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Research Article

The Solvability of a New System of Nonlinear Variational-Like Inclusions

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We introduce and study a new system of nonlinear variational-like inclusions involving s-(G,η)-maximal monotone operators, strongly monotone operators, η-strongly monotone operators, relaxed monotone operators, cocoercive operators, (λ,ξ)-relaxed cocoercive operators, (ζ,φ,ϱ)-g-relaxed cocoercive operators and relaxed Lipschitz operators in Hilbert spaces. By using the resolvent operator technique associated with s-(G,η)-maximal monotone operators and Banach contraction principle, we demonstrate the existence and uniqueness of solution for the system of nonlinear variational-like inclusions. The results presented in the paper improve and extend some known results in the literature.

1. Introduction

It is well known that the resolvent operator technique is an important method for solving various variational inequalities and inclusions [1–20]. In particular, the generalized resolvent operator technique has been applied more and more and has also been improved intensively. For instance, Fang and Huang [5] introduced the class of H-monotone operators and defined the associated resolvent operators, which extended the resolvent operators associated with η-subdifferential operators of Ding and Luo [3] and maximal η-monotone operators of Huang and Fang [6], respectively. Later, Liu et al. [17] researched a class of general nonlinear implicit variational inequalities including the H-monotone operators. Fang and Huang [4] created a class of (H,η)-monotone operators, which offered a unifying framework for the classes of maximal monotone operators, maximal η-monotone operators and H-monotone operators. Recently, Lan [8] introduced a class of (A,η)-accretive operators which further enriched and improved the class of generalized resolvent operators. Lan [10] studied a system of general mixed quasivariational inclusions involving (A,η)-accretive mappings in q-uniformly smooth Banach spaces. Lan et al. [14] constructed some iterative algorithms for solving a class of nonlinear (A,η)-monotone operator inclusion systems involving nonmonotone set-valued mappings in Hilbert spaces. Lan [9] investigated the existence of solutions for a class of (A,η)-accretive variational inclusion problems with nonaccretive set-valued mappings. Lan [11] analyzed and established an existence theorem for nonlinear parametric multivalued variational inclusion systems involving (A,η)-accretive mappings in Banach spaces. By using the random resolvent operator technique associated with (A,η)-accretive mappings, Lan [13] established an existence result for nonlinear random multi-valued variational inclusion systems involving (A,η)-accretive mappings in Banach spaces. Lan and Verma [15] studied a class of nonlinear Fuzzy variational inclusion systems with (A,η)-accretive mappings in Banach spaces. On the other hand, some interesting and classical techniques such as the Banach contraction principle and Nalder's fixed point theorems have been considered by many researchers in studying variational inclusions.

Inspired and motivated by the above achievements, we introduce a new system of nonlinear variational-like inclusions involving s-(G,η)-maximal monotone operators in Hilbert spaces and a class of (ζ,φ,ϱ)-g-relaxed cocoercive operators. By virtue of the Banach's fixed point theorem and the resolvent operator technique, we prove the existence and uniqueness of solution for the system of nonlinear variational-like inclusions. The results presented in the paper generalize some known results in the field.

2. Preliminaries

In what follows, unless otherwise specified, we assume that H_i is a real Hilbert space endowed with norm $\|\cdot\|_i$ and inner product $\langle\cdot,\cdot\rangle_i$, and 2^{H_i} denotes the family of all nonempty subsets of H_i for $i\in\{1,2\}$. Now let's recall some concepts.

Definition 2.1. Let $A : H_1 \to H_2, f, g : H_1 \to H_1, \eta : H_1 \times H_1 \to H_1$ be mappings.

(1) A is said to be *Lipschitz continuous*, if there exists a constant $\alpha > 0$ such that

$$\|Ax - Ay\|_2 \le \alpha\|x - y\|_1, \quad \forall x, y \in H_1; \tag{2.1}$$

(2) A is said to be *r-expanding,* if there exists a constant $r > 0$ such that

$$\|Ax - Ay\|_2 \ge r\|x - y\|_1, \quad \forall x, y \in H_1; \tag{2.2}$$

(3) f is said to be *δ-strongly monotone,* if there exists a constant $\delta > 0$ such that

$$\langle fx - fy, x - y\rangle_1 \ge \delta\|x - y\|_1^2, \quad \forall x, y \in H_1; \tag{2.3}$$

(4) f is said to be *δ-η-strongly monotone,* if there exists a constant $\delta > 0$ such that

$$\langle fx - fy, \eta(x, y)\rangle_1 \ge \delta\|x - y\|_1^2, \quad \forall x, y \in H_1; \tag{2.4}$$

(5) f is said to be $(\zeta, \varphi, \varrho)$-*g-relaxed cocoercive,* if there exist nonnegtive constants ζ, φ and ϱ such that

$$\langle fx - fy, gx - gy \rangle_1 \geq -\zeta \|fx - fy\|_1^2 - \varphi \|gx - gy\|_1^2 + \varrho \|x - y\|_1^2, \quad \forall x, y \in H_1; \tag{2.5}$$

(6) g is said to be ζ-*relaxed Lipschitz,* if there exists a constant $\zeta > 0$ such that

$$\langle gx - gy, x - y \rangle_1 \leq -\zeta \|x - y\|_1^2, \quad \forall x, y \in H_1. \tag{2.6}$$

Definition 2.2. Let $N : H_2 \times H_1 \times H_2 \to H_1, A, C : H_1 \to H_2, B : H_2 \to H_1$ be mappings. N is called

(1) (λ, ξ)-*relaxed cocoercive* with respect to A in the first argument, if there exist nonnegative constants λ, ξ such that

$$\begin{aligned} &\langle N(Au, x, y) - N(Av, x, y), u - v \rangle_1 \\ &\quad \geq -\lambda \|Au - Av\|_2^2 + \xi \|u - v\|_1^2, \quad \forall u, v, x \in H_1, y \in H_2; \end{aligned} \tag{2.7}$$

(2) θ-*cocoercive* with respect to B in the second argument, if there exists a constant $\theta > 0$ such that

$$\langle N(x, Bu, y) - N(x, Bv, y), u - v \rangle_1 \geq \theta \|Bu - Bv\|_1^2, \quad \forall u, v, x, y \in H_2; \tag{2.8}$$

(3) τ-*relaxed Lipschitz* with respect to C in the third argument, if there exists a constant $\tau > 0$ such that

$$\langle N(x, y, Cu) - N(x, y, Cv), u - v \rangle_1 \leq -\tau \|u - v\|_1^2, \quad \forall u, v, y \in H_1, \ x \in H_2; \tag{2.9}$$

(4) τ-*relaxed monotone* with respect to C in the third argument, if there exists a constant $\tau > 0$ such that

$$\langle N(x, y, Cu) - N(x, y, Cv), u - v \rangle_1 \geq -\tau \|u - v\|_1^2, \quad \forall u, v, y \in H_1, x \in H_2; \tag{2.10}$$

(5) *Lipschitz continuous* in the first argument, if there exists a constant $\mu > 0$ such that

$$\|N(u, x, y) - N(v, x, y)\|_1 \leq \mu \|u - v\|_1, \quad \forall u, v, y \in H_2, x \in H_1. \tag{2.11}$$

Similarly, we can define the Lipschitz continuity of N in the second and third arguments, respectively.

Definition 2.3. For $i \in \{1,2\}, j \in \{1,2\} \setminus \{i\}$, let $M_i : H_j \times H_i \to 2^{H_i}$, $\eta_i : H_i \times H_i \to H_i$ be mappings. For each given $(x_2, x_1) \in H_1 \times H_2$ and $i \in \{1,2\}, M_i(x_i, \cdot) : H_i \to 2^{H_i}$ is said to be *s_i-η_i-relaxed monotone,* if there exists a constant $s_i > 0$ such that

$$\langle x^* - y^*, \eta_i(x,y) \rangle_i \geq -s_i \|x - y\|_i^2, \quad \forall (x, x^*), (y, y^*) \in \operatorname{graph}(M_i(x_i, \cdot)). \tag{2.12}$$

Definition 2.4. For $i \in \{1,2\}, j \in \{1,2\} \setminus \{i\}$, let $M_i : H_j \times H_i \to 2^{H_i}, G_i : H_i \to H_i$ be mappings. For any given $(x_2, x_1) \in H_1 \times H_2$ and $i \in \{1,2\}, M_i(x_i, \cdot) : H_i \to 2^{H_i}$ is said to be *s_i-(G_i, η_i)-maximal monotone,* if (B1) $M_i(x_i, \cdot)$ is s_i-η_i-relaxed monotone; (B2) $(G_i + \rho_i M_i(x_i, \cdot))H_i = H_i$ for $\rho_i > 0$.

Lemma 2.5 (see [8]). *Let H be a real Hilbert space, $\eta : H \times H \to H$ be a mapping, $G : H \to H$ be a d-η-strongly monotone mapping and $M : H \to 2^H$ be a s-(G, η)-maximal monotone mapping. Then the generalized resolvent operator $R_{M,\rho}^{G,\eta} = (G + \rho M)^{-1} : H \to H$ is singled-valued for $d > \rho s > 0$.*

Lemma 2.6 (see [8]). *Let H be a real Hilbert space, $\eta : H \times H \to H$ be a σ-Lipschitz continuous mapping, $G : H \to H$ be a d-η-strongly monotone mapping, and $M : H \to 2^H$ be a s-(G, η)-maximal monotone mapping. Then the generalized resolvent operator $R_{M,\rho}^{G,\eta} : H \to H$ is $\sigma/(d - \rho s)$-Lipschitz continuous for $d > \rho s > 0$.*

For $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, assume that $A_i, C_i : H_i \to H_j, B_i : H_j \to H_i, \eta_i : H_i \times H_i \to H_i, N_i : H_j \times H_i \times H_j \to H_i, f_i, g_i : H_i \to H_i$ are single-valued mappings, $M_i : H_j \times H_i \to 2^{H_i}$ satisfies that for each given $x_i \in H_j$, $M_i(x_i, \cdot)$ is s_i-(G_i, η_i)-maximal monotone, where $G_i : H_i \to H_i$ is d_i-η_i-strongly monotone and $\operatorname{Range}(f_i - g_i) \bigcap \operatorname{dom} M_i(x_i, \cdot) \neq \emptyset$. We consider the following problem of finding $(x, y) \in H_1 \times H_2$ such that

$$\begin{aligned} x &\in N_1(A_1 x, B_1 y, C_1 x) + M_1(y, (f_1 - g_1)x), \\ y &\in N_2(A_2 y, B_2 x, C_2 y) + M_2(x, (f_2 - g_2)y), \end{aligned} \tag{2.13}$$

where $(f_i - g_i)x = f_i(x) - g_i(x)$ for $x \in H_i$ and $i \in \{1,2\}$. The problem (2.13) is called the system of nonlinear variational-like inclusions problem.

Special cases of the problem (2.13) are as follows.

If $A_1 = B_1 = B_2 = C_2 = f_1 - g_1 = f_2 - g_2 = I$, $N_1(x, y, z) = N_1(x, y) + x$, $N_2(u, v, w) = N_2(v, w) + w$, $M_1(x, y) = M_1(y)$, $M_2(u, v) = M_2(v)$ for each $x, z, v \in H_2, y, u, w \in H_1$, then the problem (2.13) collapses to finding $(x, y) \in H_1 \times H_2$ such that

$$\begin{aligned} 0 &\in N_1(x, y) + M_1(x), \\ 0 &\in N_2(x, y) + M_2(y), \end{aligned} \tag{2.14}$$

which was studied by Fang and Huang [4] with the assumption that M_i is (G_i, η_i)-monotone for$i \in \{1,2\}$.

If $H_i = H, A_i = A, B_i = B, C_i = C, M_i = M, f_i = f, g_i = g$, and $N_i(u,v,w) = N(u,v)$, for all $u,v,w \in H$ for $i \in \{1,2\}$, then the problem (2.13) reduces to finding $x \in H$ such that

$$0 \in N(Ax,Bx) + M(x,(f-g)x), \tag{2.15}$$

which was studied in Shim et al. [19].

It is easy to see that the problem (2.13) includes a number of variational and variational-like inclusions as special cases for appropriate and suitable choice of the mappings $N_i, A_i, B_i, C_i, M_i, f_i, g_i$ for $i \in \{1,2\}$.

3. Existence and Uniqueness Theorems

In this section, we will prove the existence and uniqueness of solution of the problem (2.13).

Lemma 3.1. *Let ρ_1 and ρ_2 be two positive constants. Then $(x,y) \in H_1 \times H_2$ is a solution of the problem* (2.13) *if and only if $(x,y) \in H_1 \times H_2$ satisfies that*

$$\begin{aligned} f_1(x) &= g_1(x) + R^{G_1,\eta_1}_{M_1(y,\cdot),\rho_1}\left[x + G_1((f_1-g_1)x) - \rho_1 N_1(A_1x, B_1y, C_1x)\right], \\ f_2(y) &= g_2(y) + R^{G_2,\eta_2}_{M_2(x,\cdot),\rho_2}\left[y + G_2((f_2-g_2)y) - \rho_2 N_2(A_2y, B_2x, C_2y)\right], \end{aligned} \tag{3.1}$$

where $R^{G_1,\eta_1}_{M_1(y,\cdot),\rho_1}(u) = (G_1 + \rho_1 M_1(y,\cdot))^{-1}(u), R^{G_2,\eta_2}_{M_2(x,\cdot),\rho_2}(v) = (G_2 + \rho_2 M_2(x,\cdot))^{-1}(v)$, for all $(u,v) \in H_1 \times H_2$.

Theorem 3.2. *For $i \in \{1,2\}, j \in \{1,2\} \setminus \{i\}$, let $\eta_i : H_i \times H_i \to H_i$ be Lipschitz continuous with constant σ_i, $A_i, C_i : H_i \to H_j, B_i : H_j \to H_i, f_i, g_i : H_i \to H_i$ be Lipschitz continuous with constants $\alpha_i, \gamma_i, \beta_i, \vartheta_{f_i}, \vartheta_{g_i}$ respectively, $N_i : H_j \times H_i \times H_j \to H_i$ be Lipschitz continuous in the first, second and third arguments with constants μ_i, ν_i, ω_i respectively, let N_i be (λ_i, ξ_i)-relaxed cocoercive with respect to A_i in the first argument, and τ_i-relaxed Lipschitz with respect to C_i in the third argument, f_i be $(\zeta_i, \varphi_i, \varrho_i)$-$g_i$-relaxed cocoercive, $f_i - g_i$ be δ_{f_i,g_i}-strongly monotone, $G_i : H_i \to H_i$ be t_i-Lipschitz continuous and d_i-η_i-strongly monotone, and $G_i(f_i - g_i)$ be ζ_i-relaxed Lipschitz, $M_i : H_j \times H_i \to 2^{H_i}$ satisfy that for each fixed $x_i \in H_j$, $M_i(x_i,\cdot) : H_i \to 2^{H_i}$ is s_i-(G_i,η_i)-maximal monotone, Range$(f_i - g_i) \cap \operatorname{dom} M_i(x_i,\cdot) \neq \emptyset$ and*

$$\left\| R^{G_i,\eta_i}_{M_i(y_i,\cdot),\rho_i}(x) - R^{G_i,\eta_i}_{M_i(z_i,\cdot),\rho_i}(x) \right\|_i \le r \|y_i - z_i\|_j, \quad \forall x \in H_i,\ y_i, z_i \in H_j, i \in \{1,2\},\ j \in \{1,2\} \setminus \{i\}. \tag{3.2}$$

If there exist positive constants ρ_1, ρ_2, and k such that

$$d_i > \rho_i s_i, \quad i \in \{1,2\}, \tag{3.3}$$

$$k = \max\left\{ m_1 + \frac{\sigma_1}{d_1 - \rho_1 s_1}(c_1 + \rho_1 l_1) + \frac{\sigma_2}{d_2 - \rho_2 s_2}\chi_2,\ m_2 + \frac{\sigma_2}{d_2 - \rho_2 s_2}(c_2 + \rho_2 l_2) + \frac{\sigma_1}{d_1 - \rho_1 s_1}\chi_1 \right\} + r < 1, \tag{3.4}$$

where

$$
\begin{gathered}
m_i = \sqrt{1 - 2\delta_{f_i,g_i} + \left[\vartheta_{f_i}^2 + 2(\zeta_i\vartheta_{f_i} + \varphi_i\vartheta_{g_i} - \varrho_i) + \vartheta_{g_i}^2\right]}, \\
c_i = \sqrt{1 - 2\zeta_i + t_i^2\left(\vartheta_{f_i} + \vartheta_{g_i}\right)^2}, \\
l_i = \sqrt{\mu_i^2\alpha_i^2 + 2(\lambda_i\alpha_i - \xi_i) + 1} + \sqrt{\omega_i^2\gamma_i^2 - 2\tau_i + 1}, \\
\chi_i = \rho_i\nu_i\beta_i, \quad i \in \{1, 2\},
\end{gathered}
\tag{3.5}
$$

then the problem (2.13) *possesses a unique solution in* $H_1 \times H_2$.

Proof. For any $(x, y) \in H_1 \times H_2$, define

$$
\begin{gathered}
F_{\rho_1}(x, y) = x - (f_1 - g_1)x + R_{M_1(y,\cdot),\rho_1}^{G_1,\eta_1}\left[x + G_1((f_1 - g_1)x) - \rho_1 N_1(A_1x, B_1y, C_1x)\right], \\
F_{\rho_2}(x, y) = y - (f_2 - g_2)y + R_{M_2(x,\cdot),\rho_2}^{G_2,\eta_2}\left[y + G_2((f_2 - g_2)y) - \rho_2 N_2(A_2y, B_2x, C_2y)\right].
\end{gathered}
\tag{3.6}
$$

For each $(u_1, v_1), (u_2, v_2) \in H_1 \times H_2$, it follows from Lemma 2.6 that

$$
\begin{aligned}
&\left\| F_{\rho_1}(u_1, v_1) - F_{\rho_1}(u_2, v_2) \right\|_1 \\
&\quad \le \left\| u_1 - u_2 - \left[(f_1 - g_1)u_1 - (f_1 - g_1)u_2\right] \right\|_1 + \frac{\sigma_1}{d_1 - \rho_1 s_1} \\
&\qquad \times \left\{ \left\| u_1 - u_2 + G_1((f_1 - g_1)u_1) - G_1((f_1 - g_1)u_2) \right\|_1 \right. \\
&\qquad\quad \left. + \rho_1 \| N_1(A_1u_1, B_1v_1, C_1u_1) - N_1(A_1u_2, B_1v_2, C_1u_2) \|_1 \right\} + r\|v_1 - v_2\|_2.
\end{aligned}
\tag{3.7}
$$

Because $f_1 - g_1$ is δ_{f_1,g_1}-strongly monotone, f_1, g_1 and G_1 are Lipschitz continuous, and $G_1(f_1 - g_1)$ is ζ_1-relaxed Lipschitz, we deduce that

$$
\begin{aligned}
&\left\| u_1 - u_2 - \left[(f_1 - g_1)u_1 - (f_1 - g_1)u_2\right] \right\|_1^2 \\
&\quad \le \left(1 - 2\delta_{f_1,g_1} + \left(\vartheta_{f_1}^2 + 2(\zeta_1\vartheta_{f_1} + \varphi_1\vartheta_{g_1} - \varrho_1) + \vartheta_{g_1}^2\right)\right)\|u_1 - u_2\|_1^2,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&\left\| u_1 - u_2 + G_1((f_1 - g_1)u_1) - G_1((f_1 - g_1)u_2) \right\|_1^2 \\
&\quad \le \left(1 - 2\zeta_1 + t_1^2\left(\vartheta_{f_1} + \vartheta_{g_1}\right)^2\right)\|u_1 - u_2\|_1^2.
\end{aligned}
\tag{3.9}
$$

Since A_1, B_1, C_1 are all Lipschitz continuous, N_1 is (λ_1, ξ_1)-relaxed cocoercive with respect to A_1, τ_1-relaxed Lipschitz with respect to C_1, and is Lipschitz continuous in the first, second and third arguments, respectively, we infer that

$$\begin{aligned}&\|N_1(A_1u_1, B_1v_1, C_1u_1) - N_1(A_1u_2, B_1v_1, C_1u_1) - (u_1 - u_2)\|_1^2\\&\quad\leq \left(\mu_1^2\alpha_1^2 + 2(\lambda_1\alpha_1 - \xi_1) + 1\right)\|u_1 - u_2\|_1^2,\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\|N_1(A_1u_2, B_1v_2, C_1u_1) - N_1(A_1u_2, B_1v_2, C_1u_2) + u_1 - u_2\|_1^2\\&\quad\leq \left(\omega_1^2\gamma_1^2 - 2\tau_1 + 1\right)\|u_1 - u_2\|_1^2,\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\|N_1(A_1u_2, B_1v_1, C_1u_1) - N_1(A_1u_2, B_1v_2, C_1u_1)\|\\&\quad\leq \nu_1\beta_1\|v_1 - v_2\|_2.\end{aligned} \tag{3.12}$$

In terms of (3.7)–(3.12), we obtain that

$$\begin{aligned}&\left\|F_{\rho_1}(u_1 - v_1) - F_{\rho_1}(u_2, v_2)\right\|\\&\quad\leq m_1\|u_1 - u_2\|_1 + \frac{\sigma_1}{d_1 - \rho_1 s_1}\left[(c_1 + \rho_1 l_1)\|u_1 - u_2\|_1 + \chi_1\|v_1 - v_2\|_2\right] + r\|v_1 - v_2\|_2.\end{aligned} \tag{3.13}$$

Similarly, we deduce that

$$\begin{aligned}&\left\|F_{\rho_2}(u_1, v_1) - F_{\rho_2}(u_2, v_2)\right\|\\&\quad\leq m_2\|v_1 - v_2\|_2 + \frac{\sigma_2}{d_2 - \rho_2 s_2}\left[(c_2 + \rho_2 l_2)\|v_1 - v_2\|_2 + \chi_2\|u_1 - u_2\|_1\right] + r\|u_1 - u_2\|_1.\end{aligned} \tag{3.14}$$

Define $\|\cdot\|_*$ on $H_1 \times H_2$ by $\|(u, v)\|_* = \|u\|_1 + \|v\|_1$ for any $(u, v) \in H_1 \times H_2$. It is easy to see that $(H_1 \times H_2, \|\cdot\|_*)$ is a Banach space. Define $L_{\rho_1,\rho_2} : H_1 \times H_2 \to H_1 \times H_2$ by

$$L_{\rho_1,\rho_2}(u, v) = \left(F_{\rho_1}(u, v), F_{\rho_2}(u, v)\right), \quad \forall (u, v) \in H_1 \times H_2. \tag{3.15}$$

By virtue of (3.3),(3.4),(3.13) and (3.14), we achieve that $0 < k < 1$ and

$$\left\|L_{\rho_1,\rho_2}(u_1, v_1) - L_{\rho_1,\rho_2}(u_2, v_2)\right\|_* \leq k\|(u_1, v_1) - (u_2, v_2)\|_*, \tag{3.16}$$

which means that $L_{\rho_1,\rho_2} : H_1 \times H_2 \to H_1 \times H_2$ is a contractive mapping. Hence, there exists a unique $(x, y) \in H_1 \times H_2$ such that $L_{\rho_1,\rho_2}(x, y) = (x, y)$. That is,

$$\begin{aligned}f_1(x) &= g_1(x) + R^{G_1,\eta_1}_{M_1(y,\cdot),\rho_1}\left[x + G_1\left((f_1 - g_1)x\right) - \rho_1 N_1\left(A_1x, B_1y, C_1x\right)\right],\\f_2(y) &= g_2(y) + R^{G_2,\eta_2}_{M_2(x,\cdot),\rho_2}\left[y + G_2\left((f_2 - g_2)y\right) - \rho_2 N_2\left(A_2y, B_2x, C_2y\right)\right].\end{aligned} \tag{3.17}$$

By Lemma 3.1, we derive that (x, y) is a unique solution of the problem (2.13). This completes the proof. □

Theorem 3.3. *For $i \in \{1,2\}, j \in \{1,2\} \setminus \{i\}$, let $\eta_i, A_i, C_i, M_i, f_i, g_i, f_i - g_i, G_i$ be all the same as in Theorem 3.2, $B_i : H_j \to H_i$ be r_i-expanding, $N_i : H_j \times H_i \times H_j \to H_i$ be Lipschitz continuous in the first, second and third arguments with constants μ_i, ν_i, ω_i respectively, and N_i be (λ_i, ξ_i)-relaxed cocoercive with respect to A_i in the first argument, be θ_i-cocoercive with respect to B_i in the second argument, be τ_i-relaxed Lipschtz with respect to C_i in the third argument. If there exist constants ρ_1, ρ_2 and k such that* (3.3) *and* (3.4), *but*

$$c_i = t_i\sqrt{\vartheta_{f_i}^2 + 2(\zeta_i\vartheta_{f_i} + \varphi_i\vartheta_{g_i} - \varrho_i) + \vartheta_{g_i}^2}, \quad \chi_i = \sqrt{\rho_i^2\nu_i^2\beta_i^2 - 2\rho_i\theta_i r_i + 1}, \quad i \in \{1,2\}, \tag{3.18}$$

then the problem (2.13) *possesses a unique solution in $H_1 \times H_2$.*

Theorem 3.4. *For $i \in \{1,2\}, j \in \{1,2\} \setminus \{i\}$, let $\eta_i, A_i, B_i, C_i, M_i, f_i, g_i, f_i - g_i, G_i, G_i(f_i - g_i)$ be all the same as in Theorem 3.2, $N_i : H_j \times H_i \times H_j \to H_i$ be Lipschitz continuous in the first, second and third arguments with constants μ_i, ν_i, ω_i respectively, and N_i be (λ_i, ξ_i)-relaxed cocoercive with respect to A_i in the first argument, be θ_i-relaxed Lipschitz with respect to B_i in the second argument, be τ_i-relaxed monotone with respect to C_i in the third argument. If there exist constants ρ_1, ρ_2 and k such that* (3.3) *and* (3.4), *but*

$$l_i = \sqrt{(\mu_i\alpha_i + \omega_i\gamma_i)^2 + 2(\lambda_i\alpha_i - \xi_i + \tau_i) + 1}, \quad \chi_i = \rho_i\sqrt{\nu_i^2\beta_i^2 - 2\theta_i + 1}, \quad i \in \{1,2\}, \tag{3.19}$$

then the problem (2.13) *possesses a unique solution in $H_1 \times H_2$.*

Remark 3.5. In this paper, there are three aspects which are worth of being mentioned as follows:

(1) Theorem 3.2 extends and improves in [4, Theorem 3.1] and in [19, Theorem 4.1];

(2) the class of $(\zeta, \varphi, \varrho)$-$g$-relaxed cocoercive operators includes the class of (α, ξ)-relaxed cocoercive operators in [8] as a special case;

(3) the class of s-(G, η)-maximal monotone operators is a generalization of the classes of η-subdifferential operators in [3], maximal η-monotone operators in [6], H-monotone operators in [5] and (H, η)-monotone operators in [4].

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