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Research Article

Weak and Strong Convergence Theorems for Nonexpansive Mappings in Banach Spaces

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The purpose of this paper is to introduce two implicit iteration schemes for approximating fixed points of nonexpansive mapping T and a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$, respectively, in Banach spaces and to prove weak and strong convergence theorems. The results presented in this paper improve and extend the corresponding ones of H.-K. Xu and R. Ori, 2001, Z. Opial, 1967, and others.

1. Introduction and preliminaries

Let E be a real Banach space, K a nonempty closed convex subset of E, and $T : K \to K$ a mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. T is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$. In this paper, $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively. $\overline{\text{co}}(A)$ denotes the closed convex hull of A, where A is a subset of E.

In 2001, Xu and Ori [1] introduced the following implicit iteration scheme for common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in Hilbert spaces:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \ge 1, \tag{1.1}$$

where $T_n = T_{n \bmod N}$, and they proved weak convergence theorem.

In this paper, we introduce a new implicit iteration scheme:

$$x_n = \alpha_n x_{n-1} + \beta_n T x_{n-1} + \gamma_n T x_n, \quad n \ge 1, \tag{1.2}$$

for fixed points of nonexpansive mapping T in Banach space and also prove weak and strong convergence theorems. Moreover, we introduce an implicit iteration scheme:

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \gamma_n T_n x_n, \quad n \geq 1, \tag{1.3}$$

where $T_n = T_{n \bmod N}$, for common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in Banach spaces and also prove weak and strong convergence theorems.

Observe that if K is a nonempty closed convex subset of a real Banach space E and $T : K \to K$ is a nonexpansive mapping, then for every $u \in K$, $\alpha, \beta, \gamma \in [0,1]$, and positive integer n, the operator $S = S_{(\alpha,\beta,\gamma,n)} : K \to K$ defined by

$$Sx = \alpha u + \beta T u + \gamma T x \tag{1.4}$$

satisfies

$$\|Sx - Sy\| = \|\gamma T x - \gamma T y\| \leq \gamma \|x - y\| \tag{1.5}$$

for all $x, y \in K$. Thus, if $\gamma < 1$ then S is a contractive mapping. Then S has a unique fixed point $x^* \in K$. This implies that, if $\gamma_n < 1$, the implicit iteration scheme (1.2) and (1.3) can be employed for the approximation of fixed points of nonexpansive mapping and common fixed points of a finite family of nonexpansive mappings, respectively.

Now, we give some definitions and lemmas for our main results.

A Banach space E is said to satisfy *Opial's condition* if, for any $\{x_n\} \subset E$ with $x_n \rightharpoonup x \in E$, the following inequality holds:

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in E,\ x \neq y. \tag{1.6}$$

Let D be a closed subset of a real Banach space E and let $T : D \to D$ be a mapping.

T is said to be *demiclosed* at zero if $Tx_0 = 0$ whenever $\{x_n\} \subset D$, $x_n \rightharpoonup x_0$ and $Tx_n \to 0$.

T is said to be *semicompact* if, for any bounded sequence $\{x_n\} \subset D$ with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to $x^* \in D$.

Lemma 1.1 (see [2, 3]). *Let E be a uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Lemma 1.2 (see [4]). *Let E be a uniformly convex Banach space and let a, b be two constants with $0 < a < b < 1$. Suppose that $\{t_n\} \subset [a, b]$ is a real sequence and $\{x_n\}$, $\{y_n\}$ are two sequences in E. Then the conditions*

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = d, \qquad \limsup_{n\to\infty} \|x_n\| \leq d, \qquad \limsup_{n\to\infty} \|y_n\| \leq d \tag{1.7}$$

imply that $\lim_{n\to\infty}\|x_n - y_n\| = 0$, where $d \geq 0$ is a constant.

2. Main results

Theorem 2.1. *Let E be a real uniformly convex Banach space which satisfies Opial's condition, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be a nonexpansive mapping with nonempty fixed points set F. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < a \leq \gamma_n \leq b < 1$, where a, b are some constants. Then implicit iteration process $\{x_n\}$ defined by* (1.2) *converges weakly to a fixed point of T.*

Proof. Firstly, the condition of Theorem 2.1 implies $\gamma_n < 1$, so that (1.2) can be employed for the approximation of fixed point of nonexpansive mapping.

For any given $p \in F$, we have

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n T x_{n-1} + \gamma_n T x_n - p\| \\ &= \|\alpha_n (x_{n-1} - p) + \beta_n (T x_{n-1} - p) + \gamma_n (T x_n - p)\| \\ &\le \alpha_n \|x_{n-1} - p\| + \beta_n \|T x_{n-1} - p\| + \gamma_n \|T x_n - p\| \\ &\le (\alpha_n + \beta_n) \|x_{n-1} - p\| + \gamma_n \|x_n - p\| \end{aligned} \tag{2.1}$$

which leads to

$$(1 - \gamma_n) \|x_n - p\| \le (\alpha_n + \beta_n) \|x_{n-1} - p\| = (1 - \gamma_n) \|x_{n-1} - p\|. \tag{2.2}$$

It follows from the condition $\gamma_n \le b < 1$ that

$$\|x_n - p\| \le \|x_{n-1} - p\|. \tag{2.3}$$

Thus $\lim_{n\to\infty} \|x_n - p\|$ exists, and so let

$$\lim_{n\to\infty} \|x_n - p\| = d. \tag{2.4}$$

Hence $\{x_n\}$ is a bounded sequence. Moreover, $\overline{\text{co}}(\{x_n\})$ is a bounded closed convex subset of K. We have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p\| &= \lim_{n\to\infty} \|\alpha_n (x_{n-1} - p) + \beta_n (T x_{n-1} - p) + \gamma_n (T x_n - p)\| \\ &= \lim_{n\to\infty} \left\| (1 - \gamma_n) \left[\frac{\alpha_n}{1 - \gamma_n} (x_{n-1} - p) + \frac{\beta_n}{1 - \gamma_n} (T x_{n-1} - p) \right] + \gamma_n (T x_n - p) \right\| t = d, \\ \limsup_{n\to\infty} \|T x_n - p\| &\le \limsup_{n\to\infty} \|x_n - p\| = d. \end{aligned} \tag{2.5}$$

Again, it follows from the condition $\alpha_n + \beta_n + \gamma_n = 1$ that

$$\begin{aligned} &\limsup_{n\to\infty} \left\| \frac{\alpha_n}{1 - \gamma_n} (x_{n-1} - p) + \frac{\beta_n}{1 - \gamma_n} (T x_{n-1} - p) \right\| \\ &\quad \le \limsup_{n\to\infty} \left(\frac{\alpha_n}{1 - \gamma_n} \|x_{n-1} - p\| + \frac{\beta_n}{1 - \gamma_n} \|T x_{n-1} - p\| \right) \\ &\quad \le \limsup_{n\to\infty} \left(\frac{\alpha_n + \beta_n}{1 - \gamma_n} \|x_{n-1} - p\| \right) = d. \end{aligned} \tag{2.6}$$

By Lemma 1.2, the condition $0 < a \le \gamma_n \le b < 1$, and (2.5)–(2.6), we get

$$\lim_{n\to\infty} \left\| \frac{\alpha_n}{1 - \gamma_n} (x_{n-1} - p) + \frac{\beta_n}{1 - \gamma_n} (T x_{n-1} - p) - (T x_n - p) \right\| = 0. \tag{2.7}$$

This means that

$$\lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}x_{n-1}+\frac{\beta_n}{1-\gamma_n}Tx_{n-1}-Tx_n\right\| = \lim_{n\to\infty}\left(\frac{1}{1-\gamma_n}\right)\|\alpha_n x_{n-1}+\beta_n Tx_{n-1}-(1-\gamma_n)Tx_n\| = 0. \tag{2.8}$$

Since $0<a\le\gamma_n\le b<1$, we have $1/(1-a)\le 1/(1-\gamma_n)\le 1/(1-b)$. Hence,

$$\lim_{n\to\infty}\|\alpha_n x_{n-1}+\beta_n Tx_{n-1}-(1-\gamma_n)Tx_n\| = 0. \tag{2.9}$$

Because

$$\begin{aligned}\lim_{n\to\infty}\|\alpha_n x_{n-1}+\beta_n Tx_{n-1}-(1-\gamma_n)Tx_n\| &= \lim_{n\to\infty}\|x_n-\gamma_n Tx_n-(1-\gamma_n)Tx_n\|\\ &= \lim_{n\to\infty}\|x_n-Tx_n\|,\end{aligned} \tag{2.10}$$

by (2.9), we get

$$\lim_{n\to\infty}\|x_n-Tx_n\| = 0. \tag{2.11}$$

Since E is uniformly convex, every bounded closed convex subset of E is weakly compact, so that there exists a subsequence $\{x_{n_k}\}$ of sequence $\{x_n\}\subseteq\overline{\text{co}}(\{x_n\})$ such that $x_{n_k}\rightharpoonup q\in K$. Therefore, it follows from (2.11) that

$$\lim_{k\to\infty}\|Tx_{n_k}-x_{n_k}\| = 0. \tag{2.12}$$

By Lemma 1.1, we know that $I-T$ is demiclosed at zero; it is esay to see that $q\in F$.

Now, we show that $x_n\rightharpoonup q$. In fact, this is not true; then there must exist a subsequence $\{x_{n_i}\}\subset\{x_n\}$ such that $x_{n_i}\rightharpoonup q_1\in K$, $q_1\neq q$. Then, by the same method given above, we can also prove that $q_1\in F$.

Because, for any $p\in F$, the limit $\lim_{n\to\infty}\|x_n-p\|$ exists. Then we can let

$$\lim_{n\to\infty}\|x_n-q\| = d_1, \qquad \lim_{n\to\infty}\|x_n-q_1\| = d_2. \tag{2.13}$$

Since E satisfies Opial's condition, we have

$$\begin{aligned}d_1 &= \limsup_{k\to\infty}\|x_{n_k}-q\| < \limsup_{k\to\infty}\|x_{n_k}-q_1\| = d_2,\\ d_2 &= \limsup_{i\to\infty}\|x_{n_i}-q_1\| < \limsup_{i\to\infty}\|x_{n_i}-q\| = d_1.\end{aligned} \tag{2.14}$$

This is a contradiction and hence $q = q_1$. This implies that $\{x_n\}$ converges weakly to a fixed point q of T. This completes the proof. □

From the proof of Theorem 2.1, we give the following strong convergence theorem.

Theorem 2.2. *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, let $T : K \to K$ be a nonexpansive mapping with nonempty fixed points set F, and let T be semicompact. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n = 1$ and $0<a\le\gamma_n\le b<1$, where a, b are some constants. Then implicit iteration process $\{x_n\}$ defined by* (1.2) *converges strongly to a fixed point of T.*

Proof. From the proof of Theorem 2.1, we know that there exists subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup q \in K$ and satisfies (2.11). By the semicompactness of T, there exists a subsequence of $\{x_{n_k}\}$ (we still denote it by $\{x_{n_k}\}$) such that $\lim_{n\to\infty}\|x_{n_k} - q\| = 0$. Because the limit $\lim_{n\to\infty}\|x_n - q\|$ exists, thus we get $\lim_{n\to\infty}\|x_n - q\| = 0$. This completes the proof. □

Next, we study weak and strong convergence theorems for common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in Banach spaces.

Theorem 2.3. *Let E be a real uniformly convex Banach space which satisfies Opial's condition, let K be a nonempty closed convex subset of E, and let $\{T_i\}_{i=1}^N : K \to K$ be N nonexpansive mappings with nonempty common fixed points set F. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in* $[0,1]$ *satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \le \gamma_n \le b < 1$, and $\alpha_n - \beta_n > c > 0$, where a, b, c are some constants. Then implicit iteration process $\{x_n\}$ defined by* (1.3) *converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Proof. Substituing T_i $(1 \le i \le N)$ to T in the proof of Theorem 2.1, we know that for all i $(1 \le i \le N)$,

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0. \tag{2.15}$$

Now we show that, for any $l = 1, 2, \ldots, N$,

$$\lim_{n\to\infty}\|x_n - T_l x_n\| = 0. \tag{2.16}$$

In fact,

$$\begin{aligned}
\|x_n - x_{n-1}\| &= \|\beta_n T_n x_{n-1} + \gamma_n T_n x_n - (\beta_n + \gamma_n)x_{n-1}\| \\
&= \|\beta_n T_n x_{n-1} - \beta_n x_n + \gamma_n T_n x_n - \gamma_n x_n + (\beta_n + \gamma_n)(x_n - x_{n-1})\| \\
&\le \beta_n\|T_n x_{n-1} - x_n\| + \gamma_n\|T_n x_n - x_n\| + (\beta_n + \gamma_n)\|x_n - x_{n-1}\| \\
&\le \beta_n\|T_n x_{n-1} - T_n x_n\| + \beta_n\|T_n x_n - x_n\| + \gamma_n\|T_n x_n - x_n\| + (\beta_n + \gamma_n)\|x_n - x_{n-1}\| \\
&\le (\beta_n + \gamma_n)\|T_n x_n - x_n\| + (2\beta_n + \gamma_n)\|x_n - x_{n-1}\| \\
&= (\beta_n + \gamma_n)\|T_n x_n - x_n\| + (\beta_n + 1 - \alpha_n)\|x_n - x_{n-1}\|.
\end{aligned} \tag{2.17}$$

By removing the second term on the right of the above inequality to the left, we get

$$(\alpha_n - \beta_n)\|x_n - x_{n-1}\| \le (\beta_n + \gamma_n)\|T_n x_n - x_n\|. \tag{2.18}$$

It follows from the condition $\alpha_n - \beta_n > c > 0$ and (2.15) that

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{2.19}$$

So, for any $i = 1, 2, \ldots, N$,

$$\lim_{n\to\infty}\|x_n - x_{n+i}\| = 0. \tag{2.20}$$

Since, for any $i = 1, 2, 3, \ldots, N$,

$$\begin{aligned}\|x_n - T_{n+i}x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\leq 2\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\|,\end{aligned} \tag{2.21}$$

it follows from (2.15) and (2.20) that

$$\lim_{n\to\infty}\|T_{n+i}x_n - x_n\| = 0, \quad i = 1, 2, 3, \ldots, N. \tag{2.22}$$

Because $T_n = T_{n \bmod N}$, it is easy to see, for any $l = 1, 2, 3, \ldots, N$, that

$$\lim_{n\to\infty}\|T_l x_n - x_n\| = 0. \tag{2.23}$$

Since E is uniformly convex, so there exists a subsequence $\{x_{n_k}\}$ of bounded sequence $\{x_n\}$ such that $x_{n_k} \rightharpoonup q \in K$. Therefore, it follows from (2.23) that

$$\lim_{k\to\infty}\|T_l x_{n_k} - x_{n_k}\| = 0, \quad \forall\, l = 1, 2, 3, \ldots, N. \tag{2.24}$$

By Lemma 1.1, we know that $I - T_l$ is demiclosed, it is easy to see that $q \in F(T_l)$, so that $q \in F = \bigcap_{l=1}^{N} F(T_l)$. Because E satisfies Opial's condition, we can prove that $\{x_n\}$ converges weakly to a common fixed point q of $\{T_l\}_{l=1}^{N}$ by the same method given in the proof of Theorem 2.1. □

Remark 2.4. If $N = 1$, implicit iteration scheme (1.3) becomes (1.2), so from Theorem 2.1, we know that assumption $\alpha_n - \beta_n > c > 0$ in Theorem 2.3 can be removed.

Theorem 2.5. *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, let $\{T_i\}_{i=1}^{N} : K \to K$ be N nonexpansive mappings with nonempty common fixed points set F, and there exists an $l \in \{1, 2, \ldots, N\}$ such that T_l is semicompact. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \leq \gamma_n \leq b < 1$, and $\alpha_n - \beta_n > c > 0$, where a, b , c are some constants. Then implicit iteration process $\{x_n\}$ defined by (1.3) converges strongly to a common fixed point of $\{T_i\}_{i=1}^{N}$.*

Proof. From the proof of Theorem 2.3, we know that there exists subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to some $q \in K$ and satisfies (2.23). By the semicompactness of T_l, there exists a subsequence of $\{x_{n_k}\}$ (we still denote it by $\{x_{n_k}\}$) such that $\lim_{n\to\infty}\|x_{n_k} - q\| = 0$. Because the limit $\lim_{n\to\infty}\|x_n - q\|$ exists, thus we get $\lim_{n\to\infty}\|x_n - q\| = 0$. This completes the proof. □

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