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Research Article

A New Extension Theorem for Concave Operators

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We present a new and interesting extension theorem for concave operators as follows. Let X be a real linear space, and let (Y, K) be a real order complete PL space. Let the set $A \subset X \times Y$ be convex. Let X_0 be a real linear proper subspace of X, with $\theta \in (A_X - X_0)^{\mathrm{ri}}$, where $A_X = \{x \mid (x, y) \in A$ for some $y \in Y\}$. Let $g_0 : X_0 \to Y$ be a concave operator such that $g_0(x) \leq z$ whenever $(x, z) \in A$ and $x \in X_0$. Then there exists a concave operator $g : X \to Y$ such that (i) g is an extension of g_0, that is, $g(x) = g_0(x)$ for all $x \in X_0$, and (ii) $g(x) \leq z$ whenever $(x, z) \in A$.

1. Introduction

A very important result in functional analysis about the extension of a linear functional dominated by a sublinear function defined on a real vector space was first presented by Hahn [1] and Banach [2], which is known as the Hahn-Banach extension theorem. The complex version of Hahn-Banach extension theorem was proved by Bohnenblust and Sobczyk in [3]. Generalizations and variants of the Hahn-Banach extension theorem were developed in different directions in the past. Weston [4] proved a Hahn-Banach extension theorem in which a real-valued linear functional is dominated by a real-valued convex function. Hirano et al. [5] proved a Hahn-Banach theorem in which a concave functional is dominated by a sublinear functional in a nonempty convex set. Chen and Craven [6], Day [7], Peressini [8], Zowe [9–12], Elster and Nehse [13], Wang [14], Shi [15], and Brumelle [16] generalized the Hahn-Banach theorem to the partially ordered linear space. Yang [17] proved a Hahn-Banach theorem in which a linear map is weakly dominated by a set-valued map which is convex. Meng [18] obtained Hahn-Banach theorems by using concept of efficient for K-convex set-valued maps. Chen and Wang [19] proved a Hahn-Banach theorems in which a linear map is dominated by a K-set-valued map. Peng et al. [20] proved some Hahn-Banach theorems in which a linear map or an affine map is dominated by a K-set-valued map. Peng et al. [21] also proved a Hahn-Banach theorem in which an affine-like set-valued map is dominated by a K-set-valued map. The various geometric forms of Hahn-Banach theorems (i.e., Hahn-Banach separation theorems) were presented by Eidelheit [22], Rockafellar [23], Deumlich et al. [24], Taylor and Lay [25], Wang [14], Shi [15], and Elster and Nehse [26] in different spaces.

Hahn-Banach theorems play a central role in functional analysis, convex analysis, and optimization theory. For more details on Hahn-Banach theorems as well as their applications, please also refer to Jahn [27–29], Kantorovitch and Akilov [30], Lassonde [31], Rudin [32], Schechter [33], Aubin and Ekeland [34], Yosida [35], Takahashi [36], and the references therein.

The purpose of this paper is to present some new and interesting extension results for concave operators.

2. Preliminaries

Throughout this paper, unless other specified, we always suppose that X and Y are real linear spaces, θ is the zero element in both X and Y with no confusion, $K \subset Y$ is a pointed convex cone, and the partial order $\leq$ on a partially ordered linear space (in short, PL space) (Y, K) is defined by $y_1, y_2 \in Y$, $y_1 \leq y_2$ if and only if $y_2 - y_1 \in K$. If each subset of Y which is bounded above has a least upper bound in (Y, K), then Y is order complete. If A and B are subsets of a PL space (Y, K), then $A \leq B$ means that $a \leq b$ for each $a \in A$ and $b \in B$. Let C be a subset of X, then the algebraic interior of C is defined by

$$\operatorname{core} C = \{x \in C \mid \forall x_1 \in X, \exists \delta > 0, \text{ s.t. } \forall \lambda \in (0, \delta), \ x + \lambda x_1 \in C\}. \tag{2.1}$$

If $\theta \in \operatorname{core} C$, then C is called to be absorbed (see [14]).

The relative algebraic interior of C is denoted by C^{ri}, that is, C^{ri} is the algebraic interior of C with respect to the affine hull aff$C$ of C.

Let $F : X \to 2^Y$ be a set-valued map, then the domain of F is

$$D(F) = \{x \in X \mid F(x) \neq \varnothing\}, \tag{2.2}$$

the graph of F is a set in $X \times Y$:

$$\operatorname{Gr}(F) = \{(x, y) \mid x \in D(F), \ y \in Y, \ y \in F(x)\}, \tag{2.3}$$

and the epigraph of F is a set in $X \times Y$:

$$\operatorname{Epi}(F) = \{(x, y) \mid x \in D(F), \ y \in Y, \ y \in F(x) + K\}. \tag{2.4}$$

A set-valued map $F : X \to 2^Y$ is K-convex if its epigraph $\operatorname{Epi}(F)$ is a convex set.

An operator $f : D(f) \subset X \to Y$ is called a convex operator, if the domain $D(f)$ of f is a nonempty convex subset of X and if for all $x, y \in D(f)$ and all real number $\lambda \in [0, 1]$

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y). \tag{2.5}$$

The epigraph of f is a set in $X \times Y$:

$$\mathrm{Epi}(f) = \{(x, y) \mid x \in D(f),\, y \in Y,\, y \in f(x) + K\}. \tag{2.6}$$

It is easy to see that an operator f is convex if and only if $\mathrm{Epi}(f)$ is a convex set.

An operator $f : D(f) \subset X \to Y$ is called a concave operator if $D(f)$ is a nonempty convex subset of X and if for all $x, y \in D(f)$ and all real number $\lambda \in [0, 1]$

$$f(\lambda x + (1 - \lambda) y) \geq \lambda f(x) + (1 - \lambda) f(y). \tag{2.7}$$

An operator $f : X \to Y$ is called a sublinear operator, if for all $x, y \in X$ and all real number $\lambda \geq 0$,

$$\begin{aligned} f(\lambda x) &= \lambda f(x), \\ f(x + y) &\leq f(x) + f(y). \end{aligned} \tag{2.8}$$

It is clear that if $f : X \to Y$ is a sublinear operator, then f must be a convex operator, but the converse is not true in general.

For more detail about above definitions, please see [6–8, 16, 18, 20, 21, 27–30, 34] and the references therein.

3. An Extension Theorem with Applications

The following lemma is similar to the generalized Hahn-Banach theorem [7, page 105] and [4, Lemma 1].

Lemma 3.1. *Let X be a real linear space, and let (Y, K) be a real order complete PL space. Let the set $A \subset X \times Y$ be convex. Let X_0 be a real linear proper subspace of X, with $\theta \in$ core $(A_X - X_0)$, where $A_X = \{x \mid (x, y) \in A$ for some $y \in Y\}$. Let $g_0 : X_0 \to Y$ be a concave operator such that $g_0(x) \leq z$ whenever $(x, z) \in A$ and $x \in X_0$. Then there exists a concave operator $g : X \to Y$ such that (i) g is an extension of g_0, that is, $g(x) = g_0(x)$ for all $x \in X_0$, and (ii) $g(x) \leq z$ whenever $(x, z) \in A$.*

Proof. The theorem holds trivially if $A_X = X_0$. Assume that $A_X \neq X_0$. Since X_0 is a proper subspace of X, there exists $x_0 \in X \setminus X_0$. Let

$$X_1 = \{x + r x_0 : x \in X_0,\, r \in R\}. \tag{3.1}$$

It is clear that X_1 is a subspace of X, $X_0 \subset X_1$, $\theta \in$ core $(A_X - X_1)$, and the above representation of $x_1 \in X_1$ in the form $x_1 = x + r x_0$ is unique. Since $\theta \in$ core $(A_X - X_0)$, there exists $\lambda > 0$

such that $\pm\lambda x_0 \in A_X - X_0$. And so there exist $x_1 \in X_0$, $y_1 \in Y$ such that $(x_1 + \lambda x_0, y_1) \in A$ and $x_2 \in X_0$, $y_2 \in Y$ such that $(x_2 - \lambda x_0, y_2) \in A$. We define the sets B_1 and B_2 as follows:

$$B_1 = \left\{ \frac{y_1 - g_0(x_1)}{\lambda_1} \mid x_1 \in X_0,\ y_1 \in Y,\ \lambda_1 > 0,\ (x_1 + \lambda_1 x_0, y_1) \in A \right\},$$
$$B_2 = \left\{ \frac{g_0(x_2) - y_2}{\lambda_2} \mid x_2 \in X_0,\ y_2 \in Y,\ \lambda_2 > 0,\ (x_2 - \lambda_2 x_0, y_2) \in A \right\}. \tag{3.2}$$

It is clear that both B_1 and B_2 are nonempty.

Moreover, for all $b_1 \in B_1$ and for all $b_2 \in B_2$, we have $b_1 \geq b_2$. In fact, let $b_1 \in B_1$ and $b_2 \in B_2$, then there exist $x_1, x_2 \in X_0$, $y_1, y_2 \in Y$, $\lambda_1, \lambda_2 > 0$ such that $b_1 = (y_1 - g_0(x_1))/\lambda_1$, $b_2 = (g_0(x_2) - y_2)/\lambda_2$ and $(x_1 + \lambda_1 x_0, y_1), (x_2 - \lambda_2 x_0, y_2) \in A$. Let $\alpha = \lambda_2/(\lambda_1 + \lambda_2)$, then $\alpha\lambda_1 - (1-\alpha)\lambda_2 = 0$. Since A is a convex set, we have

$$\alpha(x_1 + \lambda_1 x_0, y_1) + (1-\alpha)(x_2 - \lambda_2 x_0, y_2) = (\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) \in A \tag{3.3}$$

and $\alpha x_1 + (1-\alpha)x_2 \in X_0$. It follows from the hypothesis that

$$g_0(\alpha x_1 + (1-\alpha)x_2) \leq \alpha y_1 + (1-\alpha)y_2. \tag{3.4}$$

It follows from the concavity of g_0 on X_0 that

$$\alpha[y_1 - g_0(x_1)] \geq (1-\alpha)[g_0(x_2) - y_2]. \tag{3.5}$$

That is,

$$\frac{y_1 - g_0(x_1)}{\lambda_1} \geq \frac{g_0(x_2) - y_2}{\lambda_2}. \tag{3.6}$$

That is, $b_1 \geq b_2$.

Since (Y, K) is an order-complete PL space, there exist the supremum of B_2 denoted by y^S and the infimum of B_1 denoted by y^I. Since $y^S \leq y^I$, taking $\overline{y} \in [y^S, y^I]$, then we have

$$\frac{y - g_0(x)}{\lambda} \geq \overline{y}, \quad \text{if } \lambda > 0,\ (x + \lambda x_0, y) \in A,\ x + \lambda x_0 \in X_1, \tag{3.7}$$

$$\overline{y} \geq \frac{g_0(x) - y}{\mu}, \quad \text{if } \mu > 0,\ (x - \mu x_0, y) \in A,\ x - \mu x_0 \in X_1. \tag{3.8}$$

By (3.7),

$$y \geq g_0(x) + \lambda\overline{y}, \quad \text{if } \lambda > 0,\ (x + \lambda x_0, y) \in A,\ x + \lambda x_0 \in X_1. \tag{3.9}$$

By (3.8),

$$y \geq g_0(x) - \mu\overline{y}, \quad \text{if } \mu > 0,\ (x - \mu x_0, y) \in A,\ x - \mu x_0 \in X_1. \tag{3.10}$$

We may relabel $-\mu$ by λ, then

$$y \geq g_0(x) + \lambda\overline{y}, \quad \text{if } \lambda < 0,\ (x + \lambda x_0, y) \in A,\ x + \lambda x_0 \in X_1. \tag{3.11}$$

Define a map g_1 from X_1 to Y as

$$g_1(x + \lambda x_0) = g_0(x) + \lambda\overline{y}, \quad \forall x + \lambda x_0 \in X_1. \tag{3.12}$$

Then $g_1(x) = g_0(x)$, $\forall x \in X_0$, that is, g_1 is an extension of g_0 to X_1. Since g_0 is a concave operator, it is easy to verify that g_1 is also a concave operator.

From (3.9) and (3.11), we know that g_1 satisfies

$$y \geq g_1(x + \lambda x_0), \quad \text{whenever } (x + \lambda x_0, y) \in A,\ x + \lambda x_0 \in X_1. \tag{3.13}$$

That is,

$$y \geq g_1(x), \quad \text{whenever } (x, y) \in A,\ x \in X_1. \tag{3.14}$$

Let Γ be the collection of all ordered pairs (X_Δ, g_Δ), where X_Δ is a subspace of X that contains X_0 and g_Δ is a concave operator from X_Δ to Y that extends g_0 and satisfies $y \geq g_\Delta(x)$ whenever $(x, y) \in A$ and $x \in X_\Delta$.

Introduce a partial ordering in Γ as follows: $(X_{\Delta_1}, g_{\Delta_1}) \prec (X_{\Delta_2}, g_{\Delta_2})$ if and only if $X_{\Delta_1} \subset X_{\Delta_2}$, $g_{\Delta_2}(x) = g_{\Delta_1}(x)$ for all $x \in X_{\Delta_1}$. If we can show that every totally ordered subset of Γ has an upper bound, it will follow from Zorn's lemma that Γ has a maximal element $(X_{\max}, g_{\max})$. We can claim that $g_{\max}$ is the desired map. In fact, we must have $X_{\max} = X$. For otherwise, we have shown in the previous proof of this lemma that there would be an $(\widetilde{X}_{\max}, \widetilde{g}_{\max}) \in \Gamma$ such that $(\widetilde{X}_{\max}, \widetilde{g}_{\max}) \succ (X_{\max}, g_{\max})$ and $(\widetilde{X}_{\max}, \widetilde{g}_{\max}) \neq (X_{\max}, g_{\max})$. This would violate the maximality of the $(X_{\max}, g_{\max})$.

Therefore, it remains to show that every totally ordered subset of Γ has an upper bound. Let M be a totally ordered subset of Γ. Define an ordered pair (X_M, g_M) by

$$\begin{gathered} X_M = \bigcup_{(X_\Delta, g_\Delta) \in M} \{X_\Delta\}, \\ g_M(x) = g_\Delta(x), \quad \forall x \in X_\Delta, \text{ where } (X_\Delta, g_\Delta) \in M. \end{gathered} \tag{3.15}$$

This definition is not ambiguous, for if $(X_{\Delta_1}, g_{\Delta_1})$ and $(X_{\Delta_2}, g_{\Delta_2})$ are any of the elements of M, then either $(X_{\Delta_1}, g_{\Delta_1}) \prec (X_{\Delta_2}, g_{\Delta_2})$ or $(X_{\Delta_2}, g_{\Delta_2}) \prec (X_{\Delta_1}, g_{\Delta_1})$. At any rate, if $x \in X_{\Delta_1} \cap X_{\Delta_1}$, then $g_{\Delta_1}(x) = g_{\Delta_2}(x)$. Clearly, $(X_M, g_M) \in \Gamma$. Hence, it is an upper bound for M, and the proof is complete. □

As a generalization of Lemma 3.1, we now present the main result asfollows.

Theorem 3.2. *Let X be a real linear space, and let (Y, K) be a real order complete PL space. Let the set $A \subset X \times Y$ be convex. Let X_0 be a real linear proper subspace of X, with $\theta \in (A_X - X_0)^{\mathrm{ri}}$, where $A_X = \{x \mid (x, y) \in A$ for some $y \in Y\}$. Let $g_0 : X_0 \to Y$ be a concave operator such that $g_0(x) \leq z$ whenever $(x, z) \in A$ and $x \in X_0$. Then there exists a concave operator $g : X \to Y$ such that (i) g is an extension of g_0, that is, $g(x) = g_0(x)$ for all $x \in X_0$, and (ii) $g(x) \leq z$ whenever $(x, z) \in A$.*

Proof. Consider $\overline{X} := \mathrm{aff}\,(A_X - X_0)$. Because $0 \in (A_X - X_0)^{\mathrm{ri}}$, $\overline{X}$ is a linear space.

If $\overline{X} = X$, then $0 \in \mathrm{core}\,(A_X - X_0)$. By Lemma 3.1, the result holds.

If $\overline{X} \neq X$. Of course, $A_X \subset \overline{X}$. Taking $x_0 \in X_0 \cap A_X$, we have that $X_0 = x_0 - X_0 \subset \overline{X}$. By Lemma 3.1, we can find $\overline{g} : \overline{X} \to Y$ a concave operator such that $\overline{g}(x) = g_0(x)$, $\forall x \in X_0$, and $\overline{g}(x) \leq y$ for all $(x, y) \in A \subset \overline{X} \times Y$. Taking $\overline{Y}$ a linear subspace of X such that $X = \overline{X} \oplus \overline{Y}$ (i.e., $X = \overline{X} + \overline{Y}$ and $\overline{X} \cap \overline{Y} = \{0\}$) and $g : X \to Y$ defined by $g(\overline{x} + \overline{y}) =: \overline{g}(\overline{x})$ for all $\overline{x} \in \overline{X}$, $\overline{y} \in \overline{Y}$, g verifies the conclusion. □

By Theorem 3.2, we can obtain the following new and interesting Hahn-Banach extension theorem in which a concave operator is dominated by a K-convex set-valued map.

Corollary 3.3. *Let X be a real linear space, and let (Y, K) be a real order complete PL space. Let $F : X \to 2^Y$ be a K-convex set-valued map. Let X_0 be a real linear proper subspace of X, with $\theta \in (D(F) - X_0)^{\mathrm{ri}}$. Let $g_0 : X_0 \to Y$ be a concave operator such that $g_0(x) \leq z$ whenever $(x, z) \in \mathrm{Gr}(F)$ and $x \in X_0$. Then there exists a concave operator $g : X \to Y$ such that (i) g is an extension of g_0, that is, $g(x) = g_0(x)$ for all $x \in X_0$, and (ii) $g(x) \leq z$ whenever $(x, z) \in \mathrm{Gr}(F)$.*

Proof. Let $A = \mathrm{Epi}(F)$. Then A is a convex set, $A_X = D(F)$, and $\theta \in (A_X - X_0)^{\mathrm{ri}}$. Since $g_0 : X_0 \to Y$ is a concave operator satisfying $g_0(x) \leq z$ whenever $(x, z) \in \mathrm{Gr}(F)$ and $x \in X_0$, we have that $g_0(x) \leq z$ whenever $(x, z) \in \mathrm{Epi}(F)$ and $x \in X_0$. Then by Theorem 3.2, there exists a concave operator $g : X \to Y$ such that (i) g is an extension of g_0, that is, $g(x) = g_0(x)$ for all $x \in X_0$, and (ii) $g(x) \leq z$ for all $(x, z) \in \mathrm{Epi}(F)$. Since $\mathrm{Gr}(F) \subset \mathrm{Epi}(F)$, we have $g(x) \leq z$ for all $(x, z) \in \mathrm{Gr}(F)$. □

Let $F : X \to 2^Y$ be replaced by a single-valued map $f : X \to Y$ in Corollary 3.3, then we have the following Hahn-Banach extension theorem in which a concave operator is dominated by a convex operator.

Corollary 3.4. *Let X be a real linear space, and let (Y, K) be a real order complete PL space. Let $f : D(f) \subset X \to Y$ be a convex operator. Let X_0 be a real linear proper subspace of X, with $\theta \in (D(f) - X_0)^{\mathrm{ri}}$. Let $g_0 : X_0 \to Y$ be a concave operator such that $g_0(x) \leq f(x)$ whenever $x \in X_0 \cap D(f)$. Then there exists a concave operator $g : X \to Y$ such that (i) g is an extension of g_0, that is, $g(x) = g_0(x)$ for all $x \in X_0$, and (ii) $g(x) \leq f(x)$ for all $x \in D(f)$.*

Since a sublinear operator is also a convex operator, so from corollary 3.4, we have the following result.

Corollary 3.5. *Let X be a real linear space, and let (Y, K) be a real order complete PL space. Let $p : X \to Y$ be a sublinear operator, and let X_0 be a real linear proper subspace of X. Let $g_0 : X_0 \to Y$ be a concave operator such that $g_0(x) \leq p(x)$ whenever $x \in X_0$. Then there exists a concave operator $g : X \to Y$ such that (i) g is an extension of g_0, that is, $g(x) = g_0(x)$ for all $x \in X_0$, and (ii) $g(x) \leq p(x)$ for all $x \in X$.*

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