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Review Article

T-Stability Approach to Variational Iteration Method for Solving Integral Equations

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We consider T-stability definition according to Y. Qing and B. E. Rhoades (2008) and we show that the variational iteration method for solving integral equations is T-stable. Finally, we present some text examples to illustrate our result.

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1. Introduction and Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space and T a self-map of X. Let $x_{n+1} = f(T, x_n)$ be some iteration procedure. Suppose that $F(T)$, the fixed point set of T, is nonempty and that x_n converges to a point $q \in F(T)$. Let $\{y_n\} \subseteq X$ and define $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$. If $\lim \epsilon_n = 0$ implies that $\lim y_n = q$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable. Without loss of generality, we may assume that $\{y_n\}$ is bounded, for if $\{y_n\}$ is not bounded, then it cannot possibly converge. If these conditions hold for $x_{n+1} = Tx_n$, that is, Picard's iteration, then we will say that Picard's iteration is T-stable.

Theorem 1.1 (see [1]). *Let $(X, \|\cdot\|)$ be a Banach space and T a self-map of X satisfying*

$$\left\|Tx - Ty\right\| \leq L\|x - Tx\| + \alpha\left\|x - y\right\| \quad (1.1)$$

for all $x, y \in X$, where $L \geq 0$, $0 \leq \alpha < 1$. Suppose that T has a fixed point p. Then, T is Picard T-stable.

Various kinds of analytical methods and numerical methods [2–10] were used to solve integral equations. To illustrate the basic idea of the method, we consider the general nonlinear system:

$$L[u(t)] + N[u(t)] = g(t), \tag{1.2}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a given continuous function. The basic character of the method is to construct a functional for the system, which reads

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda(s)\left[Lu_n(s) + N\tilde{u}_n(s) - g(s)\right]ds, \tag{1.3}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory, u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation; that is, $\delta\tilde{u}_n = 0$.

Now, we consider the Fredholm integral equation of second kind in the general case, which reads

$$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt, \tag{1.4}$$

where $K(x,t)$ is the kernel of the integral equation. There is a simple iteration formula for (1.4) in the form

$$u_{n+1}(x) = f(x) + \lambda\int_a^b K(x,t)u_n(t)dt. \tag{1.5}$$

Now, we show that the nonlinear mapping T, defined by

$$u_{n+1}(x) = T(u_n(x)) = f(x) + \lambda\int_a^b K(x,t)u_n(t)dt, \tag{1.6}$$

is T-stable in $L^2[a,b]$.

First, we show that the nonlinear mapping T has a fixed point. For $m,n \in \mathbb{N}$ we have

$$\begin{aligned} \|T(u_m(x)) - T(u_n(x))\| &= \|u_{m+1}(x) - u_{n+1}(x)\| \\ &= \left\|\lambda\int_a^b K(x,t)(u_m(t) - u_n(t))dt\right\| \\ &\leq |\lambda|\left[\iint_a^b K^2(x,t)dxdt\right]^{1/2}\|u_m(x) - u_n(x)\|. \end{aligned} \tag{1.7}$$

Therefore, if

$$|\lambda| < \left[\iint_a^b K^2(x,t)\,dx\,dt\right]^{-1/2}, \tag{1.8}$$

then, the nonlinear mapping T has a fixed point.

Second, we show that the nonlinear mapping T satisfies (1.1). Let (1.6) hold. Putting $L = 0$ and $\alpha = |\lambda|[\iint_a^b K^2(x,t)dxdt]^{1/2}$ shows that (1.1) holds for the nonlinear mapping T.

All of the conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is T-stable. As a result, we can state the following theorem.

Theorem 1.2. *Use the iteration scheme*

$$\begin{aligned} u_0(x) &= f(x), \\ u_{n+1}(x) &= T(u_n(x)) = f(x) + \lambda\int_a^b K(x,t)u_n(t)\,dt, \end{aligned} \tag{1.9}$$

for $n = 0, 1, 2, \ldots$, to construct a sequence of successive iterations $\{u_n(x)\}$ to the solution of (1.4). *In addition, if*

$$|\lambda| < \left[\iint_a^b K^2(x,t)\,dx\,dt\right]^{-1/2}, \tag{1.10}$$

$L = 0$ and $\alpha = |\lambda|[\int_a^b\int_a^b K^2(x,t)dxdt]^{1/2}$. Then the nonlinear mapping T, in the norm of $L^2(a,b)$, is T-stable.

Theorem 1.3 (see [11]). *Use the iteration scheme*

$$\begin{aligned} u_0(x) &= f(x), \\ u_{n+1}(x) &= f(x) + \lambda\int_a^b K(x,t)u_n(t)\,dt, \end{aligned} \tag{1.11}$$

for $n = 0, 1, 2, \ldots$, to construct a sequence of successive iteration $\{u_n(x)\}$ to the solution of (1.4). *In addition, let*

$$\iint_a^b K^2(x,t)\,dx\,dt = B^2 < \infty, \tag{1.12}$$

and assume that $f(x) \in L^2(a,b)$. Then, if $|\lambda| < 1/B$, the above iteration converges, in the norm of $L^2(a,b)$ to the solution of (1.4).

Corollary 1.4. *Consider the iteration scheme*

$$u_0(x) = f(x),$$
$$u_{n+1}(x) = T(u_n(x)) = f(x) + \lambda\int_a^b K(x,t)u_n(t)dt, \tag{1.13}$$

for $n = 0, 1, 2, \ldots$. *If*

$$|\lambda| < \left[\iint_a^b K^2(x,t)dxdt\right]^{-1/2}, \tag{1.14}$$

$L = 0$ *and* $\alpha = |\lambda|[\int_a^b\int_a^b K^2(x,t)dxdt]^{1/2}$, *then stability of the nonlinear mapping* T *in the norm of* $L^2(a,b)$ *is a coefficient condition for the above iteration to converge in the norm of* $L^2(a,b)$, *and to the solution of* (1.4).

2. Test Examples

In this section we present some test examples to show that the stability of the iteration method is a coefficient condition for the convergence in the norm of $L^2(a,b)$ to the solution of (1.4). In fact the stability interval is a subset of converges interval.

Example 2.1 (see [12]). Consider the integral equation

$$u(x) = \sqrt{x} + \lambda\int_0^1 xtu(t)dt. \tag{2.1}$$

The iteration formula reads

$$u_{n+1}(x) = \sqrt{x} + \lambda\int_0^1 xtu_n(t)dt, \tag{2.2}$$

$$u_0(x) = \sqrt{x}. \tag{2.3}$$

Substituting (2.3) into (2.2), we have the following results:

$$u_1(x) = \sqrt{x} + \lambda\int_0^1 xt\sqrt{t}dt = \sqrt{x} + \frac{2\lambda x}{5},$$
$$u_2(x) = \sqrt{x} + \lambda\int_0^1 xt\left[\sqrt{t} + \frac{2\lambda t}{5}\right]dt = \sqrt{x} + \left[\frac{2\lambda}{5} + \frac{2\lambda^2}{15}\right]x, \tag{2.4}$$
$$u_3(x) = \sqrt{x} + \lambda\int_0^1 xt\left[\sqrt{t} + \left(\frac{2\lambda}{5} + \frac{2\lambda^2}{15}\right)t\right]dt = \sqrt{x} + \left[\frac{2\lambda}{5} + \frac{2\lambda^2}{15} + \frac{2\lambda^3}{45}\right]x.$$

Continuing this way ad infinitum, we obtain

$$u_n(x) = \sqrt{x} + \left[\frac{2}{5.3^0}\lambda + \frac{2}{5.3^1}\lambda^2 + \frac{2}{5.3^2}\lambda^3 + \cdots\right]x, \tag{2.5}$$

then

$$u_n(x) = \sqrt{x} + \left(\frac{2}{5}\sum_{i=1}^{n}\frac{\lambda^i}{3^{i-1}}\right)x. \tag{2.6}$$

The above sequence is convergent if $|\lambda| < 3$, and the exact solution is

$$\lim_{n\to\infty} u_n(x) = \sqrt{x} + \frac{6\lambda}{5(3-\lambda)}x = u(x). \tag{2.7}$$

On the other hand we have

$$\left[\iint_a^b K^2(x,t)dxdt\right]^{1/2} = \left[\iint_0^1 (xt)^2 dxdt\right]^{1/2} = \frac{1}{3}. \tag{2.8}$$

Then if $|\lambda| < 3$ for mapping

$$u_{n+1}(x) = T(u_n(x)) = \sqrt{x} + \lambda\int_0^1 xtu_n(t)dt, \tag{2.9}$$

we have

$$\begin{aligned}\|T(u_m(x)) - T(u_n(x))\| &= \|u_{m+1}(x) - u_{n+1}(x)\| \\ &= \left\|\lambda\int_0^1 xt(u_m(t) - u_n(t))dt\right\| \\ &\leq |\lambda|\left[\iint_0^1 (xt)^2 dxdt\right]^{1/2}\|u_m(x) - u_n(x)\| \\ &\leq \frac{|\lambda|}{3}\|u_m(x) - u_n(x)\|,\end{aligned} \tag{2.10}$$

which implies that T has a fixed point. Also, putting $L = 0$ and $\alpha = |\lambda|/3$ shows that (1.1) holds for the nonlinear mapping T. All of the conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is T-stable.

Example 2.2 (see [12]). Consider the integral equation

$$u(x) = x + \lambda \int_0^1 (1 - 3xt) u(t) dt, \tag{2.11}$$

its iteration formula reads

$$\begin{gathered} u_{n+1}(x) = x + \lambda \int_0^1 (1 - 3xt) u_n(t) dt, \\ u_0(x) = x. \end{gathered} \tag{2.12}$$

Then we have

$$u_n(x) = x + \sum_{j=1}^{n} \lambda^j \int\int_0^1 \cdots \int_0^1 (1 - 3xt_1)(1 - 3t_1t_2) \cdots (1 - 3t_{j-1}t_j) t_j dt_j \cdots dt_1. \tag{2.13}$$

By (2.13), we have the following results:

$$\begin{aligned} u_1(x) &= x + \lambda \int_0^1 (1 - 3xt) t dt = (1 - \lambda)x + \frac{1}{2}\lambda, \\ u_2(x) &= x + \lambda \int_0^1 (1 - 3xt) \left[(1 - \lambda)t + \frac{1}{2}\lambda \right] dt \\ &= (1 - \lambda)x + \frac{1}{2}\lambda + \frac{\lambda^2}{4}x, \\ u_3(x) &= x + \lambda \int_0^1 (1 - 3xt) \left[(1 - \lambda)t + \frac{1}{2}\lambda + \frac{\lambda^2}{4}t \right] dt \\ &= (1 - \lambda)x + \frac{\lambda^2}{4}(1 - \lambda)x + \frac{1}{2}\lambda + \frac{\lambda^3}{8}. \end{aligned} \tag{2.14}$$

Continuing this way ad infinitum, we obtain

$$u_n(x) = \sum_{j=0}^{n} \frac{3(-1)^j - 1}{2} \left(\frac{\lambda}{2} \right)^j x + \left(\frac{1 + (-1)^{i+1}}{2} \right) \left(\frac{\lambda}{2} \right)^j. \tag{2.15}$$

The above sequence is convergent if $|\lambda/2| < 1$, that is, $-2 < \lambda < 2$ and the exact solution is

$$\lim_{n \to \infty} u_n(x) = \frac{2\lambda}{4 - \lambda^2} + \frac{4(1 - \lambda)}{4 - \lambda^2} x = u(x). \tag{2.16}$$

On the other hand we have

$$\left[\iint_a^b K^2(x,t)\,dx\,dt\right]^{1/2} = \left[\iint_0^1 (1-3xt)^2\,dx\,dt\right]^{1/2} = \frac{1}{\sqrt{2}}. \tag{2.17}$$

Then if $|\lambda| < \sqrt{2}$, for mapping

$$u_{n+1}(x) = T(u_n(x)) = x + \lambda\int_0^1 (1-3xt)u_n(t)\,dt, \tag{2.18}$$

we have

$$\begin{aligned}\|T(u_m(x)) - T(u_n(x))\| &= \|u_{m+1}(x) - u_{n+1}(x)\| \\ &= \left\|\lambda\int_0^1 xt(u_m(t) - u_n(t))\,dt\right\| \\ &\le |\lambda|\left[\iint_0^1 (1-3xt)^2\,dx\,dt\right]^{1/2}\|u_m(x) - u_n(x)\| \\ &\le \frac{|\lambda|}{\sqrt{2}}\|u_m(x) - u_n(x)\|,\end{aligned} \tag{2.19}$$

which implies that T has a fixed point. Also, putting $L = 0$ and $\alpha = |\lambda|/\sqrt{2}$ shows that (1.1) holds for the nonlinear mapping T. All of conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is T-stable.

Example 2.3. Consider the integral equation

$$u(x) = \sin ax + \lambda\frac{a}{2}\int_0^{\pi/2a} \cos(ax)u(t)\,dt, \tag{2.20}$$

its iteration formula reads

$$u_{n+1}(x) = \sin ax + \lambda\frac{a}{2}\int_0^{\pi/2a} \cos(ax)u_n(t)\,dt, \tag{2.21}$$

$$u_0(x) = \sin ax. \tag{2.22}$$

Substituting (2.22) into (2.21), we have the following results:

$$u_1(x) = \sin ax + \lambda\frac{a}{2}\int_0^{\pi/2a}\cos(ax)\sin(at)dt = \sin(ax) + \frac{\lambda}{2}\cos(ax),$$

$$u_2(x) = \sin(ax) + \lambda\frac{a}{2}\int_0^{\pi/2a}\cos(ax)\left[\sin(at) + \frac{\lambda}{2}\cos(at)\right]dt$$

$$= \sin(ax) + \cos(ax)\left[\frac{\lambda}{2} + \frac{\lambda^2}{4}\right], \tag{2.23}$$

$$u_3(x) = \sin(ax) + \lambda\frac{a}{2}\int_0^{\pi/2a}\cos(ax)\left[\sin(at) + \left[\frac{\lambda}{2} + \frac{\lambda^2}{4}\right]\cos(at)\right]dt$$

$$= \sin(ax) + \cos(ax)\left[\frac{\lambda}{2} + \frac{\lambda^2}{4} + \frac{\lambda^3}{8}\right].$$

Continuing this way ad infinitum, we obtain

$$u_n(x) = \sin(ax) + \cos(ax)\sum_{i=1}^{\infty}\left(\frac{\lambda}{2}\right)^i. \tag{2.24}$$

The above sequence is convergent if $|\lambda/2| < 1$; that is, $-2 < \lambda < 2$, and the exact solution is

$$\lim_{n\to\infty} u_n(x) = \sin(ax) + \frac{\lambda}{2-\lambda}\cos(ax) = u(x). \tag{2.25}$$

On the other hand we have

$$\left[\iint_a^b K^2(x,t)dxdt\right]^{1/2} = \left[\iint_0^{\pi/2a}\left(\frac{a}{2}\cos(ax)\right)^2 dxdt\right]^{1/2} = \sqrt{\frac{\pi^2}{32}}. \tag{2.26}$$

Then if $|\lambda| < 1/\sqrt{\pi^2/32} \cong 1.800$, for mapping

$$u_{n+1}(x) = T(u_n(x)) = x + \lambda\frac{a}{2}\int_0^{\pi/2a}\cos(ax)u_n(t)dt, \tag{2.27}$$

we have

$$\begin{aligned} \|T(u_m(x)) - T(u_n(x))\| &= \|u_{m+1}(x) - u_{n+1}(x)\| \\ &= \left\| \lambda \int_0^1 xt(u_m(t) - u_n(t))dt \right\| \\ &\le |\lambda| \left[\iint_0^{\pi/2a} \left(\frac{a}{2}\cos(ax)\right)^2 dxdt \right]^{1/2} \|u_m(x) - u_n(x)\| \\ &\le |\lambda| \sqrt{\frac{\pi^2}{32}} \|u_m(x) - u_n(x)\|, \end{aligned} \tag{2.28}$$

which implies that T has a fixed point. Also, putting $L = 0$ and $\alpha = |\lambda|\sqrt{\pi^2/32}$ shows that (1.1) holds for the nonlinear mapping T. All of the conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is T-stable.

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