

Research Article

A Continuation Method for Weakly Contractive Mappings under the Interior Condition

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Recently, Frigon proved that, for weakly contractive maps, the property of having a fixed point is invariant by a certain class of homotopies, obtaining as a consequence a Leray-Schauder alternative for this class of maps in a Banach space. We prove here that the Leray-Schauder condition in the aforementioned result can be replaced by a modification of it, the interior condition. We also show that our arguments work for a certain class of generalized contractions, thus complementing a result of Agarwal and O'Regan.

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1. Introduction

Suppose that X is a Banach space, that $U \subset X$ is an open bounded subset of X , containing the origin, and that $f : \bar{U} \rightarrow X$ is a mapping. It is well known that if f satisfies the Leray-Schauder condition defined as

$$f(x) \neq \lambda x, \quad \text{for } x \in \partial U, \lambda > 1 \tag{L-S}$$

and f is a strict set-contraction or, more generally, condensing, then f has a fixed point in \bar{U} (see, e.g., [1] or [2]). The first continuation method in the setting of a complete metric space for contractive maps comes from the hands of Granas [3], in 1994, who gave a homotopy result for contractive maps (for more information on this topic see, e.g., [4, 5] or [6]).

On the other hand, it has been recently shown in [7] that, for condensing mappings, the condition (L-S) can be replaced by a modification of it which we call the interior condition,

and is defined as follows: a mapping $f : \overline{U} \rightarrow X$ satisfies the *Interior Condition* (I-C), if there exists $\delta > 0$ such that

$$f(x) \neq \lambda x, \quad \text{for } x \in U_\delta, \lambda > 1, f(x) \notin \overline{U}, \quad (\text{I-C})$$

where $U_\delta = \{x \in U : \text{dist}(x, \partial U) < \delta\}$ (some generalizations of this result can be found in [8, 9]).

We remark that the condition (I-C) by itself cannot be a substitute for the condition (L-S), and an additional assumption on the domain of f needs to be made in order to guarantee the existence of a fixed point for f . The class of sets that we need is defined as follows: suppose that $U \subset X$ is an open neighborhood of the origin. We say that U is strictly star shaped if for any $x \in \partial U$ we have that $\{\lambda x : \lambda > 0\} \cap \partial U = \{x\}$. It was shown in [7] that if U is bounded and strictly star shaped and $f : \overline{U} \rightarrow X$ is a condensing mapping satisfying the condition (I-C), then f has a fixed point. Of course, this result includes the case of a contractive map (i.e., a map f for which there exists $k \in [0, 1)$ such that $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in \overline{U}$), but our aim in this note is, following the pattern of Granas [3] and Frigon et al. [10], to give a continuation method for weakly contractive mappings, in the setting of a complete metric space, under some conditions on the homotopy which are the counterpart of the condition (I-C) and the notion of a strictly star shaped set in a space without a vector structure. Finally, in the last section we show that our arguments also work for a class of generalized contractions, thus complementing a result of Agarwal and O'Regan [11].

2. Weakly Contractive Maps

In this chapter we deal with the concept of weakly contractive maps, as it was introduced by Dugundji and Granas in [12].

Definition 2.1. Let (X, d) be a complete metric space and U an open subset of X . A function $f : \overline{U} \rightarrow X$ is said to be weakly contractive if there exists $\psi : X \times X \rightarrow (0, \infty)$ compactly positive (i.e., $\inf\{\psi(x, y) : a \leq d(x, y) \leq b\} = \theta(a, b) > 0$ for every $0 < a \leq b$) such that

$$d(f(x), f(y)) \leq d(x, y) - \psi(x, y). \quad (2.1)$$

If ψ is a compactly positive function, we define for $0 < a \leq b$

$$\gamma(a, b) = \min\{a, \theta(a, b)\}. \quad (2.2)$$

It was shown in [12] that any weakly contractive map $f : X \rightarrow X$ defined on a complete metric space X has a unique fixed point. Some years later, Frigon [5] proved that, for weakly contractive maps, the property of having a fixed point is invariant by a certain class of homotopies, obtaining as a consequence a Leray-Schauder alternative for weakly contractive maps in the setting of a Banach space. We prove here that the Leray-Schauder condition in the aforementioned result can be replaced by the condition (I-C), and it will also be obtained as a consequence of a continuation method. The definition of homotopy that we need for our purposes is the following.

Definition 2.2. Let (X, d) be a complete metric space, and U an open subset of X . Let $f, g : \overline{U} \rightarrow X$ be two weakly contractive maps. We say that f is (I-C)-homotopic to g if there exists $H : \overline{U} \times [0, 1] \rightarrow X$ with the following properties:

- (P1) $H(x, 1) = f(x)$ and $H(x, 0) = g(x)$ for every $x \in \overline{U}$;
- (P2) there exists $\delta > 0$ such that $x \neq H(x, t)$ for every $x \in U_\delta$, with $f(x) \notin \overline{U}$, and $t \in [0, 1]$, where $U_\delta = \{x \in U : \text{dist}(x, \partial U) < \delta\}$;
- (P3) there exists a compactly positive function $\psi : X \times X \rightarrow (0, \infty)$ such that $d(H(x, t), H(y, t)) \leq d(x, y) - \psi(x, y)$ for every $x, y \in \overline{U}$, and $t \in [0, 1]$;
- (P4) there exists a continuous function $\phi : [0, 1] \rightarrow \mathbb{R}$ such that, for every $x \in \overline{U}$ and $t, s \in [0, 1]$, $d(H(x, t), H(x, s)) \leq |\phi(t) - \phi(s)|$;
- (P5) if $x \in \partial U$ and $0 \leq \lambda < 1$, with $H(x, \lambda) \in \partial U$, then $H(x, 1) \notin \overline{U}$.

In the proof of the main result of this chapter we shall make use of the following lemma (see Frigon [5]).

Lemma 2.3. *Let $x_0 \in X$, $r > 0$, and $h : \overline{B(x_0, r)} \rightarrow X$ weakly contractive. If $d(x_0, h(x_0)) < \gamma(r/2, r)$, then h has a fixed point.*

Theorem 2.4. *Let $f, g : \overline{U} \rightarrow X$ be two weakly contractive maps. Suppose that f is homotopic to g and $g(\overline{U})$ is bounded. If g has a fixed point in U , then f has a fixed point in \overline{U} .*

Proof. We argue by contradiction. Suppose that f does not have any fixed point in \overline{U} , and let H be a homotopy between f and g , in the sense of Definition 2.1. Consider the set

$$A = \{\lambda \in [0, 1] : x = H(x, \lambda) \text{ for some } x \in U\}, \quad (2.3)$$

and notice that A is nonempty since g has a fixed point in U , that is, $0 \in A$. We will show that A is both open and closed in $[0, 1]$, and hence, by connectedness, we will have that $A = [0, 1]$. As a result, f will have a fixed point in U , which establishes a contradiction.

To show that A is closed, suppose that $\{\lambda_n\}$ is a sequence in A converging to $\lambda \in [0, 1]$ and let us show that $\lambda \in A$. Since $\lambda_n \in A$, there exists $x_n \in U$ with $x_n = H(x_n, \lambda_n)$. Fix $\varepsilon > 0$. Using that $g(\overline{U})$ is bounded and that ϕ is continuous on the compact interval $[0, 1]$, it is easy to show that there exists $M > \varepsilon$ such that $\text{diam } H(\overline{U} \times [0, 1]) \leq M$, and hence $d(x_n, x_m) \leq M$ for all $n, m \in \mathbb{N}$. Define $\mu = \theta(\varepsilon, M)$ and let $n_0 \in \mathbb{N}$ be such that for all $n, m \geq n_0$, $|\phi(\lambda_n) - \phi(\lambda_m)| < \mu$. Then $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$ because, otherwise, we would have $d(x_n, x_m) \geq \varepsilon$ for some $n, m \geq n_0$, and then

$$\begin{aligned} d(x_n, x_m) &= d(H(x_n, \lambda_n), H(x_m, \lambda_m)) \\ &\leq d(H(x_n, \lambda_n), H(x_n, \lambda_m)) + d(H(x_n, \lambda_m), H(x_m, \lambda_m)) \\ &\leq |\phi(\lambda_n) - \phi(\lambda_m)| + d(x_n, x_m) - \psi(x_n, x_m) \\ &< \mu + d(x_n, x_m) - \psi(x_n, x_m) \\ &\leq d(x_n, x_m), \end{aligned} \quad (2.4)$$

which is a contradiction. Then $\{x_n\}$ is a Cauchy sequence and, since (X, d) is complete, there exists $x_0 \in \overline{U}$ such that $x_n \rightarrow x_0$ as $n \rightarrow \infty$. In addition, $x_0 = H(x_0, \lambda)$ since for all $n \in \mathbb{N}$ we have that

$$\begin{aligned} d(x_n, H(x_0, \lambda)) &= d(H(x_n, \lambda_n), H(x_0, \lambda)) \\ &\leq d(H(x_n, \lambda_n), H(x_n, \lambda)) + d(H(x_n, \lambda), H(x_0, \lambda)) \\ &\leq |\phi(\lambda_n) - \phi(\lambda)| + d(x_n, x_0) - \psi(x_n, x_0) \\ &\leq |\phi(\lambda_n) - \phi(\lambda)| + d(x_n, x_0). \end{aligned} \tag{2.5}$$

Observe that $0 \leq \lambda < 1$, because if $\lambda = 1$, then $x_0 = H(x_0, 1) = f(x_0)$, which contradicts the fact that f does not have any fixed point in \overline{U} . Notice that $x_0 \in U$, because, otherwise, we would have $x_0 \in \partial U$, that is, $H(x_0, \lambda) \in \partial U$, and since $0 \leq \lambda < 1$, by (P5), we have that $H(x_0, 1) \notin \overline{U}$. However, since $x_0 \in \partial U$, $\{x_n\} \rightarrow x_0$ and $x_n \in U$ for all $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U_\delta$ for all $n \geq n_0$. Hence, since $x_n = H(x_n, \lambda_n)$ for all $n \geq n_0$, applying (P2), we have that $f(x_n) \in \overline{U}$ for all $n \geq n_0$, that is, $H(x_n, 1) \in \overline{U}$ for all $n \geq n_0$. Taking limits, we arrive to the contradiction $H(x_0, 1) \in \overline{U}$.

Therefore, $x_0 \in U$ and, consequently, $\lambda \in A$.

Next we show that A is open in $[0, 1]$. Let $\lambda_0 \in A$. Then there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Let $r > 0$ be such that $B(x_0, r) \subset U$, and let $\delta > 0$ such that $|\phi(\lambda) - \phi(\lambda_0)| < \gamma(r/2, r)$ for every $\lambda \in [0, 1]$ with $|\lambda_0 - \lambda| < \delta$. Then, if $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1]$,

$$\begin{aligned} d(x_0, H(x_0, \lambda)) &= d(H(x_0, \lambda_0), H(x_0, \lambda)) \\ &\leq |\phi(\lambda_0) - \phi(\lambda)| \\ &< \gamma\left(\frac{r}{2}, r\right). \end{aligned} \tag{2.6}$$

Using Lemma 2.3, we obtain that $H(\cdot, \lambda)$ has a fixed point in U for every $\lambda \in [0, 1]$ such that $|\lambda_0 - \lambda| < \delta$. Thus $\lambda \in A$ for any $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1]$, and therefore A is open in $[0, 1]$. \square

As an immediate consequence of the previous theorem, we obtain the following fixed point result of the Leray-Schauder type for weakly contractive maps under the condition (I-C).

Theorem 2.5. *Suppose that U is an open and strictly star shaped subset of a Banach space $(X, \|\cdot\|)$, with $0 \in U$, and that $f : \overline{U} \rightarrow X$ is a weakly contractive map with $f(\overline{U})$ being bounded. If f satisfies the condition (I-C), then f has a fixed point in \overline{U} .*

Proof. Since f satisfies the condition (I-C), there exists $\delta > 0$ such that $f(x) \neq \lambda x$ for $\lambda > 1$ and $x \in U_\delta$ with $f(x) \notin \overline{U}$. We may assume that $x \neq f(x)$ for every $x \in U_\delta$, because otherwise we are finished. Define $H : \overline{U} \times [0, 1] \rightarrow X$ as $H(x, t) = tf(x)$, and let g be the zero map. Notice that g has a fixed point in U , that is, $0 = g(0)$ and also that f and g are two weakly contractive mappings. So, the result will follow from Theorem 2.4 once we prove that f is (I-C)-homotopic to g . Let us check it.

- (P1) For all $x \in \overline{U}$, $H(x, 0) = 0 \cdot f(x) = 0 = g(x)$ and $H(x, 1) = 1 \cdot f(x) = f(x)$.
- (P2) Since f satisfies the condition (I-C), we have that $f(x) \neq \lambda x$ for $x \in U_\delta$ with $f(x) \notin \overline{U}$ and $\lambda > 1$. Hence, $x \neq H(x, t)$ for every $x \in U_\delta$, with $f(x) \notin \overline{U}$, and $t \in [0, 1]$.
- (P3) Since f is weakly contractive, there exists a compactly positive function $\psi : X \times X \rightarrow (0, \infty)$ such that $d(f(x), f(y)) \leq d(x, y) - \psi(x, y)$ for every $x, y \in \overline{U}$. Then, if $x, y \in \overline{U}$ and $t \in [0, 1]$,

$$\begin{aligned} d(H(x, t), H(y, t)) &= t \|f(x) - f(y)\| \\ &\leq d(f(x), f(y)) \\ &\leq d(x, y) - \psi(x, y). \end{aligned} \quad (2.7)$$

- (P4) Since $f(\overline{U})$ is bounded, there exists $M \geq 0$ such that $\|f(x)\| \leq M$ for all $x \in \overline{U}$. Hence,

$$\begin{aligned} d(H(x, t), H(x, s)) &= \|f(x)\| |t - s| \\ &\leq M |t - s| \\ &= |\phi(t) - \phi(s)|, \end{aligned} \quad (2.8)$$

where $\phi : [0, 1] \rightarrow \mathbb{R}$ is the continuous function defined as $\phi(t) = Mt$.

- (P5) Suppose that for some $x \in \partial U$ and $\lambda < 1$ we have that $H(x, \lambda) \in \partial U$. Then, $f(x) \neq 0$ since $H(x, \lambda) = \lambda f(x)$, $0 \in U$ and U is open. Let us see that $H(x, 1) \notin \overline{U}$: suppose, on the contrary, that $H(x, 1) \in \overline{U}$, that is, $f(x) \in \overline{U}$ and define

$$\hat{\lambda} := \sup \{ t \geq 1 : tf(x) \in \overline{U} \}. \quad (2.9)$$

Then, it is easy to see that $\hat{\lambda}f(x) \in \partial U$, which contradicts that U is strictly star shaped, since we also have that $\lambda f(x) \in \partial U$. \square

3. A Class of Generalized Contractions

A multitude of generalizations and variants of Banach's contractive condition have been given after Banach's theorem (see, e.g., Rhoades [13]) and, recently, Agarwal and O'Regan [11] have given a homotopy result (thus generalizing a fixed point theorem of Hardy and Rogers [14]) under the following generalized contractive condition: there exists $a \in (0, 1)$ such that for all $x, y \in X$

$$d(f(x), f(y)) \leq a \max \left\{ d(x, y), d(x, f(x)), d(y, f(y)), \frac{1}{2} [d(x, f(y)) + d(y, f(x))] \right\}. \quad (3.1)$$

In this section we give a homotopy result for this class of mappings under the condition (I-C). In the proof of our theorem we shall use the following result [11].

Lemma 3.1. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$, and $h : \overline{B(x_0, r)} \rightarrow X$. Suppose that there exists $a \in (0, 1)$ such that for $x, y \in \overline{B(x_0, r)}$ one has*

$$d(h(x), h(y)) \leq a \max \left\{ d(x, y), d(x, h(x)), d(y, h(y)), \frac{1}{2} [d(x, h(y)) + d(y, h(x))] \right\}, \quad (3.2)$$

$$d(x_0, h(x_0)) < (1 - a)r.$$

Then there exists $x \in \overline{B(x_0, r)}$ with $x = h(x)$.

The proof of the following theorem is very similar to the proof of Theorem 2.4, and we give a sketch of it.

Theorem 3.2. *Let (X, d) be a complete metric space, and U an open subset of X . Let $f, g : \overline{U} \rightarrow X$ be two maps such that there exists $H : \overline{U} \times [0, 1] \rightarrow X$ with the following properties:*

- (P1) $H(x, 1) = f(x)$ and $H(x, 0) = g(x)$ for every $x \in \overline{U}$;
- (P2) there exists $\delta > 0$ such that $x \neq H(x, t)$ for every $x \in U_\delta$, with $f(x) \notin \overline{U}$, and $t \in [0, 1]$, where $U_\delta = \{x \in U : \text{dist}(x, \partial U) < \delta\}$;
- (P3) there exists $a \in (0, 1)$ such that for all $x, y \in \overline{U}$ and $\lambda \in [0, 1]$ one has

$$d(H(x, \lambda), H(y, \lambda)) \leq a \max \left\{ d(x, y), d(x, H(x, \lambda)), d(y, H(y, \lambda)), \frac{1}{2} [d(x, H(y, \lambda)) + d(y, H(x, \lambda))] \right\}; \quad (3.3)$$

- (P4) there exists a continuous function $\phi : [0, 1] \rightarrow \mathbb{R}$ such that, for every $x \in \overline{U}$ and $t, s \in [0, 1]$, $d(H(x, t), H(x, s)) \leq |\phi(t) - \phi(s)|$;
- (P5) if $x \in \partial U$ and $0 \leq \lambda < 1$, with $H(x, \lambda) \in \partial U$, then $H(x, 1) \notin \overline{U}$.

If g has a fixed point in U , then f has a fixed point in \overline{U} .

Proof. Suppose that f does not have any fixed point in \overline{U} and consider the nonempty set

$$A = \{\lambda \in [0, 1] : H(x, \lambda) = x \text{ for some } x \in U\}. \quad (3.4)$$

We will arrive to a contradiction by showing that $A = [0, 1]$, and for this we only need prove that A is closed and open in $[0, 1]$.

To show that A is closed in $[0, 1]$, consider a sequence $\{\lambda_n\}$ in A , with $\lambda_n \rightarrow \lambda \in [0, 1]$ as $n \rightarrow \infty$, and show that $\lambda \in A$; that is, that there exists $x_0 \in U$ with $H(x_0, \lambda) = x_0$. To prove that x_0 exists, take any sequence $\{x_n\}$ in U with $x_n = H(x_n, \lambda_n)$, prove that $\{x_n\}$ is Cauchy, and define x_0 as the limit of $\{x_n\}$, as $n \rightarrow \infty$.

That $\{x_n\}$ is a Cauchy sequence, as well as $x_0 = H(x_0, \lambda)$, follows from standard arguments which can be seen in [11, Theorem 3.1]. It remains to show that $x_0 \in U$.

To prove this, suppose that it is not true and arrive to a contradiction as follows: we have that $H(x_0, \lambda) = x_0 \in \overline{U} \setminus U = \partial U$, and also that $0 \leq \lambda < 1$, because f does not have any fixed point in \overline{U} . Then, by (P5) $f(x_0) \notin \partial U$. On the other hand, $f(x_0) = \lim f(x_n) \in \overline{U}$ because $f(x_n) \in \overline{U}$ for n large enough. To be convinced of it, just apply (P2): since $x_0 \in \partial U$, $\{x_n\} \rightarrow x_0$ and $x_n \in U$ for all $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U_\delta$ for all $n \geq n_0$. Then, $f(x_n) \in \overline{U}$ for all $n \geq n_0$ since $x_n = H(x_n, \lambda_n)$.

To prove that A is open argue as in Theorem 2.4, use Lemma 3.1 instead of Lemma 2.3. \square

As an immediate consequence, we obtain the following result, whose proof is omitted because it is analogous to the proof of Theorem 2.5.

Theorem 3.3. *Suppose that U is an open and strictly star shaped subset of a Banach space $(X, \|\cdot\|)$, with $0 \in U$, and that $f : \overline{U} \rightarrow X$ is map with $f(\overline{U})$ being bounded. Assume also that there exists $a \in (0, 1)$ such that for all $x, y \in \overline{U}$ and $\lambda \in [0, 1]$ one has*

$$\begin{aligned} & d(\lambda f(x), \lambda f(y)) \\ & \leq a \max \left\{ d(x, y), d(x, \lambda f(x)), d(y, \lambda f(y)), \frac{1}{2} [d(x, \lambda f(y)) + d(y, \lambda f(x))] \right\}. \end{aligned} \quad (3.5)$$

If f satisfies the condition (I-C), then f has a fixed point in \overline{U} .

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