that is, for each $x,y,z\in C$,

$$\limsup_{t\to 0^+} f\big(tz+(1-t)x,y\big)\le f(x,y); \tag{5.2}$$

(A4) $f(x,\cdot)$ is convex and lower semicontinuous for each $x\in C$.

The following lemma is shown in [11, Corollary 1] and [12, Lemma 2.12].

Lemma 5.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let f be a bifunction from $C\times C$ into $\mathbb{R}$ satisfies (A1)–(A4), and let $r>0$ and $x\in H$. Then, there exists a unique $x^*\in C$ such that*

$$f(x^*,y)+\frac{1}{r}\langle y-x^*,x^*-x\rangle\ge 0\quad \forall y\in C. \tag{5.3}$$

Moreover, let T_r be a mapping of H into C defined by

$$T_r(x)=x^*\quad \forall x\in H. \tag{5.4}$$

Then, the following conditions hold:

(i) *T_r is firmly nonexpansive, that is, for any $x,y\in H$,*

$$\|T_rx-T_ry\|^2\le\|x-y\|^2-\|T_rx-x-(T_ry-y)\|^2; \tag{5.5}$$

(ii) *$F(T_r)=EP(f)$;*

(iii) *$EP(f)$ is closed and convex.*

Lemma 5.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping of C into H, and let T be a firmly nonexpansive mapping from H into C such that $F(S)\cap F(T)\ne\varnothing$. Then, ST is a nonexpansive mapping from H into itself and*

$$F(ST)=F(S)\cap F(T). \tag{5.6}$$

Proof. Since T is firmly nonexpansive, there exists a nonexpansive mapping U such that $T=(1/2)(I+U)$ and $F(U)=F(T)$. As in the proof of Lemma 2.10, the conclusion holds. □

Motivated by Tada and Takahashi [16] and S. Takahashi and W. Takahashi [17], we prove weak and strong convergence theorems for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in a Hilbert space. Using Theorem 3.4 and Lemmas 5.1 and 5.2, we have Theorem 5.3.

Theorem 5.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4), and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be two sequences generated by $x_0 \in H$ and*

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C, \\ x_n = \alpha_n x_{n-1} + (1 - \alpha_n) S u_n \tag{5.7}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$, and $\{r_n\}$ is a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$ converges weakly to $w \in F(S) \cap EP(f)$. Moreover, $\lim_{n\to\infty} P_{F(S)\cap EP(f)} x_n = w$.

Proof. It is noted that the iteration scheme is well defined. As in the proof of [14, Theorem 16], it follows from $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$ that

$$\sum_{n=1}^{\infty} \sup\left\{\left\|T_{r_{n+1}} z - T_{r_n} z\right\| : z \in B\right\} < \infty \tag{5.8}$$

for any bounded subset B of H. Moreover, by Lemma 2.5, the mapping T defined by

$$Tx = \lim_{n\to\infty} T_{r_n} x \quad \forall x \in H \tag{5.9}$$

satisfies

$$F(T) = \bigcap_{n=1}^{\infty} F\left(T_{r_n}\right) = \mathrm{EP}(f). \tag{5.10}$$

It is easy to see that T is a firmly nonexpansive mapping of H into C. Write $T_n \equiv S T_{r_n}$ then, by Lemma 5.2, we have T_n is a nonexpansive mapping from H into itself, and

$$F(T_n) = F\left(S T_{r_n}\right) = F(S) \cap F\left(T_{r_n}\right) = F(S) \cap \mathrm{EP}(f) = F(ST) \tag{5.11}$$

for all $n \in \mathbb{N}$ and so,

$$\bigcap_{n=1}^{\infty} F(T_n) = F(ST) = F(S) \cap \mathrm{EP}(f). \tag{5.12}$$

Since S is nonexpansive, (5.8) and (5.9), we have

$$\sum_{n=1}^{\infty} \sup\left\{\left\|T_{n+1} z - T_n z\right\| : z \in B\right\} < \infty \tag{5.13}$$

for any bounded subset B of H, and

$$STx = S\left(\lim_{n\to\infty} T_{r_n} x\right) = \lim_{n\to\infty} S T_{r_n} x = \lim_{n\to\infty} T_n x \quad \forall x \in H. \tag{5.14}$$

Applying Theorem 3.4, $\{x_n\}$ converges weakly to $w = \lim_{n\to\infty} P_{F(S)\cap \mathrm{EP}(f)} x_n$. □

Similarly, we have the following strong convergence theorem. We safely suppress the proof.

Theorem 5.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4), and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be two sequences generated by $x_0 \in H$ and*

$$\begin{aligned} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle &\geq 0 \quad \forall y \in C, \\ y_n &= \alpha_n x_{n-1} + (1-\alpha_n) S u_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{5.15}$$

where $\{\alpha_n\}$ is a sequence in $(0, a)$ for some $a \in (0,1)$, and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap EP(f)} x_0$.

5.2. Convergence theorem for monotone mappings

Let H be a real Hilbert space, and C be a nonempty closed convex subset of H. Let $A : C \to H$ be a mapping. The classical variational inequality is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \quad \forall y \in C. \tag{5.16}$$

The set of solutions of classical variational inequality is denoted by $\text{VIP}(C, A)$. The variational inequality has been extensively studied in the literatures (see [7, 18–23] and the references therein). We recall that a mapping $A : C \to H$ is said to be

(a) monotone if

$$\langle Au - Av, u - v\rangle \geq 0 \quad \forall u, v \in C; \tag{5.17}$$

(b) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2 \quad \forall u, v \in C; \tag{5.18}$$

(c) r-strongly monotone if there exists a constant $r > 0$ such that

$$\langle Au - Av, u - v\rangle \geq r\|u - v\|^2 \quad \forall u, v \in C; \tag{5.19}$$

(d) relaxed (γ, r)-cocoercive if there exist constants $\gamma, r > 0$ such that

$$\langle Au - Av, u - v\rangle \geq -\gamma\|Au - Av\|^2 + r\|u - v\|^2 \quad \forall u, v \in C; \tag{5.20}$$

(e) μ-Lipschitzian if there exists a constant $\mu > 0$ such that

$$\|Au - Av\| \leq \mu\|u - v\| \quad \forall u, v \in C. \tag{5.21}$$

Remark 5.5. (1) Every α-inverse-strongly monotone mapping is monotone and $1/\alpha$-Lipschitzian.

(2) Every r-strongly monotone is monotone.

(3) Every relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping with $\gamma\mu^2 \leq r$ is monotone.

Lemma 5.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H. Define a bifunction $f : C \times C \to \mathbb{R}$ as follows:*

$$f(x,y) = \langle Ax, y - x\rangle \quad \forall x, y \in C. \tag{5.22}$$

Then,

(i) *[14, Lemma 19] f satisfies (A1)–(A4) and $VIP(C, A) = EP(f)$;*

(ii) *[14, Lemma 20] If $x \in H$, $u \in C$, and $r > 0$, then*

$$f(u,y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0 \quad \forall y \in C \Longleftrightarrow u = P_C(x - rAu). \tag{5.23}$$

Using Theorem 5.3 and Lemma 5.6, we have Theorem 5.7.

Theorem 5.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C, and let S be a nonexpansive mapping of C into H such that $F(S) \cap VIP(C, A) \neq \varnothing$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$ and*

$$\begin{aligned} u_n &= P_C(x_n - r_n A u_n), \\ x_n &= \alpha_n x_{n-1} + (1 - \alpha_n) S u_n \end{aligned} \tag{5.24}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$, and $\{r_n\}$ is a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$ converges weakly to $w \in F(S) \cap VIP(C, A)$. Moreover, $\lim_{n\to\infty} P_{F(S)\cap VIP(C,A)} x_n = w$.

Using Theorem 5.4 and Lemma 5.6, we also have Theorem 5.8.

Theorem 5.8. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C, and let S be a nonexpansive mapping of C into H such that $F(S) \cap VIP(C, A) \neq \varnothing$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$ and*

$$\begin{aligned} u_n &= P_C(y_n - r_n A u_n), \\ y_n &= \alpha_n x_{n-1} + (1 - \alpha_n) S u_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{5.25}$$

where $\{\alpha_n\}$ is a sequence in $(0, a]$ for some $a \in (0,1)$, and $\{r_n\}$ is a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap VIP(C,A)} x_0$.

Remark 5.9. (1) By Remark 5.5, we obtain a strong convergence theorem for α-inverse-strongly monotone mappings, r-strongly monotone and continuous mappings and relaxed (γ, r)-cocoercive and μ-Lipschitzian mappings with $\gamma\mu^2 \leq r$.

(2) Some weak and strong convergence theorems for monotone Lipschitzian mappings were established by several authors [7, 18–23]. However, there is a monotone continuous mapping which is not Lipschitzian (see [14, Remark 23]). Therefore, Theorems 5.7 and 5.8 provide a new convergence theorem for a wider class of mappings.

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